INEQUALITIES FOR MEROMORPHICALLY P-VALENT FUNCTIONS^{*}

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Abstract – The aim of this paper is to prove some inequalities for p-valent meromorphic functions in the punctured unit disk Δ^* and find important corollaries.

Keywords - p-valent functions, meromorphic starlike function, convex, close-to-convex

1. INTRODUCTION

Let Σ_p denote the class of functions f(z) of the form

$$f(z) = z^{-p} + \sum_{k=p} a_k z^k$$
 (1)

which are analytic meromorphic multivalent in the punctured unit disk

$$\Delta^* = \{ z : 0 < \mid z \mid < 1 \}.$$

We say that f(z) is p-valently starlike of order $\gamma(0 \le \gamma \le p)$ if and only if for $z \le \Delta *$

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \tag{2}$$

Also, f(z) is p-valently convex of order γ ($0 \le \gamma < p$) if and only if

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \ (z \in \Delta^*).$$
(3)

Definition 1.1: A function $f(z) \in \Sigma_p$ is said to be in the subclass $X_p^*(j)$ if it satisfies the inequality

$$\left|\frac{(p-1)!}{(-1)^{j}(p+j-1)!}\frac{f^{(j)}(z)}{z^{-p-j}}-1\right| < 1$$
(4)

where

$$f^{(j)}(z) = (-1)^{j} \frac{(p+j-1)!}{(p-1)! z^{p+j}} + \sum_{k=p}^{\infty} \frac{k!}{(k-j)!} a_{k} z^{k-j}$$
(5)

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is the j-th differential of f(z) and a function $f(z) \in \Sigma_p$ is said to be in the subclass $Y_p^*(j)$ if it satisfies the inequality

$$\left| -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j) \right| < p+j.$$
(6)

To establish our main results we need the following lemma due to Jack [1].

Lemma 1.2. Let w(z) be analytic in $\Delta = \{z : |z| < 1\}$ with w(0) = 0. If | w (z) | attains its maximum value on the circle | z | = r < 1 at a point z₀, then

$$z_0 w'(z_0) = c w(z_0)$$

where *c* is a real number and $c \ge 1$.

Some different inequalities on p-valent holomorphic and p-valent meromorphic functions by using operators were studied in [2-5].

2. MAIN RESULTS

In the first theorem we give a sufficient condition for $f \in \Sigma_p$ to be in the chass $X_p^*(j)$.

Theorem 2.1. If $f(z) \in \Sigma_p$ satisfies the inequality

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j\right\} > 1 - \frac{1}{2p}$$
(7)

then $f(z) \in X_p^*(j)$.

Proof: Let $f(z) \in \Sigma_{p, we}$ define the function w(z) by

$$\frac{(p-1)!}{(-1)^{j}(p+j-1)!} \frac{f^{(j)}(z)}{z^{-p-j}} = 1 - w(z), \ (z \in \Delta^{*}).$$
(8)

It is easy to verify that w(0) = 0. From (8) we obtain

$$f^{(j)}(z) = \frac{(1)^{j}(p+j-1)!}{(p-1)!} z^{-p-j} - \frac{(-1)^{j}(p+j-1)!}{(p-1)!} z^{-p-j} w(z)$$

or

$$[f^{(j)}(z)]' = (-1)^{j+1} (p+j) z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} + (-1)^{j} (p+j) z^{-p-j-1} \frac{(p+j-1)!}{(p-1)!} \frac{(p+j-1)!}{(p-1)!} x^{-p-j} w'(z).$$

After a simple calculation we obtain

$$\frac{zw'(z)}{1-w(z)} = \frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + (p+j).$$
(9)

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Spring 2009 www.SID.ir Now, suppose that there exists a point $z_0 \in \Delta^*$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by letting $w(z_0) = e^{i\theta} (w(z_0) \neq 1)$ and using the Jack's lemma in the equation (9), we have

$$-\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} + p + j\right\} = \operatorname{Re}\left\{\frac{z_0w'(z_0)}{1 - w(z_0)}\right\} = \operatorname{Re}\left\{\frac{cw(z_0)}{1 - w(z_0)}\right\}$$
$$= c\operatorname{Re}\left\{\frac{e^{i\theta}}{1 - e^{i\theta}}\right\} = \frac{-C}{2} < \frac{-1}{2},$$

which contradicts the hypothesis (7). Hence, we conclude that for all z, |w(z)| < 1 and from (8) we have

$$\left|\frac{(p-1)!f^{(j)}(z)}{(-1)^{j}(p+j-1)!z^{-p-j}} - 1\right| = |w(z)| < 1$$

and this gives the result.

Theorem 2.2. If $f(z) \in \Sigma_p$ satisfies the inequality

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right)\right\} > \frac{2p+1}{2(p+1)},\tag{10}$$

then $f(z) \in Y_p^*(j)$.

Proof: Let $f(z) \in \Sigma_{p}$. We consider the function w(z) as follows:

$$-\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} = (p+j)(1-w(z)).$$
(11)

It is easy to see that w(0)=0. Furthermore, by differentiating both sides of (11) we get

$$-\left[1+\frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right]=(p+j)(1-w(z))+\frac{zw'(z)}{1-w(z)}.$$

Now suppose there exists a point $z_0 \in \Delta^*$ such that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$. Then by letting $w(z_0) = e^{i\theta}$ and using Jack's lemma we have

$$\operatorname{Re}\left\{\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - \left(1 + \frac{z[f^{(j)}(z)]''}{[f^{(j)}(z)]'}\right)\right\} = \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{1 - w(z_0)}\right\}$$
$$= c \operatorname{Re}\left\{\frac{e^{i\theta}}{1 - e^{i\theta}}\right\} = -\frac{c}{2} < -\frac{1}{2}$$

which contradicts the condition (10). So we conclude that $|w(z)| \le 1$ for all $z \in \Delta^*$. Hence, from (11) we obtain

$$\left| -\frac{z[f^{(j)}(z)]'}{f^{(j)}(z)} - (p+j) \right| < p+j.$$

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This completes the proof.

By taking j = 0 in Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 1. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf'}{f}+p\right\} > 1 - \frac{1}{2p}$$

then

$$\left|\frac{f(z)}{z^{-p}} - 1\right| < 1.$$

Corollary 2. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf'}{f}-\left(1+\frac{zf''}{f'}\right)\right\}>\frac{2p+1}{2(p+1)},$$

then $\left| -\frac{zf'}{f} - p \right| < p$ or equivalently f(z) is meromorphically p-valent starlike with respect to the origin.

By taking j = 1 in theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 3. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf''}{f'}+p+1\right\}>1-\frac{1}{2p}$$

Then $\left| -\frac{f'(z)}{z^{-p-1}} - p \right| < p$ or equivalently f(z) is meromorphically p-valent close-to-convex with respect to the origin.

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Corollary 4. If $f(z) \in \Sigma_p$ satisfies the inequality

$$-\operatorname{Re}\left\{\frac{zf''}{f'}-\left(1+\frac{zf'''}{f''}\right)\right\}>\frac{2p+1}{2(p+1)},$$

then

or equivalently f(z) is meromorphically multivalent convex.

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 $\left| \frac{zf''}{f'} - (p+1) \right| < p+1$

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