Iranian Journal of Science & Technology, Transaction A, Vol. 34, No. A4
Printed in the Islamic Republic of Iran, 2010

SUBORBITAL GRAPHS FOR A SPECIAL SUBGROUP OF THE NORMALIZER OF $\Gamma_0(m)$*

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Abstract – In this paper, we find the number of sides of circuits in suborbital graph for the normalizer of $\Gamma_0(m)$ in PSL(2,ℝ), where *m* will be of the form $2p^2$, *p* is a prime and $p \equiv 1 \pmod 4$. In addition, we give a number theoretical result which says that the prime divisors *p* of $2u^2 \pm 2u + 1$ are of the form $p \equiv 1 \pmod 4$.

Keywords – Normalizer, imprimitive action, suborbital graph, circuits

1. INTRODUCTION

Let PSL(2,ℝ) denote the group of all linear fractional $T : z \to \dfrac{az+b}{cz+d}$, where *a, b, c, d* are real and $ad - bc = 1$. The modular group Γ is the subgroup of PSL(2,ℝ) such that a, *b, c* and *d* are integers. For any natural number *m*, $\Gamma_0(m)$ is the subgroup of Γ with $m \mid c$. The elements of PSL(2,ℝ) are represented as

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1.$$

We will omit the symbol ± and identify each matrix with its negative.

$\Gamma_1(m)$ will denote the normalizer of $\Gamma_0(m)$ in PSL(2,ℝ). The elements of $\Gamma_1(m)$ are of the form by [1]

$$\begin{pmatrix} ae & b/h \\ cm/h & de \end{pmatrix}$$

where all letters are integers, $e \,\|\, m/h^2$ and *h* is the largest divisor of 24 for which $h^2 \mid m$ with the understanding that the determinant is *e* > 0, and that $r \,\|\, s$ means that $r \mid s$ and $\left(r, \frac{s}{r}\right) = 1$.

Here, *m* will be $2p^2$, where *p* is a prime such that $p \equiv 1 \pmod 4$. All circuits in suborbital graph for the normalizer of $\Gamma_0(m)$ in PSL(2,ℝ) where *m* is a square-free positive integer was studied in [2, 3].

*Received by the editor February 21, 2009 and in final revised form December 18, 2010
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Our main idea is that we investigate a case in which m is not square-free. Similar studies were done for the modular group and some Hecke groups [4-6]. In this case, h will be 1 and e is 1, 2, p^2 or 2 p^2.

2. THE ACTION OF $\Gamma_1(2p^2)$ ON $\widehat{\mathbb{Q}}$

Any element of $\widehat{\mathbb{Q}}$ can be given as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}.$$

Therefore, the action of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ is identical. If the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 1 and $(x, y) = 1$, then $(ax+by, cx+dy) = 1$. A necessary and sufficient condition for $\Gamma_1(m)$ to act transitively on $\widehat{\mathbb{Q}}$ is given in [7].

Lemma 2.1. Let m be any integer and $m = 2^{\alpha_1}.3^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r}$, the prime power decomposition of m. Then $\Gamma_1(m)$ is transitive on $\widehat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7, \alpha_2 \le 3$ and $\alpha_i \le 1$ for $i = 3, ..., r$.

Corollary 2.2. The action of the normalizer $\Gamma_1(2p^2)$ is not transitive on $\widehat{\mathbb{Q}}$.

Since the action is not transitive on $\widehat{\mathbb{Q}}$ we now find a maximal subset of $\widehat{\mathbb{Q}}$ on which the normalizer acts transitively. First we start with

Lemma 2.3. The orbits of the action of $\Gamma_0(2p^2)$ on $\widehat{\mathbb{Q}}$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix}, ..., \begin{pmatrix} p-1 \\ p \end{pmatrix};$ $\begin{pmatrix} 1 \\ 2p \end{pmatrix}, \begin{pmatrix} 3 \\ 2p \end{pmatrix}, ..., \begin{pmatrix} p-2 \\ 2p \end{pmatrix}, \begin{pmatrix} p+2 \\ 2p \end{pmatrix}, \begin{pmatrix} p+4 \\ 2p \end{pmatrix}, ..., \begin{pmatrix} 2p-1 \\ 2p \end{pmatrix}; \begin{pmatrix} 1 \\ p^2 \end{pmatrix}; \begin{pmatrix} 1 \\ 2p^2 \end{pmatrix}$, where $\begin{pmatrix} x \\ y \end{pmatrix} := \left\{ \frac{k}{l} \in \widehat{\mathbb{Q}} \middle| \ (2p^2, l) = y, x \equiv k\frac{l}{y} mod\left(y, \frac{2p^2}{y}\right) \right\}$.

Proof: It is well known that if $\frac{k}{s} \in \widehat{\mathbb{Q}}$ is given, then there exists some $T \in \Gamma_0(2p^2)$ such that $T\begin{pmatrix} k \\ s \end{pmatrix} = \begin{pmatrix} k_1 \\ s_1 \end{pmatrix}$ with $s_1 \mid 2p^2$. And furthermore, for $d \mid 2p^2$, $\begin{pmatrix} a_1 \\ d \end{pmatrix} = \begin{pmatrix} a_2 \\ d \end{pmatrix}$ if and only if $a_1 \equiv a_2 \bmod \left(d, \frac{2p^2}{d} \right)$. So the result follows.

Lemma 2.4. The orbits of the action of $\Gamma_1(2p^2)$ are as follows. Let $l \in \{1, 2, ..., p-1\}$. Then

(a) If l is odd then

$$\begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} p-l \\ p \end{pmatrix} \cup \begin{pmatrix} l \\ 2p \end{pmatrix} \cup \begin{pmatrix} 2p-l \\ 2p \end{pmatrix}$$

(b) If l is even then

$$\begin{pmatrix} l \\ p \end{pmatrix} \cup \begin{pmatrix} p-l \\ p \end{pmatrix} \cup \begin{pmatrix} p+l \\ 2p \end{pmatrix} \cup \begin{pmatrix} 2p-l+1 \\ 2p \end{pmatrix}$$

(c) $$\begin{pmatrix}1\\1\end{pmatrix}\cup\begin{pmatrix}1\\2\end{pmatrix}\cup\begin{pmatrix}1\\p^2\end{pmatrix}\cup\begin{pmatrix}1\\2p^2\end{pmatrix}$$

Proof: We prove only (a). The rest are similar.

Let $T=\begin{pmatrix}ae & b\\ 2p^2c & de\end{pmatrix}$ be an arbitrary element in $\Gamma_1(2p^2)$. Then e must be $1, 2, p^2$ or $2p^2$.

Case 1. Let e = 1. Then $T\in\Gamma_0(2p^2)$. Therefore T fixes $\begin{pmatrix}l\\p\end{pmatrix}$.

Case 2. Let e = 2. Then $\begin{pmatrix}2a & b\\ 2p^2c & 2d\end{pmatrix}\begin{pmatrix}l\\p\end{pmatrix}=\begin{pmatrix}2al+bp\\ 2p^2cl+2dp\end{pmatrix}$.

Since $\begin{pmatrix}2a & b\\ p^2c & d\end{pmatrix}\begin{pmatrix}l\\p\end{pmatrix}=\begin{pmatrix}2al+bp\\ p^2cl+dp\end{pmatrix}$ and $2ad-p^2bc=1$, we conclude that $(2al+bp, 2p^2cl+2dp)=1$.

Therefore,

$$\begin{pmatrix}2al+bp\\ 2p(pcl+d)\end{pmatrix}=\begin{pmatrix}x\\ 2p\end{pmatrix}, \text{ where } x\equiv(2al+bp)(pcl+d)\bmod p.$$

This shows that $\begin{pmatrix}\ell\\p\end{pmatrix}$ and $\begin{pmatrix}\ell\\2p\end{pmatrix}$ must be in a single orbit of $\Gamma_1(2p^2)$.

Case 3. Let $e=p^2$. Then $T=\begin{pmatrix}ap^2 & b\\ 2p^2c & dp^2\end{pmatrix}$, $adp^4-2p^2bc=p^2$.

$$T\begin{pmatrix}l\\p\end{pmatrix}=\begin{pmatrix}ap^2l+bp\\ 2p^2cl+dp^3\end{pmatrix}=\begin{pmatrix}apl+b\\ 2pcl+dp^2\end{pmatrix}$$

and as in Case 2, $(apl+b, 2pcl+dp^2)=1$. Therefore,

$$T\begin{pmatrix}l\\p\end{pmatrix}=\begin{pmatrix}x\\p\end{pmatrix}, \text{ where } x\equiv(apl+b)(2cl+dp)\bmod p \text{ or } x\equiv 2bcl\,(\bmod p).$$

Since $2bc\equiv-1(\bmod p)$, $x\equiv p-l(\bmod p)$. Therefore $\begin{pmatrix}l\\p\end{pmatrix}$ and $\begin{pmatrix}p-l\\p\end{pmatrix}$ must be in a single orbit of $\Gamma_1(2p^2)$.

Case 4. Let $e=2p^2$. Then we easily find that T sends $\begin{pmatrix}l\\p\end{pmatrix}$ to $\begin{pmatrix}2p-l\\2p\end{pmatrix}$. So we consequently have the orbit $\begin{pmatrix}l\\p\end{pmatrix}\cup\begin{pmatrix}p-l\\p\end{pmatrix}\cup\begin{pmatrix}l\\2p\end{pmatrix}\cup\begin{pmatrix}2p-l\\2p\end{pmatrix}$.

Corollary 2.6. The action of $\Gamma_1(2p^2)$ on $\hat{\mathbb{Q}}(2p^2)=\begin{pmatrix}1\\1\end{pmatrix}\cup\begin{pmatrix}1\\2\end{pmatrix}\cup\begin{pmatrix}1\\p^2\end{pmatrix}\cup\begin{pmatrix}1\\2p^2\end{pmatrix}$ is transitive.

Lemma 2.7. The stabilizer of a point in $\hat{\mathbb{Q}}(2p^2)$ is an infinite cyclic group.

Proof: Since the action is transitive, stabilizers of any two points are conjugate. Therefore, we can only look at the stabilizer of ∞ in $\Gamma_1(2p^2)$.

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}ae & b\\ 2p^2c & de\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}ae\\ 2p^2c\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},$$

then $c=0$. In this case $e=1$ and since $ad=1$, $T=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$. This shows that stabilizer $\left(\Gamma_1(2p^2)\right)_\infty$ of ∞ is $\left\langle\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right\rangle$.

We know from [7] (see also [8]) that the orders of the elliptic elements of $\Gamma_1(2p^2)$ may be 2, 3, 4, or 6. Therefore, we give the following:

Lemma 2.8. Let *p* be a prime and $p\equiv 1(\text{mod } 4)$. Then the normalizer $\Gamma_1(2p^2)$ contains an elliptic element *E* of order 4 and that *E* is of the form $\begin{pmatrix}2a & b\\ 2p^2c & 2(1-a)\end{pmatrix}$, det *E* = 2.

Let (G,X) be transitive permutation group, and suppose that *R* is an equivalence relation on *X*. *R* is said to be *G*-invariant if $(x,y)\in R$ implies $(g(x),g(y))\in R$ for all $g\in G$. The equivalence classes of a G-invariant relation are called *block*s. We give the following from [9].

Lemma 2.9. Suppose that (G,X) is a transitive permutation group, and *H* is a subgroup of *G* such that, for some $x\in X$, $G_x\subset H$. Then $R=\{(g(x),gh(x)) : g\in G, h\in H\}$ is an equivalence relation.

Lemma 2.10. Let (G,X) be a transitive permutation group, and ≈ the *G*-invariant equivalence relation defined in Lemma 2.9; then $g_1(\alpha)=g_2(\alpha)$ if and only if $g_1\in g_2H$. Furthermore, the number of blocks is $|G:H|$.

To apply the ideas, we take $\left(\Gamma_1(2p^2),\hat{\mathbb{Q}}(2p^2)\right)$, $\left\langle\Gamma_0(2p^2),\begin{pmatrix}2a & b\\ 2p^2c & 2(1-a)\end{pmatrix}\right\rangle$ and the stabilizer $\left(\Gamma_1(2p^2)\right)_\infty$ of ∞ in $\Gamma_1(2p^2)$ instead of (G,X), *H* and G_x. In this case the number of blocks is 2 and these blocks are

$$[\infty]:=\begin{pmatrix}1\\ p^2\end{pmatrix}\cup\begin{pmatrix}1\\ 2p^2\end{pmatrix} \text{ and } [0]:=\begin{pmatrix}1\\ 1\end{pmatrix}\cup\begin{pmatrix}1\\ 2\end{pmatrix}.$$

3. SUBORBITAL GRAPHS OF $\Gamma_1(2p^2)$ ON $\hat{\mathbb{Q}}(2p^2)$

Let (G,X) be a transitive permutation group. Then *G* acts on $X\times X$ by

$$g(\alpha,\beta)=(g(\alpha),g(\beta)),\ (g\in G;\ \alpha,\beta\in X).$$

The orbits of this action are called suborbitals of the normalizer *G*. The orbit containing (α,β) is denoted by $O(\alpha,\beta)$. From $O(\alpha,\beta)$ we can form a suborbital graph $G(\alpha,\beta)$: its vertices are the elements of *X*, and there is a directed edge from γ to δ if $(\gamma,\delta)\in O(\alpha,\beta)$. A directed edge from γ to δ is denoted by $\gamma\to\delta$. If $(\gamma,\delta)\in O(\alpha,\beta)$, then we will say that there exists an edge $\gamma\to\delta$ in $G(\alpha,\beta)$.

If $\alpha=\beta$, the corresponding suborbital graph $G(\alpha,\alpha)$, called the trivial suborbital graph, is self-paired: it consists of a loop based at each vertex $x\in X$. We will mainly be interested in the remaining non-trivial suborbital graphs. These ideas were first introduced by Sims [10].

We now investigate the suborbital graphs for the action of $\Gamma_1(2p^2)$ on $\hat{\mathbb{Q}}(2p^2)$. Since the action of $\Gamma_1(2p^2)$ on $\hat{\mathbb{Q}}(2p^2)$ is transitive, $\Gamma_1(2p^2)$ permutes the blocks transitively; so the subgraphs are all isomorphic. Hence, it is sufficient to study with only one block. On the other hand, it is clear that each non-trivial suborbital graph contains a pair $\left(\infty, u/p^2\right)$ for some $u/p^2 \in \hat{\mathbb{Q}}(2p^2)$. Therefore, we work on the following case: We denote by $F\left(\infty, u/p^2\right)$ the subgraph of $G\left(\infty, u/p^2\right)$ such that its vertices are in the block $[\infty]$.

Theorem 3.1. Let r/s and x/y be in the block $[\infty]$. Then there is an edge $r/s \to x/y$ in $F\left(\infty, u/p^2\right)$ if and only if

(i) If $p^2 \mid s$ but $2p^2 \nmid s$, then $x \equiv \pm 2ur \left(\bmod p^2\right)$, $y \equiv \pm 2us \left(\bmod 2p^2\right)$, $ry - sx = \pm p^2$

(ii) If $2p^2 \mid s$, then $x \equiv \pm ur \left(\bmod p^2\right)$, $y \equiv \pm us \left(\bmod p^2\right)$, $ry - sx = \pm p^2$.

Proof: Assume first that $r/s \to x/y$ is an edge in $F\left(\infty, u/p^2\right)$ and that $p^2 \mid s$ but $2p^2 \nmid s$. Therefore, there exists some T in the normalizer $\Gamma_1(2p^2)$ such that T sends the pair $\left(\infty, u/p^2\right)$ to the pair $\left(r/s, x/y\right)$, that is $T(\infty) = r/s$ and $T\left(u/p^2\right) = x/y$. Since $2p^2 \nmid s$, T must be of the form $\begin{pmatrix} 2a & b \\ 2p^2c & 2d \end{pmatrix}$. $T(\infty) = \dfrac{2a}{2p^2c} = \begin{pmatrix} (-1)^i r \\ (-1)^i s \end{pmatrix}$ gives that $r = (-1)^i a$ and $s = (-1)^i p^2 c$, for $i = 0,1$.

$$T\left(u/p^2\right) = \begin{pmatrix} 2a & b \\ 2p^2c & 2d \end{pmatrix}\begin{pmatrix} u \\ p^2 \end{pmatrix} = \begin{pmatrix} 2au + bp^2 \\ 2p^2cu + 2dp^2 \end{pmatrix} = \begin{pmatrix} (-1)^j x \\ (-1)^j y \end{pmatrix} \text{ for } j = 0,1.$$

Since the matrix $\begin{pmatrix} 2a & b \\ p^2c & d \end{pmatrix}$ has determinant 1 and $(u, p^2) = 1$, then $(2au + bp^2, p^2cu + dp^2) = 1$. And therefore, $(2au + bp^2, 2p^2cu + 2dp^2) = 1$. So

$$x = (-1)^j (2au + bp^2), y = (-1)^j (2p^2cu + 2dp^2).$$

That is, $x \equiv (-1)^{i+j} 2au \left(\bmod p^2\right)$, $y \equiv (-1)^{i+j} 2su \left(\bmod 2p^2\right)$. Finally, since

$$\begin{pmatrix} 2a & b \\ 2p^2c & 2d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} (-1)^i 2r & (-1)^j x \\ (-1)^i 2s & (-1)^j y \end{pmatrix}, \text{ for } i,j = 0,1,$$

we get $ry - sx = \pm p^2$. This proves (i).

Secondly, let $r/s \to x/y$ be an edge in $F\left(\infty, u/p^2\right)$ and $2p^2 \mid s$. In this case T must be of the form $\begin{pmatrix} a & b \\ 2p^2c & d \end{pmatrix}$, det T=1. Therefore, since $T(\infty) = \begin{pmatrix} a \\ 2p^2c \end{pmatrix} = \begin{pmatrix} (-1)^i r \\ (-1)^i s \end{pmatrix}$ we get $a = r$ and $s = 2p^2c$, by taking i to be 0. Likewise, since

$$\begin{pmatrix} a & b \\ 2p^2c & d \end{pmatrix}\begin{pmatrix} u \\ p^2 \end{pmatrix} = \begin{pmatrix} au + bp^2 \\ 2p^2cu + dp^2 \end{pmatrix} = \begin{pmatrix} (-1)^j x \\ (-1)^j y \end{pmatrix},$$

we have $x \equiv ur\left(\bmod p^2\right)$ and $y \equiv us\left(\bmod p^2\right)$ and that $ry - sx = p^2$. In the case where $i = 0$ and $j = 1$, the minus sign holds.

In the opposite direction we do calculations only for (i) and the plus sign. The other are likewise done. So suppose $x \equiv 2ur \left(\bmod p^2\right)$, $y \equiv 2us \left(\bmod 2p^2\right)$, $ry - sx = p^2$, $p^2 \mid s$ and $2p^2 \nmid s$. Therefore there exists b, d in $\mathbb{Z}$ such that $x = 2ur + p^2 b$ and $y = 2us + 2p^2 d$. Since $ry - sx = p^2$, we get $2rd - bs = 1$, or $4rd - bs = 2$. Hence the element $T := \begin{pmatrix} 2r & b \\ 2s & 2d \end{pmatrix}$ is not only in the normalizer $\Gamma_1(2p^2)$, but also *H*. It is obvious that $T(\infty) = \begin{pmatrix} r \\ s \end{pmatrix}$ and $T\begin{pmatrix} u \\ p^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Theorem 3.2. If we present edges of $F\left(\infty, u/p^2\right)$ as hyperbolic geodesics in the upper half-plane $\mathbb{H}$, no edges of the subgraph $F\left(\infty, u/p^2\right)$ of $\Gamma_1(2p^2)$ cross in $\mathbb{H}$.

Proof: Without loss of generality, since the action on $\hat{\mathbb{Q}}(2p^2)$ is transitive, suppose that $\infty \to u/p^2$, $x_1/y_1p^2 \to x_2/y_2p^2$ and $x_1/y_1p^2 < u/p^2 < x_2/y_2p^2$, where all letters are positive integers. Since $x_1/y_1p^2 \to x_2/y_2p^2$ and $x_1/y_1p^2 < u/p^2 < x_2/y_2p^2$, then $x_1y_2 - x_2y_1 = -1$ and $x_1/y_1 < u < x_2/y_2$, respectively. Therefore

$$\left(x_1/y_1\right) - \left(x_2/y_2\right) < u - \left(x_2/y_2\right) < 0.$$

Then $\left(x_1y_2 - x_2y_1\right)/y_1y_2 < \left(uy_2 - x_2\right)/y_2 < 0$. So $-1/y_2 < uy_2 - x_2 < 0$, a contradiction [11].

4. THE NUMBER OF SIDES OF CIRCUITS

Let (G, X) be a transitive permutation group and $G(\alpha, \beta)$ be a suborbital graph. By a directed circuit in $G(\alpha, \beta)$, we mean a sequence $v_1 \to v_2 \to \ldots \to v_m \to v_1$, where $m \geq 3$; an anti-directed circuit will denote a configuration like the above with at least one arrow (not all) reversed. If $m = 2, 3$ or 4 then the circuit, directed or not, is called a self-paired, a triangle or a rectangle, respectively.

Theorem 4.1. $F\left(\infty, u/p^2\right)$ has a self-paired edge if and only if $2u^2 \equiv -1 \left(\bmod p^2\right)$.

Proof: Without loss of generality, from transitivity, we can suppose that the self-paired edge be $\frac{1}{0} \to \frac{u}{p^2} \to \frac{1}{0}$. Applying Theorem 3.1, the proof then follows.

Theorem 4.2. $F\left(\infty, u/p^2\right)$ contains no triangles.

Proof: Suppose contrary $F\left(\infty, u/p^2\right)$ contains a triangle. From transitivity and Theorem 3.1 the form of such a triangle $\frac{1}{0} \to \frac{u}{p^2} \to \frac{x}{2p^2} \to \frac{1}{0}$. But, to be $\frac{x}{2p^2} \to \frac{1}{0}$ gives a contradiction to Theorem 3.1(ii).

Theorem 4.3. The normalizer $\Gamma_1(2p^2)$ does not contain period 3.

Proof: Suppose the converse that $\Gamma_1(2p^2)$ does have a period 3. Then it has an elliptic element *T* of order 3. *T* must be of the form $\begin{pmatrix} a & b \\ 2p^2c & d \end{pmatrix}$, $\det T = 1$ and $a + d = \pm 1$. Take $a + d = 1$. Then $a + d = 1 \left(\bmod 2p^2\right)$, and since $a + d = 1$, then $a\left(1 - a\right) = 1 \left(\bmod 2p^2\right)$, or $a^2 - a + 1 = 0 \left(\bmod 2p^2\right)$, which is a contradiction.

Theorem 4.4. The subgraph $F\left(\infty, u/p^2\right)$ contains a rectangle if and only if $2u^2 \pm 2u + 1 \equiv 0 \left(\bmod p^2\right)$.

Proof: Assume first that $F\left(\infty, u/p^2\right)$ has a rectangle $\frac{k_0}{l_0} \to \frac{m_0}{n_0} \to \frac{s}{t} \to \frac{x_0}{y_0} \to \frac{k_0}{l_0}$. It can be easily shown that *H* permutes the vertices and edges of $F\left(\infty, u/p^2\right)$ transitively. Therefore we suppose that the above rectangle is transformed under *H* to the rectangle $\frac{1}{0} \to \frac{m}{p^2} \to \frac{x}{y} \to \frac{k}{l} \to \frac{1}{0}$.

Furthermore, without loss of generality, suppose $\frac{m}{p^2} < \frac{x}{y} < \frac{k}{l}$. From the first edge and Theorem 3.1 we get $m \equiv u \left(\bmod p^2\right)$. The second edge gives $x \equiv -2um \left(\bmod p^2\right)$ and $2ym - x = -1$; and that from the third edge we have $k \equiv -ux \left(\bmod p^2\right)$ and $x - 2ky = -1$. If we combine these we obtain

$$2u^2 + 2ym + 1 \equiv 0 \left(\bmod p^2\right) \text{ or } 2u^2 + 2uy + 1 \equiv 0 \left(\bmod p^2\right).$$

Since $x = 2ym + 1 = 2ky - 1$, then $y(m-k) = -1$. This gives that *y*=1. Therefore $2u^2 + 2u + 1 \equiv 0 \left(\bmod p^2\right)$.

If $\frac{m}{p^2} > \frac{x}{y} > \frac{k}{l}$ holds then we conclude that $2u^2 - 2u + 1 \equiv 0 \left(\bmod p^2\right)$, and furthermore, if $2u^2 - 2u + 1 \equiv 0 \left(\bmod p^2\right)$ then we get the rectangle

$$\frac{1}{0} \to \frac{u}{p^2} \to \frac{2u-1}{2p^2} \to \frac{u-1}{p^2} \to \frac{1}{0}.$$

Secondly suppose that $2u^2 \pm 2u + 1 \equiv 0 \bmod p^2$. Then, using Theorem 3.1, we see that $\frac{1}{0} \to \frac{u}{p^2} \to \frac{2u \pm 1}{2p^2} \to \frac{u \pm 1}{p^2} \to \frac{1}{0}$ is a rectangle.

As an example, $\infty \to 3/25 \to 7/50 \to 4/25 \to \infty$ is a rectangle in $G\left(\infty, 3/25\right)$.

Corollary 4.5. For some *u* in ℤ, $F\left(\infty, u/p^2\right)$ contains a rectangle if and only if the group *H* has a period 4.

Proof: Firstly suppose $F\left(\infty, u/p^2\right)$ contains a rectangle. Then, Theorem 4.4 shows that $2u^2 \pm 2u + 1 \equiv 0 \left(\bmod p^2\right)$. So we have the elliptic element $\begin{pmatrix} -2u & \frac{2u^2 \pm 2u + 1}{p^2} \\ -2p^2 & 2u \pm 2 \end{pmatrix}$ of order 4 in *H*. Since the index of *H* is 2 in $\Gamma_1(2p^2)$, the elements of this form must be in *H*.

Conversely, suppose that *H* has a period for order 4, so *H* contains an elliptic element of order 4. Let this element be $\begin{pmatrix} 2a & b \\ 2p^2 & -2a \pm 2 \end{pmatrix}$, det = 2. From this we get $p^2 \mid (2u^2 \pm 2u + 1)$. Therefore $F\left(\infty, u/p^2\right)$ contains a rectangle.

We predict from the above lemmas that the elliptic elements of $\Gamma_1(2p^2)$ correspond to the circuit in $F\left(\infty, u/p^2\right)$. To support this idea we have

Theorem 4.6. The set $H \backslash \Gamma_0(2p^2)$ has a period for order 2 if and only if there exists some $u \in \mathbb{Z}$, $(u, p) = 1$ such that $F\left(\infty, u/p^2\right)$ has a self-paired edge.

Proof: First suppose that the set has such an elliptic element *T*. Then *T* must be of the form $\begin{pmatrix} 2a & -b \\ 2p^2 & -2a \end{pmatrix}$, det *T* = 2. Therefore we have $2a^2+1 \equiv 0 \pmod{p^2}$. So, Theorem 3.1 shows that $\frac{1}{0} \to \frac{a}{p^2} \to \frac{1}{0}$ is a self-paired edge in $F\left(\infty, u/p^2\right)$.

Secondly, let $F\left(\infty, u/p^2\right)$ have a self-paired edge. Without loss of generality, from transitivity, we can suppose that the self-paired edge be $\frac{1}{0} \to \frac{u}{p^2} \to \frac{1}{0}$. So we have, by Theorem 3.1, $2u^2 \equiv -1 \pmod{p^2}$. This showes that there exists some $b \in \mathbb{Z}$ such that $b = \frac{-(2u^2+1)}{p^2}$. Therefore $\begin{pmatrix} 2a & -b \\ 2p^2 & -2a \end{pmatrix}$ is an elliptic element of order 2 in the set $H \setminus \Gamma_0(2p^2)$.

Notice that $H \setminus \Gamma_0(2 \cdot 5^2)$ has no period for order 2, and therefore $F\left(\infty, u/25\right)$ does not have a self-paired edge.

Finally, as a finishing point, we give a number theoretical result as follows:

Theorem 4.7. The prime divisors *p* of $2u^2+2u+1$, for any $u \in \mathbb{Z}$, are of the form $p \equiv 1 \pmod 4$.

Proof: Let *u* be any integer and *p* a prime divisor of $2u^2+2u+1$. Then, without any difficulty, it can be easily seen that the normalizer $\Gamma_1(2p)$, like $\Gamma_1(2p^2)$, has the elliptic element $\begin{pmatrix} -2u & \frac{2u^2+2u+1}{p} \\ -2p & 2u+2 \end{pmatrix}$ of order 4. From Lemma 2.8 we get that $p \equiv 1 \pmod 4$.

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