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NONINNER AUTOMORPHISMS OF FINITE p-GROUPS LEAVING THE CENTER ELEMENTWISE FIXED

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Abstract. A longstanding conjecture asserts that every finite nonabelian p-group admits a noninner automorphism of order p. Let G be a finite nonabelian p-group. It is known that if G is regular or of nilpotency class 2 or the commutator subgroup of G is cyclic, or $G/Z(G)$ is powerful, then G has a noninner automorphism of order p leaving either the center $Z(G)$ or the Frattini subgroup $\Phi(G)$ of G elementwise fixed. In this note, we prove that the latter noninner automorphism can be chosen so that it leaves $Z(G)$ elementwise fixed.

1. Introduction

One of the most widely known, although nontrivial, properties of finite p-groups of order greater than p is that they always have a noninner automorphism α of p-power order. This fact was first proved by Gaschütz in 1966 [5]. Schmid [8] extended Gaschütz's result by showing that if G is a finite nonabelian p-group, then the automorphism α can be chosen to act trivially on the center. A longstanding conjecture that had been raised even before Gaschütz's result is the following

Conjecture 1. *Every finite nonabelian p-group admits a noninner automorphism of order p.*

Indeed, in 1964 Liebeck [7] proved that if p is an odd prime and G is a finite p-group of class 2 then G has a noninner automorphism of order p acting trivially on the Frattini subgroup $\Phi(G)$. The corresponding result for 2-groups is false in general, as Liebeck himself produced an example of a 2-group G of class 2 with the property that all automorphisms of order two leaving $\Phi(G)$ elementwise

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fixed are inner. By a cohomological result of Schmid [9], it follows that finite regular nonabelian p-groups admit a noninner automorphism leaving the Frattini subgroup elementwise fixed. Deaconescu and Silberberg [4] proved that if $C_G(Z(\Phi(G))) \neq \Phi(G)$, then the noninner automorphism can be chosen to act trivially on $\Phi(G)$. Hence the main result of [4] reduced the verification of Conjecture 1 to finite nonabelian p-groups G satisfying the condition $C_G(Z(\Phi(G))) = \Phi(G)$. In [1, 2, 3] it is proved that if G is a finite nonabelian p-group of class at most 3 or $G/Z(G)$ is powerful, then G has a noninner automorphism of order p leaving either $\Phi(G)$ or $\Omega_1(Z(G))$ elementwise fixed. Jamali and Viseh [6] proved that every nonabelian finite 2-group with cyclic commutator subgroup has a noninner automorphism of order two leaving either $\Phi(G)$ or $Z(G)$ elementwise fixed. They have also observed that the results of [1, 2] can be improved, that is, if G is of nilpotency class 2 or $G/Z(G)$ is powerful, then G has a noninner automorphism of order p leaving either the center $Z(G)$ or Frattini subgroup elementwise fixed. Therefore the following result holds.

Proposition 1.1. *Let G be a finite nonabelian p-group satisfying one of the following conditions:*

(1) *G is regular;*
(2) *G is nilpotent of class 2;*
(3) *the commutator subgroup of G is cyclic;*
(4) *$G/Z(G)$ is powerful.*

Then G has a noninner automorphism of order p leaving either $Z(G)$ or $\Phi(G)$ elementwise fixed.

The main result of our paper is the following.

Theorem 1.2. *Let G be a finite nonabelian p-group satisfying one of the following conditions:*

(1) *G is regular;*
(2) *G is nilpotent of class 2;*
(3) *the commutator subgroup of G is cyclic;*
(4) *$G/Z(G)$ is powerful.*

Then G has a noninner automorphism of order p leaving $Z(G)$ elementwise fixed.

2. Proof of the main result

We need the following result which may be well-known. We prove it for the reader's convenience.

Lemma 2.1. *Let G be any finite p-group. Then $G = AH$ for some subgroups A and H such that $A \leq Z(G)$ and $Z(H) \leq \Phi(H)$.*

Proof. We prove Lemma by induction on $|G|$. If G is abelian then the assertion is clear, take $A = G$ and $H = 1$. Now let G be a finite nonabelian p-group and assume that the assertion holds for all p-groups of order less than $|G|$. Moreover we may assume that $Z(G) \not\leq \Phi(G)$, otherwise one may take $A = 1$ and $H = G$ to complete the proof. Thus there exist some element $a \in Z(G)$ and a maximal subgroup M of G such that $a \notin M$. By induction hypothesis $M = BH$ for some subgroups B and H of M such that $B \leq Z(M)$ and $Z(H) \leq \Phi(H)$. Let $A = \langle a, B\rangle$. Therefore $A \leq Z(G)$ and $G = AH$. This completes the proof. □

Remark 2.2 ([4, Remark 4.]). *Let G be a central product of subgroups A and B; i.e., $G = AB$ and $[A, B] = 1$. Suppose that $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ agree on $A \cap B$. Then α and β admit a common extension $\gamma \in \mathrm{Aut}(G)$. In particular, if A has a noninner automorphism of order p which fixes $Z(A)$ elementwise, then G has a noninner automorphism of order p leaving both $Z(A)$ and B elementwise fixed.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a finite nonabelian p-group. By Lemma 2.1, we have $G = AH$ for some subgroups A and H of G such that $A \leq Z(G)$ and $Z(H) \leq \Phi(H)$. If G is regular, or of nilpotency class 2, or with cyclic commutator subgroup, then so is H. Now, suppose that $G/Z(G)$ is powerful. If $p > 2$, then $H'Z(G)/Z(G) \leq G'Z(G)/Z(G) \leq G^pZ(G)/Z(G)$. Thus $H' \leq G^pZ(G) = H^pZ(G)$, since $G^p = A^pH^p$. Now if $c \in H'$, then $c = ba$ for some $b \in H^p$ and $a \in Z(G)$. But $b^{-1}c = a \in Z(H)$. Therefore $H' \leq H^pZ(H)$ and this means that $H/Z(H)$ is powerful. A similar argument shows that $H/Z(H)$ is powerful for $p = 2$. Then, by Proposition 1.1, H has a noninner automorphism of order p fixing $Z(H)$ elementwise. Now it follows from Remark 2.2 that G has a noninner automorphism of order p leaving $AZ(H) = Z(G)$ elementwise fixed. This completes the proof. □

We finish the paper with the following conjecture.

Conjecture 2. *Every finite nonabelian p-group admits a noninner automorphism of order p leaving the center elementwise fixed.*

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