Analysis of Time Delay Systems Via New Triangular Functions

Sayyed Mohammad Hoseini [*], Khosro Soleimani
Department of Mathematics, Engineering School, Islamic Azad University, NajafAbad Branch, NajafAbad, Iran.

Abstract

This paper presents a numerical method for finding the solution of time-delay systems using triangular functions. We present the properties of the triangular functions. The operational matrices of integration and delay are utilized to reduce the solution of time-delay systems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Delay System, Numerical Method, Triangular Function, Operational Matrix.

1 Introduction

Analysis, identification and optimal control of systems with time-delay has been of considerable concern. Delays occur frequently in biological, chemical, transportation, electronic, communication, manufacturing and power systems [7]. Time-delay and multi-delay systems are therefore very important classes of systems whose control and optimization have been of interest to many investigators [1].

The available sets of orthogonal functions can be divided into three classes. The first includes set of piecewise constant basis functions (PCBFs) (e.g., Walsh, block-pulse, etc.). The second consists of a set of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used set of sine–cosine functions in Fourier series. While orthogonal polynomials and sine–cosine functions together form a class of continuous basis functions, PCBFs have inherent discontinuities or jumps. It is worth mentioning that approximating a continuous function with PCBFs results in an approximation that is piecewise constant. On the other hand, if a discontinuous function is approximated by continuous basis functions, the discontinuities are not properly modeled. Signals frequently have mixed features of continuity and jumps. These signals are continuous over certain segments of time, with discontinuities or jump occurring at the transitions of the segments. In such situations, neither the continuous basis functions nor PCBFs taken alone would form an efficient basis in the representation of such signals.

Much progress has been made towards the solution of delay systems by using orthogonal functions. The approach is to convert the delay-differential equation to an algebraic form through the use of operational matrices of integration and delay. These

[*]Corresponding author

matrices can be uniquely determined based on the particular choices of basis functions. Special attention has been given to applications of Walsh functions [2], block pulse functions [15], Laguerre polynomials [9], Legendre polynomials [11], Chebyshev polynomials [14], Haar wavelets [6] and Fourier series [5]. In general, the computed response of the delay systems via orthogonal functions is not in good agreement with the exact response of the system [3]. The superiority of the chosen orthogonal function approach depends on the nature of signals to be dealt with, as certain class of orthogonal functions fit certain signals more accurately than others [13]. Moreover, Marzban and Razzaghi [12] used hybrid functions of block-pulse and Legendre polynomials and obtained an excellent results for the solution of delay systems for the cases where the exact solution are of the form of polynomials in different intervals.

In the present paper, we introduce a new direct computational method to solve time-delay systems. The method consists of reducing the delay problem to a set of algebraic equations by first expanding the candidate function as a triangular functions with unknown coefficients. These triangular functions, which evolved from a simple dissection of block pulse functions are first introduced. The operational matrices of integration and delay are given. These matrices are then used to evaluate the coefficients of the triangular functions for the solution of delay systems.

The paper is organized as follows: in Section 2, we describe the basic properties of the block pulse functions and triangular functions required for our subsequent development. In this section, we drive the delay operational matrix of the triangular functions. Section 3 is devoted to the formulation of linear time-delay systems. In Section 4, we apply the proposed numerical method to delay systems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2 Review of Block Pulse Functions and The Triangular Functions

A set of block pulse functions $\Psi(t)$ containing m component functions in the interval $[0,t_f)$ is given by [8]

$$\Psi(t)=[\psi_0(t),\psi_1(t),\ldots,\psi_{m-1}(t)]^T,$$

where $[\ldots]^T$ denotes transpose and

$$\psi_i(t)=\begin{cases}1, & i\frac{t_f}{m}\le t<(i+1)\frac{t_f}{m},\\ 0, & \text{otherwise,}\end{cases}$$

for $i=0,1,\ldots,m-1$.

The block pulse functions are orthogonal in the interval $[0,t_f)$ and we have

$$\int_0^{t_f}\psi_i(t)\psi_j(t)\,dt=h\delta_{ij},\quad t\in[0,t_f),$$

where δ_{ij} is the Kronecker delta and $h=\frac{t_f}{m}$. A square integrable time function $f(t)$ of Lebesgue measure may

be expanded into an m-term block pulse functions series in $t\in[0,t_f)$ as

$$f(t)\approx[c_0,c_1,\ldots,c_{m-1}]\Psi(t)=C^T\Psi(t), \tag{1}$$

where

$$c_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)dt, \tag{2}$$

The operational matrix for integration of block pulse functions, has been derived as following upper triangular matrix [8]

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3}$$

where P is a $m\times m$ matrix and performs as an integrator in the block pulse functions domain and it is pivotal in any block pulse functions domain analysis. Thus, approximate integration of a function $f(t)$ using Eqs. (1) and (3) is

$$\int_0^t f(s)ds \approx C^T P\Psi(t).$$

Though block pulse functions are effective for analysis and synthesis of various control systems, it is not a baseless hunch to think that a staircase solution provided by the block pulse functions domain analysis may introduce relatively more error than an equivalent piecewise linear solution.

Now, we demonstrate the construction of triangular functions according to [4]

$$\psi_i(t) = T1_i(t) + T2_i(t),$$

where

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

For the whole set of block pulse function, $\Psi(t)$, we can thus generate two sets of orthogonal triangular functions, namely $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ such that

$$\Psi(t) = \mathbf{T1}(t) + \mathbf{T2}(t).$$

It could be said that these two sets are complementary to each other as far as block pulse functions are considered. For convenience, we call $\mathbf{T1}(t)$ the left-handed triangular function (LHTF) vector and $\mathbf{T2}(t)$ the right-handed triangular function (RHTF) vector [4].

A square integrable time function $f(t)$ of lebesgue measure may be expanded into an m-term triangular functions series $t \in [0, t_f)$ as [4]

$$\begin{aligned} f(t) &\approx [c_0, c_1, \ldots, c_{m-1}]\, \mathbf{T1}(t) + [d_0, d_1, \ldots, d_{m-1}]\, \mathbf{T2}(t) \\ &= C^T\, \mathbf{T1}(t) + D^T\, \mathbf{T2}(t), \end{aligned} \tag{4}$$

where the constant coefficient c_i's and d_i's are given by

$$c_i = f(ih), \quad d_i = f((i+1)h). \tag{5}$$

The following relation between the coefficients are also noted

$$c_{i+1} = d_i, \quad i = 0, 1, \ldots, m-1.$$

The advantage of choosing the coefficients as different samples of $f(t)$ for obtaining a piecewise linear solution, instead of conventional integration formula, is obvious. It is apparent from Eqs. (2) and (5) that unlike block pulse functions, the triangular functions representation does not need any integration to evaluate the coefficients, thereby reducing a lot of computational burden.

The orthogonality of LHTF set (similarly RHTF set) is resulted from mutually disjointness of LHTF (and RHTF), i. e. for $i, j = 0, 1, \ldots, m-1$, [4]

$$\int_0^{t_f} T1_i(t) T1_j(t)\, dt = \begin{cases} \frac{h}{3}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$\int_0^{t_f} T2_i(t) T2_j(t)\, dt = \begin{cases} \frac{h}{3}, & i = j, \\ 0, & i \neq j, \end{cases}$$

The operational matrix for integration can be obtained as [4]

$$\int_0^t \mathbf{T1}(s)\, ds \approx P_1 \mathbf{T1}(t) + P_2 \mathbf{T2}(t), \tag{6}$$

$$\int_0^t \mathbf{T2}(s)\, ds \approx P_1 \mathbf{T1}(t) + P_2 \mathbf{T2}(t), \tag{7}$$

where

$$P_1 = \frac{h}{2} \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$P_2 = \frac{h}{2}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

we call the matrices P_1 and P_2, the operational matrices for integration in triangular functions domain. Where P_1 and P_2 are $m \times m$ matrices and performs as an integrator in the triangular functions domain. Thus approximation integration of a function $f(t)$ using Eqs. (4), (6) and (7) is

$$\int_0^t f(s)ds \approx (C^T + D^T)(P_1\mathbf{T1}(t) + P_2\mathbf{T2}(t)),$$

It is easy to see that

$$P = P_1 + P_2.$$

2.1 The Delay Operational Matrix of Triangular Functions

The delay functions $\mathbf{T1}(t-\tau)$ and $\mathbf{T2}(t-\tau)$ are the shift of the functions $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$. The general expression is given by

$$\mathbf{T1}(t-\tau) = D\mathbf{T1}(t) \quad t > \tau, \tag{8}$$
$$\mathbf{T2}(t-\tau) = D\mathbf{T2}(t) \quad t > \tau, \tag{9}$$

where D is the delay operational matrices of triangular functions. We now derive the delay operational matrix as follows:

It is easily to see that if the delay term, τ, be multiplier of $h = \frac{t_f}{m}$ such as

$$\tau = kh = k\frac{t_f}{m}, \quad k = 1,2,\dots,m-1,$$

then

$$\begin{aligned} T1_i(t-\tau) &= T1_i(t-kh) \\ &= \begin{cases} 1 - \frac{t-kh-ih}{h}, & ih \le t-kh < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} 1 - \frac{t-(k+i)h}{h}, & (i+k)h \le t < (i+k+1)h, \\ 0, & \text{otherwise}, \end{cases} \\ &= T1_{i+k}(t), \end{aligned}$$

and similarly

$$T2_i(t-\tau) = T2_{i+k}(t).$$

Therefor we have

$$D = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{10}$$

where the first 1 on the first row is in the $(k+1)$ th column. But in other cases of τ we choose N in the following manner [12]

$$N = \begin{cases} \dfrac{t_f}{\tau}, & \text{if } \dfrac{t_f}{\tau} \text{ is an integer number,} \\ \left[\dfrac{t_f}{\tau}\right] + 1, & \text{otherwise,} \end{cases} \tag{11}$$

where (11) denotes greatest integer value. furthermore, first by using Eq. (11) we determine N. Thus we have different intervals given by

$$[0,\tau] \quad [\tau,2\tau] \quad \cdots \quad [(N-1)\tau, N\tau],$$

where $N\tau \geq t_f$. Then we choose m to be

$$m = kN \quad k = 1,2,\ldots \tag{12}$$

so

$$\tau = k\frac{N\tau}{m}.$$

Therefore D can be computed as Eq. (10).

3 Problem Statement

consider the following linear time--delay system

$$\dot{X}(t) = EX(t) + FX(t-\tau) + GU(t), \quad 0 \leq t < t_f, \tag{13}$$

$$X(0) = X_0, \tag{14}$$

$$X(t) = \phi(t), \qquad -\tau \leq t < 0 \tag{15}$$

where $X(t)\in\Re^{l}, U(t)\in\Re^{q}, E, F$ and G are constant matrices of appropriate dimensions, X_0 is a constant specified vector, and $\phi(t)$ is an arbitrary known function. The problem is to find $X(t), 0\le t<t_f$, satisfying Eqs. (13) – (15).

4 Approximation Using Triangular Functions

we approximate Eq. (13) as follows:

Let

$$X(t) = [x_1(t),\ldots,x_l(t)]^T, \tag{16}$$
$$U(t) = [u_1(t),\ldots,u_q(t)]^T, \tag{17}$$
$$\hat{T}1(t) = I_l\otimes\mathbf{T1}(t),\quad \hat{T}2(t)=I_l\otimes\mathbf{T2}(t), \tag{18}$$
$$\hat{B}1(t) = I_q\otimes\mathbf{T1}(t),\quad \hat{B}2(t)=I_q\otimes\mathbf{T2}(t), \tag{19}$$

where I_l and I_q are the $l-$ and $q-$dimensional identity matrices and $\otimes$ denotes Kronecker product [10]. $\hat{T}1(t)$ and $\hat{T}2(t)$ are $lm\times l$ matrices and $\hat{B}1(t)$ and $\hat{B}2(t)$ are $qm\times q$ matrices. By using Eq. (4) each of $x_i(t)$ and each of $u_j(t), i=1,2,\ldots,l, j=1,2,\ldots,q$ can be written in terms of triangular functions as

$$x_i(t) = X_{1i}^T\mathbf{T1}(t)+X_{2i}^T\mathbf{T2}(t), \tag{20}$$
$$u_j(t) = U_{1j}^T\mathbf{T1}(t)+U_{2j}^T\mathbf{T2}(t), \tag{21}$$

Then from Eqs. (16) and (17) we get

$$X(t) = X_1^T\hat{T}1(t)+X_2^T\hat{T}2(t), \tag{22}$$
$$U(t) = U_1^T\hat{B}1(t)+U_2^T\hat{B}2(t), \tag{23}$$

where

$$X_1 = [X_{11},X_{12},\ldots,X_{1l}]^T,\quad X_2 = [X_{11},X_{12},\ldots,X_{1l}]^T,$$
$$U_1 = [U_{11},U_{12},\ldots,U_{1q}]^T,\quad U_2 = [U_{21},U_{22},\ldots,U_{2q}]^T,$$

Similarly we have

$$X(0) = d_1^T\hat{T}1(t)+d_2^T\hat{T}2(t), \tag{24}$$
$$\phi(t-\tau) = R_1^T\hat{T}1(t)+R_2^T\hat{T}2(t), \tag{25}$$

We can also write $X(t-\tau)$ in terms of triangular functions as

$$X(t-\tau)=\begin{cases}R_1^T\hat{T}1(t)+R_2^T\hat{T}2(t), & 0\le t<\tau,\\ X_1^T\hat{D}\hat{T}1(t)+X_2^T\hat{D}\hat{T}2(t), & \tau\le t<t_f,\end{cases} \tag{26}$$

where

$$\hat{D} = I_l \otimes D,$$

and D is delay operational matrix given in Eq. (10). Moreover

$$\int_0^t \hat{T}1(s)ds = \hat{P}_1\hat{T}1(t) + \hat{P}_2\hat{T}2(t), \tag{27}$$

$$\int_0^t \hat{T}2(s)ds = \hat{P}_1\hat{T}1(t) + \hat{P}_2\hat{T}2(t), \tag{28}$$

$$\int_0^t X(s-\tau)ds = \begin{cases} (R_1^T + R_2^T)\left(\hat{P}_1\hat{T}1(t) + \hat{P}_2\hat{T}2(t)\right), & 0 \le t < \tau \\ R_1^T\hat{D}Z_1\hat{T}1(t) + R_2^T\hat{D}Z_2\hat{T}2(t) + \\ \left(X_1^T + X_2^T\right)\hat{D}\left(\hat{P}_1\hat{T}1(t) + \hat{P}_2\hat{T}2(t)\right), & \tau \le t < t_f \end{cases} \tag{29}$$

where

$$\hat{P}_1 = I_L \otimes P_1, \quad \hat{P}_2 = I_L \otimes P_2,$$

and P_1 and P_2 are operational matrices of integration given in Eqs.(6) -- (7) and

$$\int_0^\tau \hat{T}1(t)dt = Z_1\hat{T}1(t), \quad \int_0^\tau \hat{T}2(t)dt = Z_2\hat{T}2(t), \tag{30}$$

where Z_1 and Z_2 are constant matrices of order $lm \times lm$.
By integrating Eq.(13) from 0 to t and using Eqs. (16) -- (30) we have

$$\begin{cases} X_1^T - d_1^T = \left(E\left(X_1^T + X_2^T\right) + F\left(R_1^T + R_2^T\right) + F\left(X_1^T + X_2^T\right)\hat{D} + G\left(U_1^T + U_2^T\right)\right)\hat{P}_1 + FR_1^T\hat{D}Z_1 \\ X_2^T - d_2^T = \left(E\left(X_1^T + X_2^T\right) + F\left(R_1^T + R_2^T\right) + F\left(X_1^T + X_2^T\right)\hat{D} + G\left(U_1^T + U_2^T\right)\right)\hat{P}_2 + FR_2^T\hat{D}Z_2 \end{cases} \tag{31}$$

by solving the set of linear algebraic equations, Eq. (31) we obtain the coefficients vectors X_1 and X_2.

Furthermore, first by using Eq. (11) we determine N. Then we choose m, using Eq. (12). When selecting k we first choose an arbitrary number depending on the problem. We evaluate the results for two consecutive k for different t in $[0, t_f)$ until the results are similar up to a required number of decimal places.

5 Illustrative Examples

In this section three examples are given to demonstrate the applicability and accuracy of the method.

Example 5.1

Consider the delay system described by [4]

$$\dot{x}(t) = 4x\left(t-\frac{1}{4}\right), \quad 0 \le t \le 1, \tag{32}$$

$$x(0) = 1, \tag{33}$$

$$x(t) = 0, \qquad -\frac{1}{4} \le t < 0, \tag{34}$$

The exact solution is [4]

$$x(t) = \begin{cases} 1, & 0 \le t < \frac{1}{4}, \\ 1+4\left(t-\frac{1}{4}\right), & \frac{1}{4} \le t < \frac{1}{2}, \\ 1+4\left(t-\frac{1}{4}\right)+8\left(t-\frac{1}{2}\right)^2, & \frac{1}{2} \le t < \frac{3}{4}, \\ 1+4\left(t-\frac{1}{4}\right)+8\left(t-\frac{1}{2}\right)^2+\frac{32}{3}\left(t-\frac{3}{4}\right)^3, & \frac{3}{4} \le t \le 1, \end{cases}$$

Here, we solve the delay problem by using the triangular functions. Since $\tau = \frac{1}{4}$, by using Eq. (11) we select $N = 4$, also we choose $k = 1$.

Let

$$x(t) = X_1^T \mathbf{T1}(t) + X_2^T \mathbf{T2}(t), \tag{35}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are LHTF vector and RHTF vector, respectively.

By expanding $x(0)$ in terms of triangular functions we get

$$x(0) = [1,1,1,1]\mathbf{T1}(t) + [1,1,1,1]\mathbf{T2}(t) = d_1^T \mathbf{T1}(t) + d_2^T \mathbf{T2}(t), \tag{36}$$

Using Eqs. (8), (9) and (35) we get

$$x\left(t-\frac{1}{4}\right) = X_1^T D\mathbf{T1}(t) + X_2^T D\mathbf{T2}(t), \quad t > \frac{1}{4}, \tag{37}$$

where D is the delay operational matrix given by

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Integrating Eq. (32) from 0 to t and using Eqs. (33) – (37) we get

$$\begin{cases} X_1^T - d_1^T = 4\left(X_1^T + X_2^T\right)DP_1, \\ X_2^T - d_2^T = 4\left(X_1^T + X_2^T\right)DP_2, \end{cases}$$

where P_1 and P_2 are the operational matrices of integration. In table 1 a comparison is made between the exact solution and the approximation solution of $x(t)$ for $0 \le t \le 1$. The approximation value of $x(t)$ on $[0, \frac{1}{2})$ is the same as the exact solution. The results obtained via block pulse functions [3] is much inferior to that shown in Table 1. The results obtained via Walsh functions are exactly identical with those given in table 1. This is not surprising as block-pulse functions and Walsh functions have a one to one correspondence and they produce the same results if the number of basis functions is the same in both the cases [3]. The response $x(t)$ obtained via Chebyshev polynomials of the first kind (TP1) by [5] is inferior to that shown in table 1. With $m = 8$, the response $x(t)$ obtained via Chebyshev and Legendre polynomials (LeP) by [16] are much inferior to those given in table 1. The Laguerre approach even with higher values for m could not produce acceptable results in this example. This fact was also confirmed by [16]. It can be noted that, with a large value of m, the response $x(t)$ in each case, except for Laguerre polynomials (LaP) can be very much improved [3].

Table 1
Estimated and exact value of x(t)

t	LaP	TP1 m=8	LeP m=8	BPFs m=4	TFs k=1	Exact
0	1	1	1	1	1	1
0.25	1	1	1	1.5	1	1
0.50	0.70578	1.76	1.76493	2.75	2	2
0.75	-3.87165	3.14667	3.14683	4.875	3.5	3.5
1	-7.03743	5.49333	5.47503	4.875	6.25	6.1666667

Example 5.2
Consider the following delay system with delay in both control and state

$$\dot{x}(t) = -x(t) - 2x\left(t - \frac{1}{4}\right) + 2u\left(t - \frac{1}{4}\right), \quad 0 \le t \le 1, \tag{38}$$

$$x(t) = u(t) = 0, \qquad \text{for } -\frac{1}{4} \le t \le 0, \tag{39}$$

$$u(t) = 1, \qquad \text{for } t > 0, \tag{40}$$

Although the above system has a delay in control, the method described here can be used. The exact solution is [4]

$$x(t)=\begin{cases}0, & 0\le t<\frac{1}{4},\\ 2-2\exp\left(-(t-\frac{1}{4})\right), & \frac{1}{4}\le t<\frac{1}{2},\\ -2-2\exp\left(-(t-\frac{1}{4})\right)+(2+4t)\exp\left(-(t-\frac{1}{2})\right), & \frac{1}{2}\le t<\frac{3}{4},\\ 6-2\exp\left(-(t-\frac{1}{4})\right)+(2+4t)\exp\left(-(t-\frac{1}{2})\right) & \\ -(\frac{17}{4}+2t+4t^2)\exp\left(-(t-\frac{3}{4})\right), & \frac{3}{4}\le t\le 1,\end{cases}$$

Here, we solve the same problem by using the triangular functions. Since $\tau=\frac{1}{4}$, by using Eq. (11) we select $N=4$. Let

$$x(t)=X_1^T\mathbf{T1}(t)+X_2^T\mathbf{T2}(t), \tag{41}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are LHTF vector and RHTF vector, respectively. By expanding $u(t)$ in terms of triangular functions we get

$$u(t)=U_1^T\mathbf{T1}(t)+U_2^T\mathbf{T2}(t), \tag{42}$$

Using Eqs. (8) – (9) and (41) – (42) we get

$$x\left(t-\frac{1}{4}\right) = X_1^T D\mathbf{T1}(t)+X_2^T D\mathbf{T2}(t), \quad t>\frac{1}{4}, \tag{43}$$

$$u\left(t-\frac{1}{4}\right) = U_1^T D\mathbf{T1}(t)+U_2^T D\mathbf{T2}(t), \quad t>\frac{1}{4}, \tag{44}$$

where D is the delay operational matrix.
Integrating Eq. (38) from 0 to t and using Eqs. (39) – (44) we get

$$\begin{cases}X_1^T = -\left(X_1^T+X_2^T\right)P_1-2\left(X_1^T+X_2^T\right)DP_1+2\left(U_1^T+U_2^T\right)DP_1,\\ X_2^T = -\left(X_1^T+X_2^T\right)P_2-2\left(X_1^T+X_2^T\right)DP_2+2\left(U_1^T+U_2^T\right)DP_2,\end{cases}$$

where P_1 and P_2 are the operational matrices of integration. In table 2 a comparison is made between the exact solution and the approximation solution of $x(t)$ for $k=1$ and $k=4$. The approximation value of $x(t)$ on $[0,\frac{1}{4}]$ is the same as the exact solution. The results obtained via Laguerre polynomials [3] and Hermit polynomials [3] are to that shown in Table 2. The computational results for $x(t)$ using the present method for $N=4, k=2$ and $N=4, k=4$, together with the exact solution of $x(t)$ are given in Fig. 1. It is noted that by increasing the value of k, the acceptable convergence would be achieved.

Figure 1
Computational results of $x(t)$ obtained for $N=4, k=2$ and $N=4, k=4$.

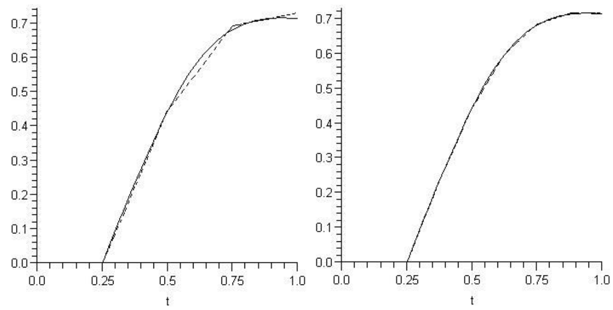


Table 2
Estimated and exact value of x(t)

t	LaP [3]	HeP [3]	TFs k=2	TFs k=4	Exact
0	0	0	0	0	0
0.125	0	0	0	0	0
0.25	0	0	0	0	0
0.375	0.23486	0.59456	0.23529	0.23508	0.23501
0.50	0.40827	1.6414	0.44291	0.44253	0.44240
0.625	0.54855	2.2788	0.59841	0.59710	0.59666
0.75	0.65916	2.387	0.68352	0.68158	0.68094
0.875	0.74333	1.8849	0.71589	0.71292	0.71194
1	0.80414	0.74092	0.71615	0.71284	0.71174

Example 5.3
Consider the time-delay system described by

$$\dot{x}(t) = 4x(t-1)+u(t), \quad 0 \le t \le 1, \tag{45}$$

$$x(0) = 1, \tag{46}$$

$$x(t) = 1, \quad -1 \le t < 0, \tag{47}$$

$$u(t) = \begin{cases} -2.1+1.05t, & 0 \le t < 1, \\ -1.05, & 1 \le t \le 2, \end{cases} \tag{48}$$

The exact solution is [3]

$$x(t) = \begin{cases} 1-1.1t+0.525t^2, & 0 \le t < 1, \\ -0.25+1.575t-1.075t^2+0.175t^3, & 1 \le t \le 2, \end{cases}$$

Since $\tau = 1$, by using Eq. (11) we select $N = 2$.
Let

$$\dot{x}(t) = X_1^T \mathbf{T1}(t) + X_2^T \mathbf{T2}(t), \tag{49}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are LHTF vector and RHTF vector, respectively. Then we have

$$x(t) = \left(\left(X_1^T + X_2^T\right)P_1 + d_1^T\right)\mathbf{T1}(t) + \left(\left(X_1^T + X_2^T\right)P_2 + d_2^T\right)\mathbf{T2}(t), \tag{50}$$

By expanding $x(t), u(t)$ and $\phi(t-1)$ in terms of triangular functions we get

$$x(0) = d_1^T \mathbf{T1}(t) + d_2^T \mathbf{T2}(t), \quad (51)$$

$$u(t) = U_1^T \mathbf{T1}(t) + U_2^T \mathbf{T2}(t), \quad (52)$$

$$\phi(t-1) = R_1^T \mathbf{T1}(t) + R_2^T \mathbf{T2}(t), \quad (53)$$

Using Eqs. (8), (9) and (50) we get

$$x(t-1) = \begin{cases} R_1^T \mathbf{T1}(t) + R_2^T \mathbf{T2}(t), & 0 \le t < 1, \\ \left(\left(X_1^T + X_2^T\right)P_1 + d_1^T\right)D\mathbf{T1}(t) + \\ \left(\left(X_1^T + X_2^T\right)P2 + d_2^T\right)D\mathbf{T2}(t), & 1 \le t < 2, \end{cases} \quad (54)$$

where D is the delay operational matrix. Using Eqs. (46) – (54) we get

$$\begin{cases} X_1^T - d_1^T = R_1^T + \left(\left(X_1^T + X_2^T\right)P_1 + d_1^T\right)D + U_1^T, \\ X_2^T - d_2^T = R_2^T + \left(\left(X_1^T + X_2^T\right)P_2 + d_2^T\right)D + U_2^T, \end{cases} \quad (55)$$

By solving set of equations (55), the derivative coefficient vectors X_1 and X_2 can be obtained. Using Eq. (49) we have

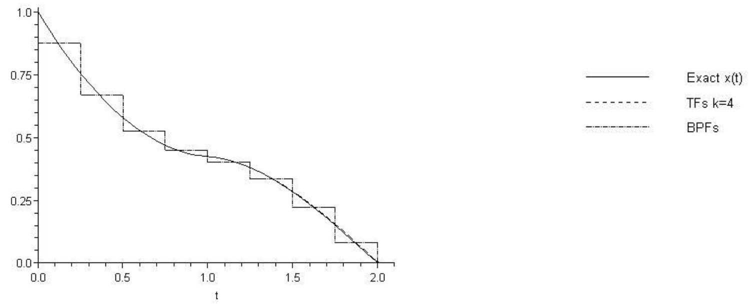
$$\dot{x}(t) = X_1^T \mathbf{T1}(t) + X_2^T \mathbf{T2}(t),$$

Here, we apply the procedure described above for obtaining $x(t)$. The approximation value of $x(t)$ on [0,1] is the same as the exact solution. We choose $k = 4$. In Table 3 a comparison is made between the exact solution and the approximation solution of $x(t)$ for $0 \le t \le 2$. The estimated $x(t)$, obtained via block pulse functions by [3] is inferior to that shown in Table 3. The computational results for $x(t)$ using the present method for $N = 2, k = 4$, together with estimated solution via block pulse functions and the exact solution of $x(t)$ are given in Fig. 2. It is noted that by increasing the value of k, the acceptable convergence for $1 \le t \le 2$ would be achieved.

Table 3
Estimated and exact value of x(t)

t	TFs k=4	Exact
0	1	1
0.25	0.757812	0.757812
0.5	0.581250	0.581250
0.75	0.470312	0.470312
1.0	0.425000	0.425000
1.25	0.382226	0.380859
1.5	0.287109	0.284375
1.75	0.156054	0.151953
2.0	0.005468	0

Figure 2
Computational results of $x(t)$ obtained for $N = 2, k = 4$.



6 Conclusions

The block-pulse functions and triangular functions and the associated operational matrices of integration and delay are applied to solve the linear time-delay systems. The method is computationally very attractive, at the same time keeping the accuracy of the solution. It is also shown that the triangular functions provide an exact solution in first subintervals for Examples (1), (2) and (3). Also in Examples (1) and (2) a comparison is made between our method and the results obtained via other numerical methods using orthogonal polynomials. The presented method reduces delay systems to the solution of algebraic equations, and so the calculation is straightforward. We are fully confident of the future development for the triangular functions, since this set is very simple in structure and calculating the coefficients for expanding an arbitrary function.

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