OPTIMAL ANALYSIS OF REGULAR STRUCTURES USING A SUBSTRUCTURING TECHNIQUE

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ABSTRACT

A structure is regular if its model can be represented as a product graph. Regular structures have certain properties that facilitate their optimal static and free vibration analysis. In this paper the concepts of rotational regular and translational regular structures are introduced, and using the well-known dynamic sub-structuring technique a method is proposed to relate the behavior of a translational regular structure to its rotational regular counterpart. It is shown that using the proposed method the analysis of a translational regular structure can be significantly accelerated compared to a direct method of solution.

The efficiency of the proposed method in approximating the requested number of natural periods and mode shapes of a translational regular structure is demonstrated through numerical examples. The accuracy of the obtained results is compared to other approximation methods.

Keywords: Optimal analysis; regular structure; dynamic substructuring; free vibrations

1. INTRODUCTION

Symmetric and regular structures commonly occur in engineering design because of ease of construction, esthetic appeal and their optimal load-carrying capabilities. A structure is said to possess symmetry if through one or more symmetry operations its configuration becomes physically indistinguishable from the initial configuration. A structure is called regular if its model can be considered as a product graph (see [1] for definition of a product graph). Physically, a regular structure consists of identical components that are repeated in a special pattern, governed by the graph model of the structure.

Symmetry and regularity can be exploited in structural mechanics problems to simplify the computations through decomposition of the structural models [2,3]. Matrices associated with regular models, such as adjacency and Laplacian matrices for graphs, and the stiffness and mass matrices for structural models, exhibit well-structured canonical forms [4]. Such

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forms have proven to be very advantageous in the computations such as eigen-solution for graph models [5-9] and free vibration analysis for structural models [10-12], due to the potential for decomposition of the corresponding matrices. An extensively investigated canonical form, for which decomposition methods both for graph and structural models are readily available now, is the block circulant form [13-14]. This form may be associated with *rotational regular* (RR) models. Structural and mechanical models having this type of regularity are also called rotational periodic or cyclic repeated structures. Different types of domes, space structures, cooling towers, pipes, blades and many others, fall in this category. The general pattern of the matrices associated with a *rotational regular structure* (RRS) can be represented by:

$$\begin{bmatrix} \mathbf{A}_m & \mathbf{B}_m & & & \mathbf{B}_m^T \\ \mathbf{B}_m^T & \mathbf{A}_m & \mathbf{B}_m & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{B}_m^T & \mathbf{A}_m & \mathbf{B}_m \\ \mathbf{B}_m & & & \mathbf{B}_m^T & \mathbf{A}_m \end{bmatrix}_{mn\times mn} . \tag{1}$$

Yet, for another type of widespread regularity pattern observed in structural models, i.e. the *translational regularity*, there has not been reported any general method of decomposition in the literature. *Translational regular structures* (TRS), also well-known as linear periodic, are more frequently encountered in engineering structures such as frames, trusses, shells and other types of structural and mechanical models. Decomposition of the matrices associated with such models is entangled due to non-commuting pattern and the occurrence of inconsistent corner blocks at the corresponding canonical forms. The general pattern of the matrices associated with a TRS can be given by:

$$\begin{bmatrix} \mathbf{C}_m & \mathbf{B}_m & & & \\ \mathbf{B}_m^T & \mathbf{A}_m & \mathbf{B}_m & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{B}_m^T & \mathbf{A}_m & \mathbf{B}_m \\ & & & \mathbf{B}_m^T & \mathbf{D}_m \end{bmatrix}_{mn\times mn} . \tag{2}$$

It would be promising to try to relate the behavior of a translational regular system to that of a rotational regular one, in order to take the computational advantages offered by the decomposable structure of the latter. For this purpose, the TRS can be represented as RRS using the established substructuring techniques. One such representation has been proposed by Garvey and Penny [15]. In this reference, a TRS is represented as an RRS by identifying the first and the last bay with each other, and hence reducing the size of the model by the size of one bay. Two classes of substructuring i.e. Kron's method [16] and Hurty method [17] have been employed for the solution of the resulting problem.

The representation provided by Garvey and Penny has two major shortcomings, so that it becomes inappropriate for the regular models considered in this paper. First, the two ends of

a TRS are identified (connected together) at nodal points, and in order for the representation to be correct, no overlapping of elements or nodal masses should be created in the identification process. The regular models studied here, may not lead to an RRS, if represented in this way. Second, the substructure representation method proposed by Garvey and Penny does not reduce the resulting matrices, and hence it is not obvious how much saving can be achieved in computational effort by using this method. Most of the structural applications require just a few lower eigen-frequencies and their corresponding modal vectors. Hence, it is essential for a substructuring technique to be able to reduce the size of the problem using suitable approximations, so that the required frequencies and mode shapes can be computed more efficiently and with acceptable accuracy.

In this paper a different substructure representation for TRS is proposed using a dual formulation. The dual formulation of general substructuring problem is due to Rixen [18]. Instead of using displacement constraints to identify the two ends of a TRS, the Lagrange multipliers or equivalently the interface forces are used to represent a TRS as a modification of its RR counterpart. The Lagrange multipliers are introduced as balancing forces to eliminate the effects of the imposed modifications. The analysis of modified regular structures using substructuring techniques was investigated by the authors in a previous work [19]. Using a dual description for substructure representation of TRS has the advantage that the response of the corresponding RRS including its natural modes and static response could be directly incorporated into a reduction basis. This basis is then used to reduce the dual system. The reduction is a key step in substructuring process and determines the efficiency of the proposed method.

2. SUBSTRUCTURE REPRESENTATION OF TRS

The generalized eigenvalue problem for free vibration of a translational regular structure may be expressed as follows:

$$\mathbf{K}_{\mathrm{TR}}\mathbf{u}-\lambda\mathbf{M}_{\mathrm{TR}}\mathbf{u}=\mathbf{0}\,, \tag{3}$$

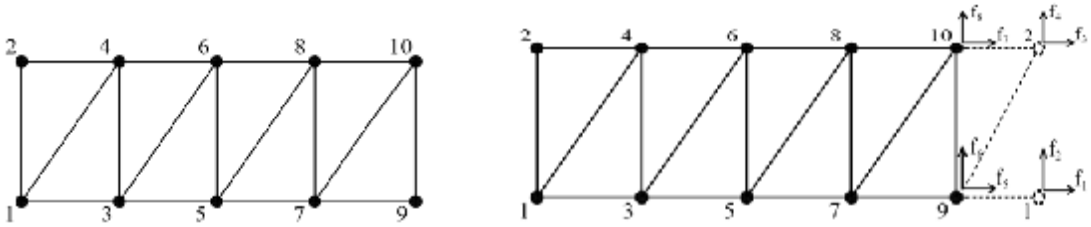
where $\mathbf{u}$ is the mode shape vector, λ the natural frequency squared, and $\mathbf{K}_{\mathrm{TR}}$ and $\mathbf{M}_{\mathrm{TR}}$ of order mn are the stiffness and mass matrices, each having a block pattern similar to (2). This matrix pattern is associated with a TRS consisting of n sequential blocks each having m DOFs. As an example, consider a 2D truss shown in Figure 1(a). In this example m=4 and n=5. Now, to construct an RR representation, let us introduce artificial members as depicted in dashed lines in Figure 1(b), to connect the nodes of the first and the last blocks. This manipulation is tantamount to modification of stiffness and mass matrices as follows:

$$\mathbf{K}_{\mathrm{RR}}=\mathbf{K}_{\mathrm{TR}}+\mathbf{E}\,\Delta\mathbf{k}\,\mathbf{E}^{\mathrm{T}};\ \ \mathbf{M}_{\mathrm{RR}}=\mathbf{M}_{\mathrm{TR}}+\mathbf{E}\,\Delta\mathbf{m}\,\mathbf{E}^{\mathrm{T}}\,. \tag{4}$$

The artificial modifications are performed in such a way that the resulting $\mathbf{K}_{\mathrm{RR}}$ and $\mathbf{M}_{\mathrm{RR}}$ matrices represent an RRS, having block patterns similar to (1). $\Delta\mathbf{k}$ and $\Delta\mathbf{m}$ are of order $2m$. $\mathbf{E}$ of order $m\times n$ by $2m$ is a Boolean matrix of association between the set of $2m$ modified

DOFs and $m\times n$ base DOFs:

$$\mathbf{E}^{\mathrm{T}}=\begin{bmatrix}\mathbf{I}_{\mathrm{m}} & 0 & 0 & \mathbf{L} & 0\\ 0 & \mathbf{L} & 0 & 0 & \mathbf{I}_{\mathrm{m}}\end{bmatrix}_{2\mathrm{m}\times\mathrm{mn}}. \tag{5}$$



(a) (b)

Figure 1. A TR structural model, (b) The RR representation

Dynamic equation (3) can still be satisfied with the modified matrices (4), introducing the vector of Lagrange multipliers **f**:

$$\mathbf{K}_{\mathrm{RR}}\mathbf{u}-\lambda\mathbf{M}_{\mathrm{RR}}\mathbf{u}+\mathbf{E}\mathbf{f}=\mathbf{0}. \tag{6}$$

f is a *2m* by 1 vector of the Lagrange multipliers or equivalently interface forces introduced at the modified nodes to balance the effects of the imposed modifications. Note that the eigenpairs (λ, **u**) are the response parameters of the free vibrating TRS, and the fabricated RRS is forced by vector **f** to exhibit such a response. Hence, vector **f** is determined solely by the artificial modification imposed on the TRS. This can be demonstrated by the equation:

$$\Delta\mathbf{k}\,\mathbf{v}-\lambda\Delta\mathbf{m}\mathbf{v}+\mathbf{f}=\mathbf{0}. \tag{7}$$

Compatibility is satisfied by:

$$\begin{bmatrix}\mathbf{E}^{\mathrm{T}} & -\mathbf{I}_{2\mathrm{m}}\end{bmatrix}\begin{Bmatrix}\mathbf{u}\\ \mathbf{v}\end{Bmatrix}=\mathbf{0}. \tag{8}$$

Putting the set of equations (6-8) altogether, the system equation can be assembled in the following block form:

$$\begin{bmatrix}\mathbf{K}_{\mathrm{RR}} & \mathbf{0} & \mathbf{E}\\ \mathbf{0} & -\Delta\mathbf{k} & -\mathbf{I}\\ \mathbf{E}^{\mathrm{T}} & -\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{f}\end{bmatrix}-\lambda\begin{bmatrix}\mathbf{M}_{\mathrm{RR}} & & \\ & -\Delta\mathbf{m} & \\ & & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{f}\end{bmatrix}=\mathbf{0}. \tag{9}$$

The order of the assembled system (9) is $mn+4m$, and has for general solution mn eigenvalues l, discarding $4m$ infinite eigenvalues introduced due to redundancy of the system equations. The order of the system should be reduced using appropriate approximations, to facilitate efficient calculation of a few requested lower eigenpairs.

3. MODAL TRUNCATION

As mentioned before, matrices associated with an RRS can be decomposed by transforming them into block-diagonal forms. Decomposition procedure involves the construction of an orthogonal matrix **T**, such that:

$$\mathbf{K}^{(\mathbf{BD})} = \mathbf{T}^{\mathrm{T}}\mathbf{K}_{\mathrm{RR}}\mathbf{T} \text{ and } \mathbf{M}^{(\mathbf{BD})} = \mathbf{T}^{\mathrm{T}}\mathbf{M}_{\mathrm{RR}}\mathbf{T}, \tag{10}$$

each have the same block-diagonal form.

Using such a transformation, the analysis of an RRS can be reduced to several smaller decoupled sub-problems. The response of the system is then obtained much more easily and quickly, by solving the reduced subsystems.

We are not going to discuss the decomposition methods here. The reader may consult the references mentioned in the introduction. For the purposes of this paper, we assume that the eigenvalues and eigenvectors of an RRS are obtained using a decomposition method. Let $\mathbf{\Phi}$ be the matrix of $\mathbf{M}_{\mathrm{RR}}$-orthonormal eigenvectors and $\mathbf{\Lambda}$ be the diagonal matrix of eigenvalues of the RRS. Hence the following relations hold:

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}_{\mathrm{RR}}\mathbf{\Phi} = \mathbf{\Lambda} \text{ and } \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}_{\mathrm{RR}}\mathbf{\Phi} = \mathbf{I}. \tag{11}$$

Let us define a cutoff eigenvalue λ_c and suppose that we are interested in those eigenvalues λ of equation (3) or the equivalent assemblage (9), such that

$$l << l_c. \tag{12}$$

Now, partition $\mathbf{\Lambda}$ into lower and higher eigenvalues based on the cutoff value λ_c, as

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_l & \\ & \mathbf{\Lambda}_h \end{bmatrix}, \tag{13}$$

where $\mathbf{\Lambda}_l$ is the set of eigenvalues less than λ_c and $\mathbf{\Lambda}_h$ is the set of eigenvalues greater than or equal to λ_c. Let the corresponding partitioning of $\mathbf{\Phi}$ be

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_l & \mathbf{\Phi}_h \end{bmatrix}. \tag{14}$$

Using modal coordinates **q** defined by

$$\mathbf{u} = \mathbf{\Phi}_l\mathbf{q}_l + \mathbf{\Phi}_h\mathbf{q}_h, \tag{15}$$

and pre-multiplying the problem (6) by $\mathbf{\Phi}_h^{\mathrm{T}}$, it follows:

$$\mathbf{\Lambda}_h \mathbf{q}_h - \lambda \mathbf{q}_h + \mathbf{\Phi}_h{}^{\mathrm{T}} \mathbf{E}\mathbf{f} = \mathbf{0}. \tag{16}$$

Premultiplying equation (16) by $\mathbf{\Lambda}_h{}^{-1}$ and putting $\lambda \mathbf{\Lambda}_h{}^{-1} \approx \mathbf{0}$ due to the assumption (12), we arrive at:

$$\mathbf{q}_h \cong -\mathbf{\Lambda}_h{}^{-1} \mathbf{\Phi}_h{}^{\mathrm{T}} \mathbf{E}\mathbf{f}\ . \tag{17}$$

Substituting into (15) from (17), we obtain the following approximation for vector **u**

$$\mathbf{u} \cong \mathbf{\Phi}_l \mathbf{q}_l - \mathbf{G}_{res} \mathbf{E}\mathbf{f}\ , \tag{18}$$

where

$$\mathbf{G}_{res} = \mathbf{\Phi}_h \mathbf{\Lambda}_h{}^{-1} \mathbf{\Phi}_h{}^{\mathrm{T}}, \tag{19}$$

is the residual flexibility of the RR model.

In summary we construct the following approximation of the coordinate vectors and the Lagrange multipliers for the reduction of eigenproblem (9):

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{f} \end{bmatrix} \cong \mathbf{T}_{\mathrm{dual}} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{v} \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_l & \mathbf{0} & -\mathbf{G}_{res}\mathbf{E} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{v} \\ \mathbf{f} \end{bmatrix}. \tag{20}$$

4. REDUCED EIGENPROBLEM

Using the approximation (20), we will reduce the assembled system (9). First, notice the following properties of the residual flexibility matrix

$$\begin{aligned} &\mathbf{G}_{res}^{\mathrm{T}} = \mathbf{G}_{res}, \quad \mathbf{G}_{res}\mathbf{K}_{\mathrm{RR}}\mathbf{G}_{res} = \mathbf{G}_{res}, \\ &\mathbf{\Phi}_l^{\mathrm{T}}\mathbf{K}_{\mathrm{RR}}\mathbf{G}_{res} = \mathbf{0}, \quad \mathbf{\Phi}_l^{\mathrm{T}}\mathbf{M}_{\mathrm{RR}}\mathbf{G}_{res} = \mathbf{0}. \end{aligned} \tag{21}$$

The reduced eigenproblem of the modified regular structure is then obtained by using transformation (20), as

$$\mathbf{\hat{K}} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{v} \\ \mathbf{f} \end{bmatrix} - \lambda \mathbf{\hat{M}} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{v} \\ \mathbf{f} \end{bmatrix} = \mathbf{0}, \tag{22}$$

with the reduced matrices

$$\mathbf{\hat{K}} = \begin{bmatrix} \mathbf{\Lambda}_l & \mathbf{0} & \mathbf{\Phi}_l^{\mathrm{T}}\mathbf{E} \\ \mathbf{0} & -\mathbf{\Delta k} & -\mathbf{I} \\ \mathbf{E}^{\mathrm{T}}\mathbf{\Phi}_l & -\mathbf{I} & \mathbf{F}_{res} \end{bmatrix} \text{ and } \mathbf{\hat{M}} = \begin{bmatrix} \mathbf{I} & & \mathbf{0} \\ & -\mathbf{\Delta m} & \\ \mathbf{0} & & \mathbf{M}_{res} \end{bmatrix}, \tag{23}$$

where

$$\mathbf{F}_{res} = -\mathbf{E}^{\mathrm{T}}\mathbf{G}_{res}\mathbf{E}, \qquad \mathbf{M}_{res} = \mathbf{E}^{\mathrm{T}}\mathbf{G}_{res}\mathbf{M}_{\mathrm{RR}}\mathbf{G}_{res}\mathbf{E}. \tag{24}$$

$\mathbf{\Lambda}_l$ and $\mathbf{\Phi}_l$ correspond to the calculated eigenvalues and eigenmodes of the RRS. $\mathbf{\Phi}_l$ is also called the *master* modes. These include rigid body modes of the RRS if they are present. Hence, $\mathbf{\Lambda}_l$ may contain zero eigenvalues corresponding to rigid body modes. The calculations differ in constructing the residual flexibility matrix, with or without the presence of rigid body modes.

5. EVALUATION OF THE RESIDUAL FLEXIBILITY MATRIX

In general case, stiffness matrix $\mathbf{K}_{\mathrm{RR}}$ of the rotational regular model may be positive semi-definite. First, we discuss the evaluation of $\mathbf{G}_{res}$ when $\mathbf{K}_{\mathrm{RR}}$ is positive definite.

5.1 Positive definite stiffness matrix

In this case, $\mathbf{K}_{\mathrm{RR}}$ is nonsingular and using the following result

$$\mathbf{K}_{\mathrm{RR}}^{-1} = \mathbf{\Phi}\mathbf{\Lambda}^{-1}\mathbf{\Phi}^{\mathrm{T}} = \mathbf{\Phi}_l\mathbf{\Lambda}_l^{-1}\mathbf{\Phi}_l^{\mathrm{T}} + \mathbf{\Phi}_h\mathbf{\Lambda}_h^{-1}\mathbf{\Phi}_h^{\mathrm{T}}, \tag{25}$$

we conclude that

$$\mathbf{G}_{res} = \mathbf{K}_{\mathrm{RR}}^{-1} - \mathbf{\Phi}_l\mathbf{\Lambda}_l^{-1}\mathbf{\Phi}_l^{\mathrm{T}}. \tag{26}$$

The inversion $\mathbf{K}_{\mathrm{RR}}^{-1}$ is constructed using the block-diagonal transform (10) as

$$\mathbf{K}_{\mathrm{RR}}^{-1} = \mathbf{T}\left[\mathbf{K}^{(\mathrm{BD})}\right]^{-1}\mathbf{T}^{\mathrm{T}}. \tag{27}$$

5.2 Positive semi-definite stiffness matrix

In this case there are rigid-body modes present in the calculated modal matrix $\mathbf{\Phi}_l$, and they should be suppressed first to obtain the elastic part of the response, from which the residual flexibility is then calculated.

Let us partition the M-orthonormal modal matrix $\mathbf{\Phi}$ as follows

$$\mathbf{\Phi} = \left[\mathbf{\Phi}_l, \mathbf{\Phi}_h\right] = \left[\mathbf{\Phi}_r, \mathbf{\Phi}_a, \mathbf{\Phi}_h\right] = \left[\mathbf{\Phi}_r, \mathbf{\Phi}_e\right], \tag{28}$$

where $\mathbf{\Phi}_r$ is the matrix of the rigid body modes, $\mathbf{\Phi}_a$ the matrix of calculated eigenvectors corresponding to nonzero eigenvalues, and $\mathbf{\Phi}_e = \left[\mathbf{\Phi}_a, \mathbf{\Phi}_h\right]$ the elastic eigenvectors.

The response of the following system

$$\mathbf{K}_{\mathrm{RR}}\mathbf{x} = \mathbf{F}_0, \tag{28}$$

is composed of a rigid body response $\mathbf{x}_r$ and an elastic response $\mathbf{x}_e$. The elastic response can be expressed in terms of eigenvalues and eigenmodes as

$$\mathbf{x}_e = \left[\boldsymbol{\Phi}_e \boldsymbol{\Lambda}_e^{-1} \boldsymbol{\Phi}_e^{\mathrm{T}}\right]\mathbf{F}_0 = \left[\boldsymbol{\Phi}_a \boldsymbol{\Lambda}_a^{-1} \boldsymbol{\Phi}_a^{\mathrm{T}} + \boldsymbol{\Phi}_h \boldsymbol{\Lambda}_h^{-1} \boldsymbol{\Phi}_h^{\mathrm{T}}\right]\mathbf{F}_0 = \mathbf{G}_e \mathbf{F}_0 . \tag{29}$$

Hence, the residual flexibility matrix $\mathbf{G}_{res}$ can be obtained, provided that we have already calculated the elastic flexibility matrix $\mathbf{G}_e$. The relation is

$$\mathbf{G}_{res} = \mathbf{G}_e - \boldsymbol{\Phi}_a \boldsymbol{\Lambda}_a^{-1} \boldsymbol{\Phi}_a^{\mathrm{T}} . \tag{30}$$

The elastic flexibility matrix $\mathbf{G}_e$ is calculated using an inertia relief procedure to remove the rigid body modes. Let $\mathbf{R}$ be the orthogonal projector onto the complement subspace of $\boldsymbol{\Phi}_r$, defined by

$$\mathbf{R} = (\mathbf{I} - \boldsymbol{\Phi}_r \boldsymbol{\Phi}_r^{\mathrm{T}} \mathbf{M}) . \tag{31}$$

It can be shown [20, 21] that

$$\mathbf{G}_e = \mathbf{R}\,\bar{\mathbf{G}}_e \mathbf{R}^{\mathrm{T}} , \tag{32}$$

where $\bar{\mathbf{G}}_e$ is the elastic flexibility matrix relative to a set of imposed constraints. $\bar{\mathbf{G}}_e$ may be obtained by taking the stiffness matrix $\mathbf{K}_{\mathrm{RR}}$ which is singular, deleting rows and columns corresponding to constrained DOFs, inverting the resulting matrix and expanding back to the original size by adding zeros. If we use the block-diagonal transform $\mathbf{K}^{(\mathrm{BD})}$ of Eq. (10) instead of $\mathbf{K}_{\mathrm{RR}}$ for inversion, then we will have

$$\mathbf{G}_e = \mathbf{R}\,\mathbf{T}\bar{\mathbf{G}}_e^{(\mathrm{BD})}\mathbf{T}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} , \tag{33}$$

where $\bar{\mathbf{G}}_e^{(\mathrm{BD})}$ is obtained form $\mathbf{K}^{(\mathrm{BD})}$ in an analogous way to that of $\bar{\mathbf{G}}_e$. It should be noted that in the actual implementations, the inverse matrices are not computed explicitly, and the calculations are performed much more efficiently using LU decomposition with partial pivoting.

6. NUMERICAL EXPERIMENTS

Example 1: Consider the regular graph model shown in Figure 2(a). The Laplacian matrix of this graph has the following block form:

$$\mathbf{L}_{\mathrm{TR}} = \begin{bmatrix} \mathbf{C} & \mathbf{B} & & & \\ \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} \\ & & & \mathbf{B}^{\mathrm{T}} & \mathbf{D} \end{bmatrix}_{20\times 20} . \tag{34}$$

It consists of 10 diagonal 2 by 2 blocks with the following submatrices:

$$\mathbf{A}=\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix},\mathbf{B}=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},\mathbf{C}=\begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix},\mathbf{D}=\begin{bmatrix}3 & -1\\ -1 & 2\end{bmatrix}. \tag{35}$$

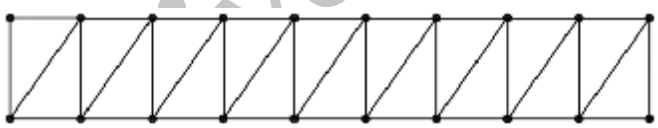
The aim is to estimate the first nonzero eigenvalue of the Laplacian matrix using the proposed method and compare the accuracy of the result with a direct method of solution. In Figure 2(b), a rotationally regular representation of the graph model is constructed by adding some artificial elements depicted in dashed lines. The corresponding modification to Laplacian $\mathbf{L}_{TR}$ is defined by:

$$\mathbf{L}_{RR}=\mathbf{L}_{TR}+\mathbf{E}\Delta\mathbf{L}\,\mathbf{E}^{T}, \tag{36}$$

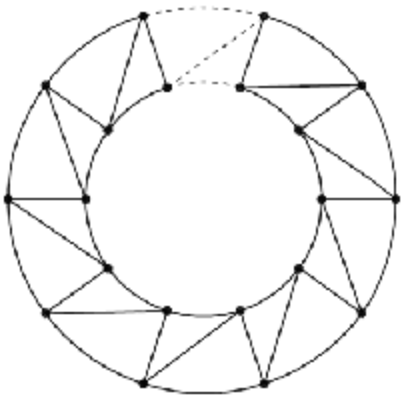
with the following submatrix:

$$\Delta\mathbf{L}=\begin{bmatrix}\mathbf{A}-\mathbf{C} & \mathbf{B}^{T}\\ \mathbf{B} & \mathbf{A}-\mathbf{D}\end{bmatrix}. \tag{37}$$

The first step of solution by the proposed method consists in obtaining necessary information from the rotational regular model, taking the advantage of its block-diagonalized Laplacian matrix. This information includes a few eigenvalues and eigenvectors together with a linear solution to get $\mathbf{G}_{res}\mathbf{E}$ using the procedure outlined in section 5.2. In Table 1, some of the eigenvalues obtained form the RR model are presented.



(a)



(b)

Figure 2. (a) A TR graph model, (b) The RR representation

The next step is to solve the reduced eigenproblem (22) with the following matrices:

$$\mathbf{K} = \begin{bmatrix} \mathbf{\Lambda}_l & \mathbf{0} & \mathbf{\Phi}_l^{\mathrm{T}}\mathbf{E} \\ \mathbf{0} & -\mathbf{\Delta L} & -\mathbf{I} \\ \mathbf{E}^{\mathrm{T}}\mathbf{\Phi}_l & -\mathbf{I} & -\mathbf{E}^{\mathrm{T}}\mathbf{G}_{res}\mathbf{E} \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \\ \mathbf{0} & & \\ \mathbf{0} & & \mathbf{E}^{\mathrm{T}}\mathbf{G}_{res}\mathbf{G}_{res}\mathbf{E} \end{bmatrix}, \quad (38)$$

in order to estimate the requested eigenvalue for the TR model. This problem is solved using different number of eigenpairs $\mathbf{\Lambda}_l$ and $\mathbf{\Phi}_l$ from the RR model, and the first nonzero eigenvalue (l_2) is obtained as shown in Table 2. It is observed that the accuracy of the result is improved by increasing the number of contributing master modes.

Table 1: Some eigenvalues of RR model-Example 1

Index	Eingenvalue
1	0.000000
2	0.479853
3	0.479853
4	1.763932
5	1.763932
6	3.442463
7	3.442463

Table 2: Comparison of the results for example 1

Number of master modes	l_2		Relative error (%)
	Present method	Direct method	
3	0.123795	0.122312	1.212135
5	0.122830	0.122312	0.422924
7	0.122612	0.122312	0.245107

Note that the relative error of an estimated eigenvalue is proportional to the ratio of that eigenvalue to the cutoff value [22]. The cutoff value is the lower eigenvalue in the set $\mathbf{\Lambda}_h$. In this example, λ_2 is estimated with about 0.25 percent error, using 7 master modes. The cutoff value for this case is the 7th eigenvalue of the RR model (3.442463, from Table 1). Hence, the relative error is proportional to 0.122312 / 3.442463 = 0.0355. The accuracy may be considered as satisfactory; however, better approximations can be obtained for large scale problems as demonstrated in the following examples.

Example 2: A translational regular model of a two dimensional truss is considered for free

vibration analysis, as shown in Figure 3. The structure is composed of pin jointed steel bar elements having cross-section area a=16.01 cm^2. The elastic modulus is E=2.1×10^{11} N/m^2. Horizontal elements are 1 m in length, and vertical elements have a length of 1.2 m. The structure is clamped at the two ends. For simplicity the mass of the structure is assumed to be concentrated at the nodal points, with a magnitude of 800 kg at each node. Each node of the structures has 2 translational DOFs and hence the total degree of freedom for the structure is 72.

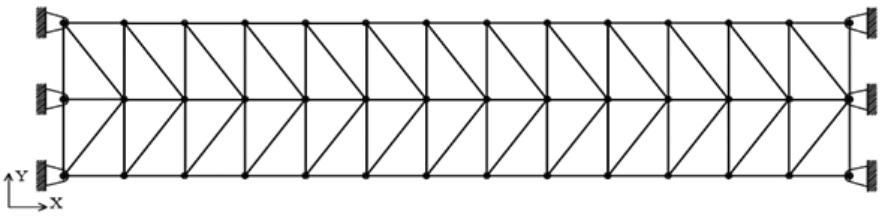


Figure 3. A TR truss model

The stiffness matrix can be put into the following canonical form:

$$\mathbf{K}_{\mathrm{TR}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \\ \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} & & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & & \\ & & \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} & \\ & & & \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \end{bmatrix}_{72\times 72} . \tag{39}$$

The blocks of the matrix are 6 by 6 each, and there are a total of 12 blocks on the diagonal. Necessary modification to the stiffness matrix in order to represent an RR is given by:

$$\Delta\mathbf{k} = \begin{bmatrix} \mathbf{0} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{0} \end{bmatrix}_{12\times 12} . \tag{40}$$

The mass matrix is diagonal and does not require any modification. The resulting stiffness matrix associated with RR representation, has the following pattern:

$$\mathbf{K}_{\mathrm{RR}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} & & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & & \\ & & \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \mathbf{B} & \\ \mathbf{B} & & & \mathbf{B}^{\mathrm{T}} & \mathbf{A} & \end{bmatrix}_{72\times 72} . \tag{41}$$

The eigenproblem associated with free vibration of the RR model is easily solved through

matrix decomposition to obtain different numbers of required eigenvectors as the master modes of RR structure. The associated isostatic modes are also obtained using the procedure outlined in section 5.2. Then, this information is utilized to form the reduced matrices in Eq. (23) using different numbers of master modes. The reduced problem in Eq. (22) is solved in each case to obtain 4 natural periods and mode shapes of the initial TR structure. The natural mode shapes are shown in Figure 4.

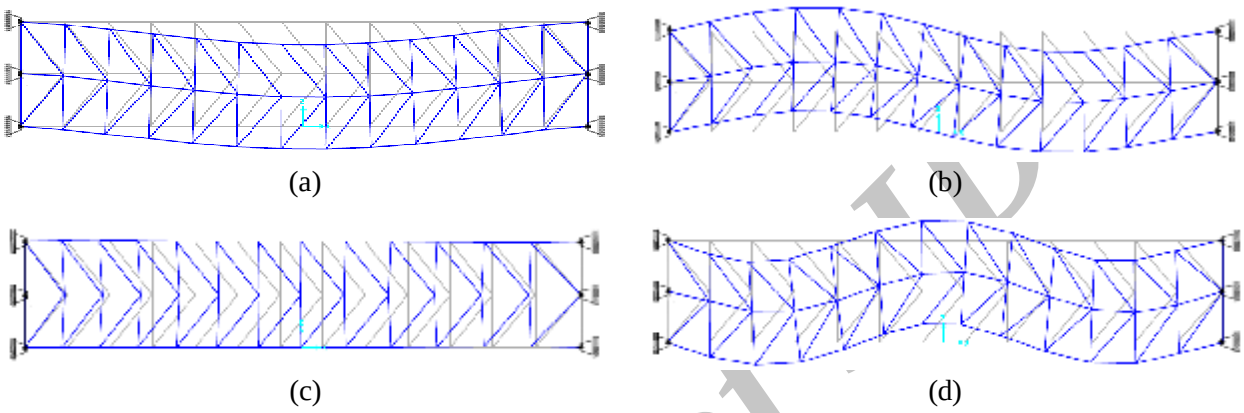


Figure 4. Natural modes of the TRS, (a) first, (b) second, (c) third, (d) fourth mode

The results of the analysis by the proposed method employing 7, 12 and 16 master modes are compared with a direct sparse eigensolver of the MATLAB software in Table 3. It is demonstrated in Figure 5 that very satisfactory results can be obtained using adequate number of master modes. According to this graph, using 12 and 16 master modes, all the four natural periods are estimated with a relative error well below 0.1 percent, which is a sufficient accuracy for usual engineering applications.

The mode shape errors are also presented in Table 3. This error is a measure of the angle between two vectors. It is calculated using the following relations:

$$\cos^2\theta = \frac{\left|\tilde{\mathbf{u}}^{\mathrm{T}}\mathbf{u}\right|^2}{\left(\tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{u}}\right)\left(\mathbf{u}^{\mathrm{T}}\mathbf{u}\right)}, \tag{42}$$

$$\text{Mode shape error} = 1-\cos^2 q\ , \tag{43}$$

in which $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are the mode shapes calculated using the direct method and the proposed method, respectively; and θ is the angle between the two modes.

With reference to Table 3, it is concluded that the obtained mode shapes are very satisfactory.

Table 3: Comparison of the natural periods and mode shapes-Example 2

		Mode number			
		1	**2**	**3**	**4**
Direct method	T(sec)	0.123632941	0.059508732	0.040227673	0.037383425
Present method with 7 master modes	T(sec)	0.123632008	0.059499666	0.039706865	0.037339264
	Relative error (%)	0.000754	0.015235	1.294650	0.118130
	Mode shape error	3.9E-06	1.1E-05	1.8E-05	1.2E-05
Present method with 12 master modes	T(sec)	0.123632645	0.059501690	0.040219806	0.037376774
	Relative error (%)	0.000239	0.011835	0.019555	0.017791
	Mode shape error	1.2E-06	1.6E-06	3.6E-06	2.0E-06
Present method with 16 master modes	T(sec)	0.123632874	0.059507609	0.040219806	0.037382283
	Relative error (%)	0.000054	0.001887	0.019555	0.003053
	Mode shape error	2.9E-08	1.1E-07	6.5E-07	3.5E-07

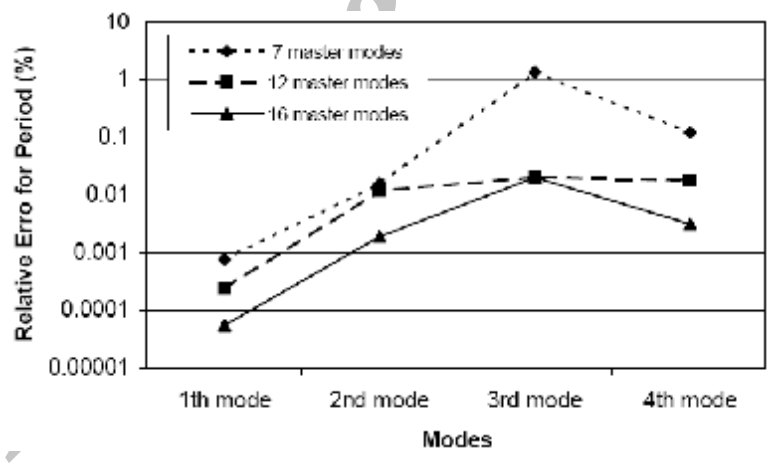


Figure 5. Relative errors of estimated natural periods

Example 3: Consider a square prismatic truss structure, shown in Figure 6(a). Horizontal and vertical members are 1 *m* and 0.5 *m* long, respectively. Members are made of steel with a mass density of $\rho=8.7\times10^3$ kg/m^3 and the modulus of elasticity $E=2.0\times10^{11}$ Pa. Each member has a cross-section area: $A = 9.14$ cm^2. Total height of structure is 10.5 m. Each node of structure has 3 translational DOFs, and the total number of DOFs for the structure amounts to 252. The structure vibrates under its own mass. Three natural periods and mode shapes of the structure are sought. A lumped mass approach is taken for evaluation of the mass matrix. Stiffness and mass matrices have the following canonical forms:

$$\mathbf{K}_{TR}=\begin{bmatrix} \mathbf{A} & \mathbf{B} & & & \\ \mathbf{B}^T & \mathbf{A} & \mathbf{B} & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{B}^T & \mathbf{A} & \mathbf{B} \\ & & & \mathbf{B}^T & \mathbf{C} \end{bmatrix}_{252\times252};\quad \mathbf{M}_{TR}=\begin{bmatrix} \mathbf{m} & & & & \\ & \mathbf{m} & & & \\ & & \mathbf{O} & & \\ & & & \mathbf{m} & \\ & & & & \bar{\mathbf{m}} \end{bmatrix}_{252\times252} \tag{44}$$

The block sub-matrices are 12 by 12 each, and 21 such blocks are present on the diagonals of each stiffness and mass matrices.

Necessary modifications to stiffness and mass matrices are imposed by using the following submatrices:

$$\Delta\mathbf{k}=\begin{bmatrix} \mathbf{0} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{A}-\mathbf{C} \end{bmatrix}_{24\times24}\quad \Delta\mathbf{m}=\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}-\bar{\mathbf{m}} \end{bmatrix}_{24\times24}, \tag{45}$$

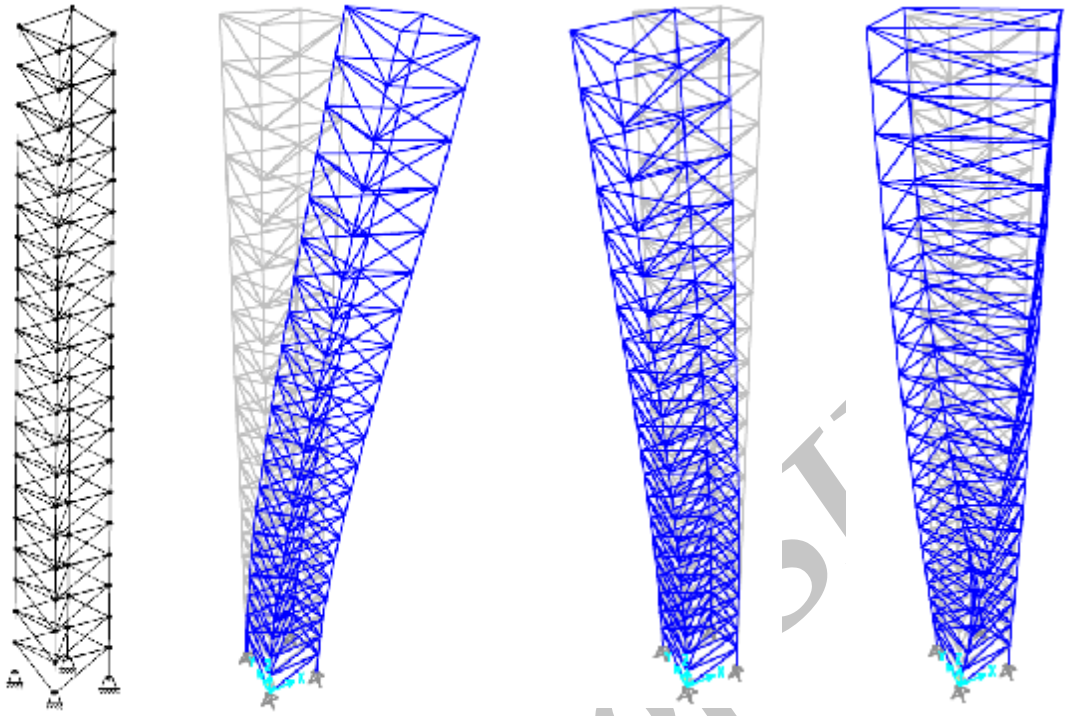
in order to turn the TRS into an RRS which has associated stiffness and mass matrices with the following patterns:

$$\mathbf{K}_{RR}=\begin{bmatrix} \mathbf{A} & \mathbf{B} & & & \mathbf{B}^T \\ \mathbf{B}^T & \mathbf{A} & \mathbf{B} & & \\ & \mathbf{O} & \mathbf{O} & \mathbf{O} & \\ & & \mathbf{B}^T & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & & & \mathbf{B}^T & \mathbf{A} \end{bmatrix}_{252\times252};\quad \mathbf{M}_{RR}=\begin{bmatrix} \mathbf{m} & & & & \\ & \mathbf{m} & & & \\ & & \mathbf{O} & & \\ & & & \mathbf{m} & \\ & & & & \mathbf{m} \end{bmatrix}_{252\times252} \tag{46}$$

Table 4 summarizes the results for the 3 natural periods obtained from the direct method and the proposed method with 7, 12 and 16 master modes form RRS. Mode shapes are also evaluated using Eqs. (43) and (44). These mode shapes are depicted in Figures 6(b-d). Calculations are performed at double precision arithmetic and on a computer with Intel® Core ™2 Duo CPU 2.33 GHz and 2 GB of RAM, which was running Microsoft Windows XP professional Service Pack 3.

With reference to Table 4, it is observed that satisfactory approximations are obtained for both natural periods and mode shapes of the TRS using the proposed method. One can conclude that using 16 master modes (about 5 times the required number of natural modes), the periods are estimated with less than 0.01 percent error, indicating that the accuracy of the obtained results is very satisfactory.

The CPU times required to accomplish the calculations using the proposed method, and the direct method using a sparse eigensolver of MATLAB software are also presented in this table. The time spent by the present method is slightly increased with the incorporation of more master modes. However, the time savings are remarkable compared with the direct method. It can be argued that the present method in the worst case performs nearly 40 times faster than the direct method to estimate a few natural periods and mode shapes of the regular structure, with an acceptable approximation.



(a) (b) (c) (d)

Figure 6. (a) A translationally regular truss, (b) first, (c) second, (d) third natural modes of vibration

Table 4: Comparison of the natural periods and mode shapes-Example 3

	Mode number		
	1	**2**	**3**
Direct method			
T(sec)	0.213472734	0.196085485	0.062959495
CPU time (sec)		4.168	
Present method with 7 master modes			
T(sec)	0.213387095	0.19602602	0.06292555
Relative error (%)	0.040117	0.030326	0.053910
Mode shape error	4.7E-06	1.0E-05	5.8E-05
CPU time (sec)		0.0671	
Present method with 12 master modes			
T(sec)	0.213447395	0.196061943	0.06294718
Relative error (%)	0.011870	0.012006	0.019555
Mode shape error	1.3E-06	7.2E-06	2.4E-05
CPU time (sec)		0.0825	
Present method with 16 master modes			
T(sec)	0.213470313	0.196079557	0.06295474
Relative error (%)	0.001134	0.003023	0.007546
Mode shape error	6.9E-07	1.8E-06	5.6E-06
CPU time (sec)		0.105	

7. CONCLUSIONS

Plenty of structural and mechanical models can be considered as translational regular (TR) systems. The behavior of a TR model can be related to another class of widely investigated regular model, i.e. rotational regular (RR) model. It is shown in this paper that the analysis of a TR structure can be significantly optimized using the similarity between the TR and the corresponding RR structure. Computational advantages offered by the decomposable matrix pattern of an RR structure are extended to TR by representing it efficiently as a substructure. Dynamic substructuring and modal approximations were used to reduce the size of the governing eigenproblem. The proposed method is very efficient for estimating a few natural periods and mode-shapes of large scale regular structures and the accuracy of the obtained results is satisfactory in usual engineering applications. The efficiency of the proposed method relies mainly on two factors: first, the information obtained from the RR model due to its decomposable structure, and second, the reduced order of the governing eigenproblem.

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