G-FRAMES AND STABILITY OF G-FRAMES
IN HILBERT SPACES

A. Rahimi
Department of Mathematics
University of Maragheh
asgharrahimi@yahoo.com

Abstract. In 2006 Wenchang Sun introduced $g$-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. We present a version of the Paley-Wiener Theorem for $g$-frames which is in spirit close to results for frames, due to Ole Christensen.

1. Introduction

There are some generalizations of frames, the most recent of these generalizations is $g$-frame. This is an extension of frames that include all of the previous extensions of frames.

Through this paper, $H$ and $K$ are Hilbert spaces and $\{ H_i : i \in I \}$ is a sequence of Hilbert spaces, where $I$ is a subset of $\mathbb{Z}$. The space $L(H, H_i)$ is the collection of all bounded linear operators from $H$ to $H_i$.

Note that for any sequence $\{ H_i : i \in I \}$, we can assume that there exists a Hilbert space $\mathcal{K}$ such that for all $i \in I, H_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = \bigoplus_{i \in I} H_i$).

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Definition 1.1. We call a sequence \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a generalized frame, or simply a g-frame for \( H \) with respect to \( \{ H_i : i \in I \} \), if there exist two positive constants \( A \) and \( B \) such that
\[
A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H.
\]
We call \( A \) and \( B \) the lower and upper g-frame bounds, respectively.

We say also a g-frame for \( H \) with respect to \( K \) whenever \( H_i = K \), for each \( i \in I \).

We say \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is a g-frame sequence, if it is a g-frame for \( \text{span}\{\Lambda_i^*(H_i)\}_{i \in I} \).

Definition 1.2. Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \). Then the synthesis operator for \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) is the operator
\[
T : \left( \sum_{i \in I} \bigoplus H_i \right)_{\ell_2} \rightarrow H
\]
defined by
\[
T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i).
\]
We call the adjoint \( T^* \) of the synthesis operator is the analysis operator.

The operator \( S = TT^* \) is called the g-frame operator.

It is easy to show that
\[
f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f,
\]
for every \( f \in H \).

2. Perturbation of g-frames

Theorem 2.1. Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \) with bounds \( A, B \) and \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a sequence of operators such that for any finite subset \( J \subseteq I \) and for each \( f \in H \),
\[
\left\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \right\|
\leq \lambda \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| + \gamma \left( \sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}},
\]
where \( \lambda, \mu, \gamma \) are positive constants.
where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a $g$-frame for $H$ with frame bounds
\begin{equation}
A \left(\frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu}\right) \quad \text{and} \quad B \left(\frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu}\right).
\end{equation}

**Corollary 2.2.** Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a $g$-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of operators. If there exists a constant $0 < R < A$ such that
\begin{equation}
\sum_{i \in I} \|\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f\| \leq R\|f\|
\end{equation}
for all $f \in H$, then $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ is a $g$-frame with $g$-frame bounds $A - R$ and $B + R + B$.

**Theorem 2.3.** Let $\{\Lambda_i \in L(H, H_i) : i \in I\}$ be a $g$-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in L(H, H_i) : i \in I\}$ be a family of operators such that for every $J \subseteq I$ with $|J| < +\infty$,
\begin{equation}
\left\|\sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i)\right\| \leq \lambda \left\|\sum_{i \in J} \Lambda_i^* f_i\right\| + \mu \left\|\sum_{i \in J} \Theta_i^* f_i\right\| + \gamma \left(\sum_{i \in J} \|f_i\|^2\right)^{\frac{1}{2}},
\end{equation}
where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\Theta_i \in L(H, H_i) : i \in I\}$ is a $g$-frame for $H$ with $g$-frame bounds
\begin{equation}
A \left(\frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu}\right)^2 \quad \text{and} \quad B \left(\frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu}\right)^2.
\end{equation}

**Proposition 2.4.** Let $\{\Lambda_i \in L(H, H_i) : i \in I\}$ be a $g$-frame for $H$ with bounds $A, B$ and let $\{\Theta_i \in L(H, H_i) : i \in I\}$ be a family of operators. If there exists an $R$ with $0 < R < A$ such that
\begin{equation}
\sum_{i \in I} \|\Lambda_i f - \Theta_i f\|^2 \leq R\|f\|^2
\end{equation}
for all $f \in H$, then $\{\Theta_i \in L(H, H_i) : i \in I\}$ is a $g$-frame for $H$ with bounds $(\sqrt{A} - \sqrt{R})^2$ and $(\sqrt{B} + \sqrt{R})^2$.

**Theorem 2.5.** Let $\{\Lambda_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a $g$-frame for $H$ with respect to $\{H_i : i \in I\}$, and $\{\Theta_i \in \mathcal{L}(H, H_i) : i \in I\}$ be a family of
operators. If
\[ K : \left( \sum_{i \in I} \bigoplus H_i \right)_{\ell_2} \to H, \quad K \left( \{f_i\}_{i \in I} \right) = \sum_{i \in I} (\Lambda_i^* - \Theta_i^*) f_i \]
is a well-defined and compact operator, then \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \)
is a g-frame sequence.

**Corollary 2.6.** Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \). Let \( J \) be a finite subset of \( I \) such that for each \( j \in J \), \( \dim H_j < \infty \). Then \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \setminus J \} \)
is a g-frame sequence.

**Theorem 2.7.** Let \( \{ \Lambda_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a g-frame for \( H \) and let \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \) be a family of operators. If
\[ K : H \to H, \quad K f = \sum_{i \in I} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \]
is a well-defined and compact operator, then \( \{ \Theta_i \in \mathcal{L}(H, H_i) : i \in I \} \)
is a g-frame sequence.

**References**


