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Congestion in DEA Model with Fuzzy Data

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Abstract Earlier methods in classic DEA investigate congestion for DMUs with crisp data. Although in real world we face data which are not crisp necessarily, available methods in this situation are not enough. In this paper for investigating congestion, a new method with fuzzy data has been promoted. In order to evaluate the performance of the method it is shown how our new method lead in exact results by an numerical example.

Keywords Data Envelopment Analysis, Fuzzy DEA Distance, Congestion, Efficiency.

1 Introduction

Data Envelopment Analysis (DEA) is a technique to measure the relative efficiency of a set of decision making units with common data. This technique was initially proposed by Charnes et al. [1] and was improved by others [2, 3]. In traditional DEA models such as CCR and BCC models and others we assume that all inputs and outputs data are exactly known. But in real world this assumption is not always true [4, 5].

But, in more general cases, the data for evaluation are stated by natural language such as good, medium, bad to reflect general situation. So we can not solve DEA models with fuzzy data by usual methods. Some researchers have proposed several fuzzy models to evaluate DMUs with fuzzy data, using concept of comparison of fuzzy numbers [6-9].

One of the most important concepts in DEA is Congestion. **"Congestion"** [10-15] is an economic state where excessively invest in inputs system. Evidence of congestion occurs when the decreasing of some inputs can indeed increase some outputs while other inputs and outputs stay unchanged. One of the problems in using classic DEA models is that they just are capable to evaluate DMUs (Decision Making Units) with crisp data. In traditional DEA models such as CCR and BCC models and others, we assume that all inputs and outputs data are exactly known.

The purpose of this paper is to introduce a new technique to investigate the issue of "congestion", where data is non crisp in specific required models. We use output additive models BCC and NEW with fuzzy inputs and outputs to identify DMUs with the property of congestion. A numerical example is presented to show the application of our method.

This paper is arranged as follows: in section 2, fuzzy numbers are introduced, in section 3 congestion in DEA along with necessary definitions and theorems are explained. In section 4

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the new method for evaluating congestion is introduced. Finally, in section 5 an example on our process is prepared.

2 Fuzzy numbers

A fuzzy set A in X is a set of ordered pairs:

$$A=\{(x,\mu_A(x))\,|\,x\in X\}$$

$\mu_A(x)$ is called the membership function of X in A.
A fuzzy number M is a convex normalized fuzzy set M of real line R such that:

1. It exists exactly one $x\in R$ with $\mu_M(x)=1$.
2. $\mu_M(x)$ is piece wise continuous.

The crisp set that belong to the fuzzy set A at least to the degree $\alpha-$cut set:

$$A_\alpha=\{x\in X:\mu_A(x)\geq\alpha\}$$

The lower and upper end points of any $\alpha-$cut set, A_α, are represented by $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ respectively.
For arbitrary fuzzy number $u=(\underline{u},\overline{u}),v=(\underline{v},\overline{v})$

$$D(u,v)=[\int_0^1(\underline{u}(r)-\underline{v}(r))^2dr+\int_0^1(\overline{u}(r)-\overline{v}(r))^2dr]^{1/2}$$

Is distance between u,v [16-18].
Theorem 1. Let u be a fuzzy number and c(u) a crisp point then the function D(u, c(u)) with respect c(u) is minimum value if c(u)=m(u) and m(u) is unique and

$m(u)=1/2\int_0^1(\underline{u}(r)+\overline{u}(r))dr.$

Proof. see [19].
Theorem 2. Let u and v are fuzzy numbers therefore, m (u+v)=m(u)+m(v).
Proof. see [5].
Theorem 3. Let u be a fuzzy number and λ be a scalar so $m(\lambda u)=\lambda m(u)$.
Proof. see [5].

For arbitrary fuzzy numbers u and v

$$u\geq v\Leftrightarrow m(u)\geq m(v)$$

and

$$u\sim v\Leftrightarrow m(u)=m(v)$$

so

$$u\geq v\Leftrightarrow u\succ v\ or\ u\sim v$$

A convex and normal fuzzy set A is said to be a fuzzy number if :

1. $\mu_A(x_0)=1$

2. μ_A is continuous

If a fuzzy number A has a membership function as following then it is called LR fuzzy number.

$$\mu_A(x) = \begin{cases} 1)\ L(\frac{M-x}{\alpha}) & x \le M \\ 2)\ R(\frac{X-M}{\beta}) & x > M \end{cases}$$

M is called number middle, and α, β are left and right expanse. Triangle fuzzy number is kind of LR in which L(x)=R(x)=1-x and usually is presented as (A, M, B).

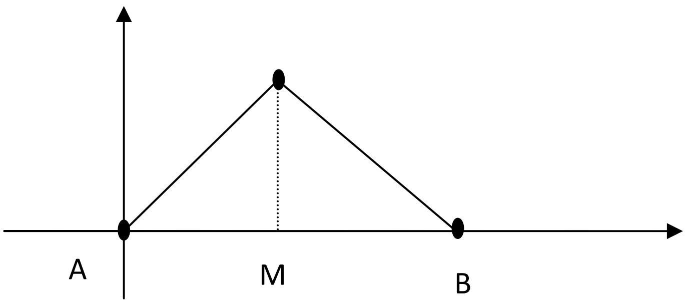

Fig. 1 Triangle fuzzy number

3 Congestion in DEA

Congestion is an economic state where one excessively invests in system inputs. Evidence of congestion occurs when the decreasing of some inputs can indeed increase some outputs while other inputs and outputs stay unchanged [14]. If we find inputs result in congestion and take a logical decision on them by decreasing them, we can observe that a huge amount of asset or man force will be saved.

We need two models BCC and NEW to identify congestion which are presented in the following.

$$\begin{aligned} \text{Max} \quad & h_{BCC} = \sum_{r=1}^{s} s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j \lambda_j \le x_o, \qquad i = 1, \dots, m, \\ & \sum_{j=1}^{n} y_j \lambda_j - s^+ = y_o, \qquad r = 1, \dots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \ge 0, \qquad j = 1, \dots, n. \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{Max} \quad & h_{NEW} = \sum_{r=1}^{s} s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_j \lambda_j = x_o, \qquad i = 1, \dots, m, \\ & \sum_{j=1}^{n} y_j \lambda_j - s^+ = y_o, \qquad r = 1, \dots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j = 1, \dots, n. \end{aligned} \tag{2}$$

Based on [14, 15] some principle definitions are required to recognize congestion.

Definition 1. Let DMU_O be (NEW) output efficient. If $(\hat{x}, \hat{y})$ be a feasible solution such that $\hat{x} \leq x_o, \hat{x} \neq x_o$ and $\hat{y} \neq y_o$, $\hat{y} \geq y_o$ then DMU_O is said to evidence congestion.

Earlier research has investigated the relationships among these models and found the following:

- Let DMU_O be (NEW) output efficient. Then, DMU_O evidences congestion if and only if DMU_O is not (BCC) output efficient.

4 Evaluating congestion in DEA with fuzzy data

In empirical cases, we encounter data which are not known; as a result, it emphasizes the necessity of considering congestion with data which are not crisp. So, we continue the steps of using mentioned models with fuzzy data.

First, we introduce two modified models BCC and NEW in the following part; then, the process of reaching this two fuzzy models is thoroughly explained.

We apply a set of DMUs that is DMU_j , j =1,…, n, with fuzzy input-output vectors $(\tilde{x}_j, \tilde{y}_j)$ in which $\tilde{x}_j \in F(R) \geq 0$ and $\tilde{y}_j \in F(R) \geq 0$ where F(R) ≥ 0 is family of all non negative fuzzy numbers.

$$
\begin{aligned}
\text{Max}\quad & h'_{NEW} = \sum_{r=1}^{s} s_r^+ \\
\text{s.t.}\quad & \sum_{j=1}^{n} \tilde{x}_j \lambda_j = \tilde{x}_o, \qquad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \tilde{y}_j \lambda_j - s^+ = \tilde{y}_o, \qquad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j = 1, \dots, n.
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
\text{Max}\quad & h'_{BCC} = \sum_{r=1}^{s} s_r^+ \\
\text{s.t.}\quad & \sum_{j=1}^{n} \tilde{x}_j \lambda_j \leq \tilde{x}_o, \qquad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \tilde{y}_j \lambda_j - s^+ = \tilde{y}_o, \qquad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j = 1, \dots, n.
\end{aligned} \tag{4}
$$

It should be attended that objective functions of models (3) and (4) are in the fuzzy form. Thereby, due to maximizing objective functions closest number to them can be used. Based on theorems 1, 2, 3 model (1) transformed as following:

$$
\begin{aligned}
\text{Max}\quad & \sum_{r=1}^{s} m(s_r^+) \\
\text{s.t.}\quad & \sum_{j=1}^{n} m(x_{ij}) \lambda_j \leq m(x_{io}), \qquad i = 1, \dots, m, \\
& \sum_{j=1}^{n} m(y_{rj}) \lambda_j - m(s_r^+) = m(y_{ro}), \quad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \qquad j = 1, \dots, n.
\end{aligned} \tag{5}
$$

(Hint: in the above model only inputs and outputs are considered in the form of fuzzy and coefficients are not fuzzy)

$$
\begin{aligned}
\text{Max}\quad & \sum_{r=1}^{s} \int_0^1 (\underline{s}_r^{+}(\alpha)+\overline{s}_r^{+}(\alpha))d\alpha \\
\text{s.t.}\quad & \sum_{j=1}^{n}\left(\int_0^1 (\underline{x}_{ij}(\alpha)+\overline{x}_{ij}(\alpha))\,d\alpha\right)\lambda_j \le m\int_0^1 (\underline{x}_{io}(\alpha)+\overline{x}_{io}(\alpha))\,d\alpha\ , \qquad i=1,\ldots,m, \\
& \sum_{j=1}^{n}\left(\int_0^1 (\underline{y}_{rj}(\alpha)+\overline{y}_{rj}(\alpha))\,d\alpha\right)\lambda_j + \int_0^1 (\underline{s}_r^{+}(\alpha)+\overline{s}_r^{+}(\alpha))d\alpha = \int_0^1 (\underline{y}_{ro}(\alpha)+\overline{y}_{ro}(\alpha))\,d\alpha,\ r=1,\ldots,s, \\
& \sum_{j=1}^{n}\lambda_j = 1, \quad \lambda_j \ge 0, \qquad j=1,\ldots,n.
\end{aligned}
\tag{6}
$$

This model can be transformed by the following substitutions:

$$\int_0^1 (\underline{x}_{ij}(\alpha)+\overline{x}_{ij}(\alpha))\,d\alpha = \hat{x}_{ij}$$

$$\int_0^1 (\underline{s}_r^{+}(\alpha)+\overline{s}_r^{+}(\alpha))d\alpha = \hat{s}_r^{+}$$

$$\int_0^1 (\underline{y}_{rj}(\alpha)+\overline{y}_{rj}(\alpha))\,d\alpha = \hat{y}_{ro}$$

After this changes the model is converted to model (4).

This process is the same for NEW model (3), the objective function of the NEW model with fuzzy data is called h'_{NEW}.

We reached BCC and NEW models with fuzzy data. Based on section three above, we are able to use definitions for evaluating congestion with surrogate fuzzy models.

5 Numerical example

To show this process precisely an example is applied. In the following part, we use fuzzy data to evaluate congestion by output additive models BCC and NEW which is altered by non crisp data.

Consider eight DMUs, A, B, C, D, E, F , G and H, each with two inputs and one output which are
triangle fuzzy numbers and $L(x)=R(x)=1-x$, given as follows:

Table 1 The set data

DMU	A	B	C	D	E	F	G	H
Input 1	(1/2, 1/2, 1/2)	(2,4,2)	(2,4,2)	(3,6,1)	(4,8,4)	(3,6,1)	(4,7,2)	(1,1,1)
Input 2	(2,4,2)	(1/2, 1/2, 1/2)	(2,4,2)	(3,6,1)	(3,6,1)	(4,8,4)	(4,7,2)	(1,1,1)
Output1	(1/2, 1/2, 1/2)	(1/2, 1/2, 1/2)	(3,5,5)	(4,7,2)	(1/2, 1/2, 1/2)	(1/2, 1/2, 1/2)	(1/2, 1/2, 1/2)	(1/4, 1/4, 1/4)

It is easy to check that for all DMUs, we have $h'_{NEW} = 1$. It means that The efficient frontier of NEW model is included in all DMUs. The results of evaluating congestion by GAMS software are shown in the following table.

Table 2 Results

DMU	A	B	C	D	E	F	G	H
h'_{BCC}	1	1	1	1	10	10	10	1
Evaluating Congestion	No	No	No	No	Congestion	Congestion	Congestion	No

6 Conclusion

In this paper we used the process of evaluating congestion based on earlier methods and definitions by Wei, Yan [17], with models that their inputs and outputs are fuzzy. As it is achieved from the numerical example DMUs E, F, G are specified as DMUs that show congestion. The reason why we insist on using fuzzy data is due to more accuracy in real cases and its application on describing the interpretive data.

Although, this method and its application has been proved in grounds such as economy, industry, organization policy, etc. to be phenomenal, the usage of this method shows that in real case studies unknown data is seen which is not an ordinary number. Thereby, we need to interpret our aim of using fuzzy data in considering congestion to qualify it to be fitted for real cases.

References

1. Charnes, A., Cooper, W. W., Rhodes, E., (1978). Measuring the efficiency of decision- making. European Journal of Operation Research, 2, 429-444.
2. Banker, R. D., Charnes, A., Cooper, W. W., (1984). Some models for estimating analysis. Management Science, 30, 1078-1092.
3. Tone, K., (2001). A slack-based measure of efficiency in data envelopment analysis. European Journal of Operation Research, 130, 498-509.
4. Sanei, M., Noori, N., Saleh, H., (2009). Sensitivity Analysis with Fuzzy Data in DEA. Applied Mathematical Science, 3(25), 1235–1241.
5. Sanei, M., Rostamy-Malkhalifeh, M., Saleh, H., (2009). A new method for solving fuzzy DEA models. Int. J. Industrial Mathematics, 1(4), 307-313.
6. Guo, P., Tanaka, H., (1998). Extended fuzzy DEA. Proceedings of the 3rd Asian Fuzzy system symposium, 517-521.
7. Guo, P., Tanaka, H., (2001). Fuzzy DEA: a perceptual, evaluation method. Fuzzy Sets and systems, 119, 149-160.
8. Saati Mohtadi, S., Memariani, A., Jahanshahloo, G. R., (2002). Efficiency analysis and ranking of DMUs with fuzzy data. Fuzzy Optimization Decision making, 3, 255-267.
9. Soleimani-damaneh, M., (2008). Fuzzy upper bounds and their applications. Chaos, Solittons and Fractal, 786-800.

10. Brockett, P. L., Cooper, W. W., Shin, H. C., Wang, Y., (1998). Inefficiency and congestion in Chinese production before and after the 1978 economic reform. Socio-Economic Planning Sciences, 32: 1–20.
11. Cooper, W. W., Deng, H., Huang, Z. M., Li, S. L., (2002). A one-model approach to congestion in data envelopment analysis. European Journal of Operational Research, 36(4), 231–8.
12. Cooper, W. W., Gu, B. S., Li, S. L., (2001). Comparisons and evaluation of alternative approaches to the treatment of congestion in DEA. European Journal of Operational Research, 132, 62–74.
13. Fare, R., Grosskopf, S., (1983). Measuring congestion in production. Zeitschrilft fur National okonomie journal of economics, 43, 257–71.
14. Wei, Q., Yan, H., (2009). Weak congestion in output additive data envelopment analysis. Socio- Economic Planning Sciences, 43, 40–54.
15. Wei, Q. L., Yan, H., (2004). Congestion and returns to scale in data envelopment analysis. European Journal of Operational Research; 153, 641–60.
16. Diamond, P., Kloeden. P., (1992). Metric spaces of fuzzy sets. Fuzzy Sets and Systems, 47, 81- 86.
17. Grzegorewski, P., Mrwka, E., (2005). Trapezoidal approximation of fuzzy number. Fuzzy Sets and Systems, 153, 115- 135.
18. Grzegorewski, P., (2002). Nearest interval approximation of fuzzy. Fuzzy Sets and Systems, 130, 321-330.
19. Zendenam, A., Asadi, B., (2007). Ranking fuzzy number by distance minimization. Applied Mathematics and Modeling, 31, 2589-259.