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Research Article

Approximate solution of the stochastic Volterra integral equations via expansion method

M. Khodabin * † , K. Maleknejad ‡ , T. Damercheli §

Abstract

In this paper, we present an efficient method for determining the solution of the stochastic second kind Volterra integral equations (SVIE) by using the Taylor expansion method. This method transforms the SVIE to a linear stochastic ordinary differential equation which needs specified boundary conditions. For determining boundary conditions, we use the integration technique. This technique gives an approximate simple and closed form solution for the SVIE. Expectation of the approximating process is computed. Some numerical examples are used to illustrate the accuracy of the method.

Keywords : Taylor series expansion; Stochastic Volterra integral equation; Itô integral.

1 Introduction

The stochastic Volterra integral equations arise in many applications such as mathematical finance, biology, medical, social sciences, etc. There is an increasing demand for studying the behavior of a number of sophisticated dynamical systems in physical, medical and social sciences, as well as in engineering and finance. These systems are often dependent on a noise source, Gaussian white noise, for example, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various the stochastic differential equations and the stochastic optimization problem [1, 2, 4, 5, 6, 10] or, in more complicated cases, the stochastic Volterra integral equations and the stochastic integro-differential equations [3, 7, 12, 13, 15, 16, 17]. Since in many problems, such equations, can not be solved explicitly, it is important to find their approximate solutions by using some numerical methods. The methods for the computational solution of stochastic integral equations are based on similar techniques for deterministic integral equations, but generalized to provide support for stochastic dynamics [4, 5, 8, 9, 10, 11].

In this paper, a novel, simple, and an efficient approach is proposed to determine the approximate solutions of the stochastic second kind Volterra integral equations. We use the Taylor series expansion of the unknown function for obtaining the solution. Then for determining specified boundary conditions, for transformed linear

*Corresponding author. m-khodabin@kiau.ac.ir
†Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
‡Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
§Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

ordinary differential equation, we employ the integration method. This method is simple and effective, and can provide an accurate approximate solution to the stochastic integral equations. The efficiently and the accuracy of the method are shown by some numerical examples from [11]. This paper is organized as follows: Section 2, describes the elementary concepts from stochastic calculus. In Section 3, we introduce the Taylor expansion method and exhibit the convergence of the proposed scheme in Section 4. The accuracy of the presented method is illustrated by some examples in Section 5. Finally, Section 6 gives some brief conclusions.

2 Stochastic Volterra integral equations

We will begin with a quick survey of the most fundamental concepts from stochastic calculus that are needed. For full details, the reader may consult Klebaner (1998), Oksendal (1998), Steele (2001).

A set of random variables X_t indexed by real numbers $t \geq 0$ is called a continuous-time stochastic process. Each instance, or realization of the stochastic process is a choice from the random variable X_t for each t, and is therefore a function of t. Any (deterministic) function $f(t)$ can be trivially considered as a stochastic process, with variance $V(f(t)) = 0$. An archetypal example that is ubiquitous in models from physics, chemistry, and finance is the Wiener process W_t, a continuous-time stochastic process with the following three properties:

Property 1. For each t, the random variable W_t is normally distributed with mean 0 and variance t.

Property 2. For each $t_1 < t_2$, the normal random variable W_{t_2}-W_{t_1} is independent of the random variable W_{t_1} , and in fact independent of all $W_t, 0 \leq t \leq t_1$.

Property 3. The Wiener process W_t can be represented by continuous paths where is not differentiable.

The Wiener process, named after Norbert Wiener, is a mathematical construct that formalizes random behavior characterized by the botanist Robert Brown in 1827, commonly called Brownian motion. It can be rigorously defined as the scaling limit of random walks as the step size and time interval between steps both go to zero. Brownian motion is crucial in the modeling of stochastic processes since it represents the integral of idealized noise that is independent of frequency, called white noise. Often, the Wiener process is called upon to represent random, external influences on an otherwise deterministic system, or more generally, dynamics that for a variety of reasons cannot be deterministically modeled.

Consider the stochastic second kind Volterra integral equation of the form

$$y(x) + \int_0^x k_1(x,t)y(t)dt + \int_0^x k_2(x,t)y(t)dW_t = f(x), 0 \leq x \leq 1. \quad (2.1)$$

In Eq. (2.1), the functions $k_1(x,t)$, $k_2(x,t)$, and $f(x)$, for $x,t \in [0,1]$, are the stochastic processes defined on the same probability space (Ω, F, P), and $y(t)$ is the unknown function. Also W_t is a Brownian process and $\int_0^x k_2(x,t)y(t)dW_t$ is called an Itô integral. The necessary and sufficient conditions for existence and uniqueness of the solution of the Eq. (2.1) could be found in [10]. So, we assume that Eq. (2.1) has a unique solution and the kernels $k_1(x,t), k_2(x,t)$ and $f(x) \in C^n[0,1]$. Under these conditions, we have $y(t) \in C^n[0,1]$. The second integral is not defined by the rules of classical calculus. This problem was overcome in the early 1950s when Ito formulated his definition of the Ito integral, for which the second integral is defined. Ito stochastic calculus exhibits many peculiarities and does not conform to the rules of classical calculus, so care must be taken when constructing methods to solve stochastic problems.

Let $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ be a grid of points on the interval [0,1]. The Riemann integral is defined as a limit

$$\int_0^1 f(x)dx = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(\acute{t}_i)\Delta t_i,$$

where $\Delta t_i = t_i - t_{i-1}$ and $t_{i-1} \leq \acute{t}_i \leq t_i$. Similarly, the Ito integral is the limit

$$\int_0^1 f(x)dW_t = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_{i-1})\Delta W_i,$$

where $\Delta W_i = W_{t_i} - W_{t_{i-1}}$, a step of Brownian motion across the interval. Note a major difference: while the $\acute{t}_i$ in the Riemann integral may be chosen at any point in the interval (t_{i-1}, t_i), the corresponding point for the Ito integral is required to be the left endpoint of that interval. To solve stochastic equations analytically, we need to introduce the chain rule for stochastic differentials, called the Ito formula:

Theorem 2.1 *(The 1-dimensional Itô formula). Let $X(t)$ be an Itô process and $g(t,x) \in C^2([0,\infty) \times R)$, then*

$$Y(t) = g(t, X(t)),$$

is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2,$$

where $(dX(t))^2 = (dX(t))(dX(t))$ is computed according to the rules

$$dt.dt = dt.dW_t = dW_t.dt = 0, dW_t.dW_t = dt.$$

Proof. see ([14], p.44).

Theorem 2.2 *(The Itô isometry). Let $f \in \nu(S,T)$, then*

$$E[(\int_S^T f(t,\omega)dW_t(\omega))^2] = E[\int_S^T f^2(t,\omega)dt].$$

Proof. *see ([14], p.29)*

The Ito formula is the stochastic analogue to the chain rule of conventional calculus. Although it is expressed in differential form for easy understanding, its meaning is precisely the equality of the Ito integral of both sides of the equation. It is proved under rather weak hypotheses by referring the equation back to the definition of Ito integral (Oksendal, 1998).

Definition 2.1 *(The Itô integral), ([14], p.29). Let $f \in \nu(S,T)$, then the Itô integral of f (from S to T) is defined by*

$$\int_S^T f(t,\omega)dB(t)(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dW_t(\omega), \quad (limit \quad in \quad L^2(P))$$

where ϕ_n is a sequence of elementary functions such that

$$E[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt] \to 0, \quad as \quad n \to \infty.$$

3 Determination of approximate solution the SVIE

Consider the Eq. (2.1). The Taylor's expansion of the unknown function $y(t)$ at x, is given by

$$y(t) = y(x) + y'(x)(t-x) + \cdots + \frac{1}{n!}y^{(n)}(x)(t-x)^n + R_n(t,x), \quad (3.2)$$

where, $R_n(t,x)$ denotes Lagrange remainder and is defined as

$$R_n(t,x) = \frac{y^{(n+1)}(\zeta)}{(n+1)!}(t-x)^{n+1}, \quad (3.3)$$

for some point ζ between x and t. In genral, the Lagrange remainder $R_n(t,x)$ becomes sufficiently small when n is large enough. In particular, if the desired solution $y(t)$ is a polynomial of the degree equal to or less than n, then $R_n(t,x) = 0$. Substituting Eq. (3.2) into Eq. (2.1) leads to

$$y(x) + \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x) \int_0^x k_1(x,t)(x-t)^j dt$$

$$+\sum_{j=0}^{n}\frac{(-1)^j}{j!}y^{(j)}(x)\int_0^x k_2(x,t)(x-t)^j dW_t = f(x). \quad (3.4)$$

In the above derivation, the Lagrange remainder has been dropped due to sufficiently small truncated error. Moreover, a notation $y^{(0)}(x)=y(x)$ is adopted. In Eq. (3.4), $y^{(j)}(x)$ for $j=0,\ldots,n$ are unknown functions. In order to obtain these unknown functions, we consider the above equation as a linear equation for $y(x)$ and its derivatives up to n. Consequently, other n independent linear equations for $y(x)$ and its derivatives up to n are required. These equations can be obtained by the integration of both sides of Eq. (2.1) n times as follows

$$\int_0^x (x-t)^{i-1}y(t)dt+ \int_0^x\int_t^x (x-s)^{i-1}k_1(s,t)y(t)dsdt + \int_0^x\int_t^x (x-s)^{i-1}k_2(s,t)y(t)dW_s dt = \int_0^x (x-t)^{i-1}f(t)dt, \quad i=1,\cdots,n, \quad (3.5)$$

Now, inserting Eq. (3.2) for $y(t)$ into Eq. (3.5), we can get

$$\int_0^x (x-t)^{i-1}\sum_{j=0}^{n}\frac{(-1)^j}{j!}y^{(j)}(x)(x-t)^j dt + \int_0^x\int_t^x (x-s)^{i-1}k_1(s,t)\sum_{j=0}^{n}\frac{(-1)^j}{j!}y^{(j)}(x)(x-t)^j dsdt + \int_0^x\int_t^x (x-s)^{i-1}k_2(s,t)\sum_{j=0}^{n}\frac{(-1)^j}{j!}y^{(j)}(x)(x-t)^j dW_s dt = \int_0^x (x-t)^{i-1}f(t)dt, \quad (3.6)$$

Hence, Eqs. (3.4) and (3.6) form a linear system of $n+1$ algebraic equation for $n+1$ unknowns $y(x)$ and its derivatives up to n which can be solved easily. Specifically, we solve the following system of linear equations for $y(x), y'(x),\ldots,y^n(x)$. The system can be rewritten as

$$A(x)Y(x)=F(x), \quad (3.7)$$

where

$$A(x)=\begin{bmatrix} c_{00}(x) & c_{01}(x) & \ldots & c_{0n}(x)\\ c_{10}(x) & c_{11}(x) & \ldots & c_{1n}(x)\\ \vdots & \vdots & \ddots & \vdots\\ c_{n0}(x) & c_{n1}(x) & \ldots & c_{nn}(x)\end{bmatrix}, \quad (3.8)$$

$$F(x)=\begin{bmatrix} f(x)\\ f^{(1)}(x)\\ \vdots\\ f^{(n-1)}(x)\end{bmatrix}, \quad Y(x)=\begin{bmatrix} y(x)\\ y'(x)\\ \vdots\\ y^{(n)}(x)\end{bmatrix}, \quad (3.9)$$

where in (3.8), the first row refers to coefficients of $y^{(j)}(x)$ in Eq. (3.4) for $j=0,\ldots,n$ and the other rows refer to coefficients of $y^{(j)}(x)$ in Eq. (3.6) for $j=0,\ldots,n$. Application of Cramers rule to the resulting system yields an approximate solution of Eq. ((3.4)). It is also noted that not only $y(x)$ but also $y^{(j)}(x)$ for $j=1,...,n$ are determined via solving the resulting system.

4 Error analysis

In this Section, we give an error analysis for this method. For convenience, we suppose $f(x)$, $k_1(x,.)$ and $k_2(x,.)\in C^\infty(I)$ where I is the interval of interest. Moreover, it is assumed that the solution to be determined is infinitely differential in the interval I. Furthermore, we assume that $k_1(x,.)$ and $k_2(x,.)$ are uniformly bounded, i.e. there are two positive constants K_1 and K_2 independent of n such that $\mid k_1(x,t)\mid\leq K_1$ and $\mid k_2(x,t)\mid\leq K_2$ for $x\in I$. For determining the approximate solution of Eq. (2.1), we consider n terms of the Taylor series $y(t)$ at x, that satisfies in Eq. (2.1) as follows

$$y_n(x)+\int_0^x k_1(x,t)y_n(t)dt+\int_0^x k_2(x,t)y_n(t)dW_t = f(x). \quad (4.10)$$

Table 1: Mean, standard deviation and mean confidence interval for error in Example 5.1 with $n = 1$.

m	$\overline{y}_E$	S_E	% 95 Confidence Interval for error mean	
			Lowerbound	*Upperbound*
30	0.00201909	0.00271682	0.00104689	0.00299129
50	0.00261748	0.00377173	0.00157201	0.00366295
100	0.00381430	0.00603494	0.00263145	0.00499714
150	0.00324601	0.00544121	0.00237523	0.00411678
200	0.00245754	0.00330552	0.00199942	0.00291566

Table 2: Mean, standard deviation and mean confidence interval for error in Example 5.1 with $n = 2$.

m	$\overline{y}_E$	S_E	% 95 Confidence Interval for error mean	
			Lowerbound	*Upperbound*
30	0.00191827	0.0024717	0.00103378	0.00280276
50	0.00696840	0.00172683	0.00121819	0.00217550
100	0.00219951	0.00276663	0.00165725	0.00274177
150	0.00221501	0.00297590	0.00173876	0.00269125
200	0.00193067	0.00240522	0.00159732	0.00226402

Table 3: Mean, standard deviation and mean confidence interval for error in Example 5.2 with $n = 1$.

m	$\overline{y}_E$	S_E	% 95 Confidence Interval for error mean	
			Lowerbound	*Upperbound*
30	0.00265150	0.00200914	0.00193254	0.00337046
50	0.00235167	0.00171831	0.00187538	0.00282796
100	0.00229577	0.00173910	0.00195490	0.00263663
150	0.00226720	0.00171843	0.00199219	0.00254220
200	0.00232418	0.00173169	0.00208418	0.00256418

Table 4: Mean, standard deviation and mean confidence interval for error in Example 5.2 with $n = 2$.

m	$\overline{y}_E$	S_E	% 95 Confidence Interval for error mean	
			Lowerbound	*Upperbound*
30	0.00248681	0.00165598	0.00189423	0.00307940
50	0.00266417	0.00192375	0.00213093	0.00319740
100	0.00279153	0.00252379	0.00229686	0.00328619
150	0.00253247	0.00221575	0.00217787	0.00288706
200	0.00252617	0.00198018	0.00225173	0.00280061

Thus, $y_n(x)$ is referred to the nth-order approximation of the exact solution $y(x)$. By subtracting Eq. (2.1) from Eq. (4.10), we get

$$y_n(x) - y(x) = \int_0^x k_1(x,t)(y(t) - y_n(t))dt + \int_0^x k_2(x,t)(y(t) - y_n(t))dW_t. \quad (4.11)$$

Since $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$E(\mid y_n(x) - y(x) \mid^2) \leq 2E(\mid \int_0^x k_1(x,t)(y(t) - y_n(t))dt \mid^2) + 2E(\mid \int_0^x k_2(x,t)(y(t) - y_n(t))dW_t \mid^2). \quad (4.12)$$

Table 5: Error between exact solution and approximate solution for Examples 5.1 and 5.2 with $n = 1, 2$

x	Example 5.1		Example 5.2	
	$n=1$	$n=2$	$n=1$	$n=2$
0.1	0.0000242909	0.0000142588	0.00036698	0.000631032
0.2	0.0001103250	0.0000693169	0.00004345	0.000444685
0.3	0.0004296230	0.0006350250	0.00126760	0.000039601
0.4	0.0006215360	0.0025920900	0.00434630	0.000880022
0.5	0.0000786668	0.0067937100	0.00989890	0.002060460

By using the Cauchy-Schwartz inequality, we obtain

$$E(|\int_0^x k_1(x,t)(y(t)-y_n(t))dt\,|^2) \leq$$

$$E(\int_0^x |\,k_1(x,t)(y(t)-y_n(t))\,|^2\,dt)$$

$$\leq K_1^2 E\int_0^x |\,y_n(x)-y(x)\,|^2\,dt$$

$$\leq K_1^2\int_0^x E(|\,y_n(x)-y(x)\,|^2)dt. \qquad (4.13)$$

Furthermore, from Theorem 2.1, we can write

$$E(|\int_0^x k_2(x,t)(y(t)-y_n(t))dW_t\,|^2) \leq$$

$$E(\int_0^x |\,k_2(x,t)(y(t)-y_n(t))\,|^2\,dt)$$

$$\leq K_2^2 E\int_0^x |\,y_n(x)-y(x)\,|^2\,dt$$

$$\leq K_2^2\int_0^x E(|\,y_n(x)-y(x)\,|^2)dt. \qquad (4.14)$$

Therefore, for some appropriate constant K we get

$$E(|\,y_n(x)-y(x)\,|^2) \leq K\int_0^x E($$

$$|\,(y(t)-y_n(t))\,|^2)dt. \qquad (4.15)$$

Thus, by using Granwall's Lemma, we have $E(|\,y_n(x)-y(x)\,|^2) \to 0$ as $n \to \infty$.

5 Numerical examples

In this Section, we present numerical results for some examples from [11] to show the efficiency and the accuracy of the presented method.

Example 5.1 *Consider the following linear stochastic Volterra integral equation,*

$$y(x) = 1+\int_0^x t^2 y(t)dt+\int_0^x ty(t)dW_t,\ \ x,t\in[0,0.5), \qquad (5.16)$$

with the exact solution $y(x) = e^{\frac{x^3}{6}+\int_0^x tdW_t}$, *for* $0 \leq x < 0.5$. *The numerical results are shown in Tables 1 and 2. In the Tables, m is the number of iterations,* $\overline{y}_E$ *is the error mean , and* s_E *is the standard deviation of error. Table 5 shows the error between the exact solution and the approximate solution for* $0 \leq x < 0.5$ *with* $n = 1, 2$.

Example 5.2 *Consider the following linear stochastic Volterra integral equation,*

$$y(x) = \frac{1}{12}+\int_0^x cos(t)y(t)dt+\int_0^x sin(t)y(t)dB(t)$$

$$x,t\in[0,0.5), \qquad (5.17)$$

with the exact solution $y(x) = \frac{1}{12}e^{-\frac{x}{4}+sin(x)+\frac{sin(2x)}{8}\int_0^x sin(t)dB(t)}$, *for* $0 \leq x < 0.5$. *The numerical results are shown in Tables 3, 4 and 5.*

6 Conclusion

Because for some SDEs that can be written as Volterra integral equations, it is impossible to find the exact solution of Eq. (2.1). It would be convenient to determine its numerical solution based on stochastic numerical analysis. This method is very simple and effective in comparison with other methods. Also, this method has least computations and cost comparing with other methods. In this paper, the applicability and the accuracy of this method were shown by two examples. The results of the numerical solution were compared with the analytical solution.

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I was born on September 4, 1969 in the city of Karaj. I receipt high school diploma in 1988 on mathematics branch. In 1989, I successfully passed the university entrance exam and started my academic studies on the statistics in the Shahid Beheshti University in Tehran. In 1994, I continue my studies at MS level on the statistics in the same university. In 1998, I became a member of the board of department of mathematics in Islamic Azad University Karaj branch. In 2000, I emprise to obtain my Ph.D. on statistics in Sciences Research University Tehran branch. I receipt my Ph.D. certification in 2004. My Ph. D. dissertation is entropy and statistical hypothesis testing. During the years when I have taught in the university, I have written some scientific papers in scientific research in various fields and also I have supervised the theses of students at MS and Ph.D levels on different subjects. Also, I am Managing Editor of the International Journal of Mathematical Sciences. Now I am associate professor in department of mathematics Islamic Azad University, Karaj branch.

Khosrow Maleknejad received his MS degree in Applied Mathematics from Tehran University, Iran, in 1972 and his PhD degree in Numerical Analysis from the University of Wales, Aberystwyth, UK in 1981. In September 1976, he joined the Faculty of the Basic Science, Department of Applied Mathematics at Iran University of Science and Technology where he has been a Professor since 2002. During 19912000, he also served as Vice-chair for graduate students. He was a Visiting Professor at the University of California at Los Angeles in 1990. His research interests include numerical analysis in solving ill-posed problems and solving Fredholm and Volterra integral equations. He has authored and coauthored more than 160 research papers on these topics. He is an Editor-in-chief of the International Journal of Mathematical Sciences and member of editorial board of some journals. He is a member of the AMS.

I was born in the city of Zanjan-Iran in 1979. I receipt B.Sc and M.Sc degrees in pure mathematics and applied mathematics in years 2003, 2007 respectively. I started the Ph.D course on numerical analysis field in Islamic Azad University Karaj branch in 2010. Currently, I taught in the Islamic Azad University Shahre-Ray Branch . My main research interest is include numerical solution of deterministic and stochastic integral equations, differential equations and dynamic systems arise of optimal control problems.