ON δ-SUPPLEMENTED MODULES

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Abstract. In this talk some characterizations of δ-supplemented and δ-lifting modules are given and are investigated some properties of these modules.

1. Introduction

Throughout this article, all rings are associative with identity, and all modules are unitary right R-modules. A submodule L of a module M is called small in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called hollow, if every proper submodule of M is small in M.

For two submodules N and K of M, N is called a supplement of K in M if N is minimal with the property $M = K + N$; equivalently $M = K + N$ and $N \cap L \ll N$. A module M is called supplemented if every submodule of M is small in M.

A submodule L of a module M is called cosmall in M (denoted by $L \ll M$), if for every proper submodule K of M, $L + K \neq M$. A module M is called coclosed in M if M has no proper cosmall submodule. A submodule K of M is called essential in M (denoted by $K \leq \text{ess} M$) if $K \cap X \neq 0$ for every proper submodule X of M. We denote by $\text{Rad}(M)$ the radical of M and $R \text{-MOD}$ the category of all R-modules. Also we write $A \leq^{m} M$ to indicate that A is a maximal submodule of M.

A submodule K of M is called essential in M (denoted by $K \leq^{\text{ess}} M$) if $K \cap X \neq 0$ for every non-zero submodule X of M. We denote by $\text{Rad}(M)$ the radical of M and $R \text{-MOD}$ the category of all R-modules. Also we write $A \leq^{m} M$ to indicate that A is a maximal submodule of M.

The singular submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{ x \in M \mid Ix = 0 \text{ for some ideal } I \leq^{\text{ess}} R \}$. A module M is called singular (nonsingular) if $Z(M) = M$ ($Z(M) = 0$).

2000 Mathematics Subject Classification: 16L30, 16E50.
Keywords and phrases: supplemented modules, δ-supplemented modules, δ-lifting modules.
Let $M$ be a module. A submodule $N$ of $M$ is said to be $\delta$-small in $M$ (notation $N \ll \delta M$) if, whenever $N + X = M$ with $M/X$ singular, $X = M$. The concept of $\delta$-small submodules was introduced by Zhou in [4]. A module $M$ is called $\delta$-hollow, if every proper submodule of $M$ is $\delta$-small in $M$.

Every small submodule of $M$ is $\delta$-small in $M$ and the converse is true whenever $M$ is singular. But as we see in the next example the converse need not be true in general.

**Example 1.1.** Let $R$ be a right semisimple ring and $M$ be a nonzero right $R$-module. Then $M$ is nonsingular and semisimple. For any nonzero $N \leq M$, $N$ is a direct summand of $M$ and hence is not small in $M$; but every submodule of $M$ (even $M$ itself) is $\delta$-small in $M$.

Let $N$ and $L$ be submodules of a module $M$. $N$ is called a (weak) $\delta$-supplement of $L$ in $M$, if $N + L = M$ and $N \cap L \ll \delta N$. A module $M$ is called (weakly) $\delta$-supplemented if every submodule of $M$ has a (weak) $\delta$-supplement in $M$. $M$ is called amply $\delta$-supplemented if, for any submodules $A$ and $B$ of $M$ with $M = A + B$, $A$ has a $\delta$-supplement contained in $B$.

### 2. Main Results

**Lemma 2.1.** Let $N$ and $L$ be submodules of a module $M$. Then the following are equivalent.

1. $N$ is a $\delta$-supplement of $L$ in $M$;
2. $N + L = M$ and for each $K \leq N$ with $K + L = M$ and $N/K$ singular, $K = N$.

**Lemma 2.2.** Let $M$ be a module and $N \leq M$. Consider the following conditions:

1. $N$ is a $\delta$-supplement submodule of $M$;
2. $N$ is weak $\delta$-coclosed in $M$;
3. For all $x \leq M$, $x \ll \delta M$ implies $x \ll \delta N$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. If $M$ is weakly $\delta$-supplemented, then (3) $\Rightarrow$ (1) holds.

**Lemma 2.3.** For $K \subseteq L \subseteq M$, the following are equivalent:

1. $K$ is a $\delta$-cosmall submodule of $L$ in $M$;
2. For any $X \leq M$ with $M/X$ singular, $L + X = M$ if and only if $K + X = M$.

**Lemma 2.4.** Let $M$ be a module. Then for any $a \in M$ we have:

$aR$ is not $\delta$-small in $M$, if and only if there exists a maximal submodule $C$ of $M$ with $M/C$ singular and $a \notin C$.

**Definition 2.5.** Let $\varphi$ be the class of all singular simple modules. For a module $M$ let $\delta(M) = \text{Rej}_M(\varphi) = \cap \{N \subseteq M | M/N \in \varphi\}$ be the reject of $\varphi$ in $M$.

From the definition we immediately have $\delta(M/\delta(M)) = 0$, for any module $M$. 

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Proposition 2.6. Given a module \( M \), each of the following sets is equal to \( \delta(M) \).

1. \( A_1 = \sum \{ A | A \ll_\delta M \} \).
2. \( A_2 = \cap \{ B | B \leq M \text{ with } M/B \text{ singular} \} \).
3. \( A_3 = \cap \{ \ker \phi | \phi \in \text{Hom}(M, N) \text{ such that } N \text{ is singular simple} \} \).
4. \( A_4 = \cap \{ \ker \phi | \phi \in \text{Hom}(M, N) \text{ such that } N \text{ is singular semisimple} \} \).

Proposition 2.7. Let \( U \) and \( V \) be submodules of a module \( M \). Assume that \( V \) is a \( \delta \)-supplement of \( U \) in \( M \). Then

1. If \( W + V = M \) for some \( W \subseteq U \), then \( V \) is a \( \delta \)-supplement of \( W \) in \( M \).
2. If \( K \ll_\delta M \), then \( V \) is a \( \delta \)-supplement of \( U + K \) in \( M \).
3. For \( K \ll_\delta M \) we have \( K \cap V \ll_\delta V \) and so \( \delta(V) = V \cap \delta(M) \).
4. For \( L \subseteq U \), \( (V + L)/L \) is a \( \delta \)-supplement of \( U/L \) in \( M/L \).
5. If \( \delta(M) \ll_\delta M \), or \( \delta(m) \subseteq U \) and if \( p : M \rightarrow M/\delta(M) \) is the canonical projection, then \( M/\delta(M) = Up \oplus Vp \).

Proposition 2.8. Let \( M \) be an amply \( \delta \)-supplemented module. Then every non \( \delta \)-small submodule \( N \) of \( M \) contains a \( \delta \)-supplement submodule \( N' \) such that \( N/N' \ll_\delta M/N' \).

Proposition 2.9. For a submodule \( U \subseteq M \), the following are equivalent.

1. There is a direct summand \( X \) of \( M \) with \( X \subseteq U \) and \( U/X \ll_\delta M/X \).
2. There is a direct summand \( X \subseteq M \) and a submodule \( Y \) of \( M \) with \( X \subseteq U \), \( U = X + Y \) and \( Y \ll_\delta M \).
3. There is a decomposition \( M = X \oplus X' \) with \( X \subseteq U \) and \( X' \cap U \ll_\delta X' \).
4. \( U \) has a \( \delta \)-supplement \( V \) in \( M \) such that \( U \cap V \) is a direct summand in \( U \).
5. There is an idempotent \( e \in \text{End}(M) \) with \( Me \subseteq U \) and \( U(1 - e) \ll_\delta M(1 - e) \).

Definition 2.10. A module \( M \) is called \( \delta \)-lifting if, for any \( A \leq M \), there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( M_1 \leq A \) and \( A/M_1 \ll_\delta M/M_1 \).

For example every \( \delta \)-hollow module is \( \delta \)-lifting and it is easy to see that every indecomposable \( \delta \)-lifting module is \( \delta \)-hollow.

The next Proposition immediately follows from Proposition 2.9 and also can be found in [2, Lemma 2.3]:

Proposition 2.11. For a module \( M \) the following are equivalent.

1. \( M \) is \( \delta \)-lifting.
2. For every submodule \( N \) of \( M \) there is a decomposition \( M = M_1 \oplus M_2 \) such that \( M_1 \subseteq N \) and \( N \cap M_2 \ll_\delta M \).
3. Every submodule \( N \) of \( M \) can be written as \( N = N_1 \oplus N_2 \) with \( N_1 \) a direct summand of \( M \) and \( N_2 \ll_\delta M \).

Corollary 2.12. Every direct summand of a \( \delta \)-lifting module is \( \delta \)-lifting.

Proposition 2.13. Let \( M \) be a \( \delta \)-lifting module. Then

1. Any \( \delta \)-coclosed submodule of \( M \) is a direct summand;
(2) $M$ is amply $\delta$-supplemented;
(3) If $N \subseteq M$ is a fully invariant submodule of $M$, then $M/N$ is a $\delta$-lifting module.

**Proposition 2.14.** Let $M$ be an amply $\delta$-supplemented module such that every $\delta$-supplement submodule of $M$ is a direct summand. Then $M$ is $\delta$-lifting.

**Proposition 2.15.** Let $M$ be a module such that every $\delta$-supplement submodule of $M$ is $\delta$-coclosed in $M$. Then $M$ is $\delta$-lifting if and only if $M$ is amply $\delta$-supplemented and every $\delta$-supplement submodule is a direct summand.

**References**