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# STRONGLY S-DENSE MONOMORPHISMS

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Abstract. Let $\mathcal{M}$ be a class of (mono)morphisms in a category $\mathcal{A}$. To study mathematical notions, such as injectivity, tensor products and flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$). In this paper we take $\mathcal{A}$ to be the category **Act-S** of $S$-acts, for a semigroup $S$, and $\mathcal{M}_{sd}$ to be the class of strongly $s$-dense monomorphisms and study the categorical properties, such as limits and colimits, of this class.

**Key Words:** Strongly s-dense, Semigroup, Limit, Colimit.

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## 1. Introduction and Preliminaries

To study mathematical notions in a category $\mathcal{A}$ with respect to a class $\mathcal{M}$ of its morphisms, one should know some of the categorical properties of the pair ($\mathcal{A}$, $\mathcal{M}$). In this paper we take $\mathcal{A}$ to be the category **Act-S** and $\mathcal{M}_{sd}$ to be a particular interesting class of monomorphisms, to be called *strongly-s-dense(st-s-dense)* monomorphisms, and investigate its categorical properties.

Let us first recall the definition and some ingredients of the category **Act-S** needed in the sequel. For more information and the notions not mentioned here see, for example, [8].

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Recall that, for a semigroup $S$, a set $A$ is a right $S$-*act* (or an $S$-*set*) if there is, a so called, an *action* $\mu : A \times S \to A$ such that, denoting $\mu(a,s) := as$, $a(st) = (as)t$ and, if $S$ is a monoid with 1, $a1 = a$.

Each semigroup $S$ can be considered as an $S$-act with the action given by its multiplication. Notice that, adjoining an external left identity 1 to a semigroup $S$, an $S$-act $S^1 := S \cup \{1\}$ is obtained.

The definitions of a *subact* $B$ of $A$, written as $B \subseteq A$, an *extension* of $A$, a *congruence* $\rho$ on $A$, a *quotient* $A/\rho$ of $A$, and a *homomorphism* between $S$-acts are clear. The category of all (right) $S$-acts and homomorphisms between them is denoted by **Act-S**.

The class of all $S$-acts is an equational class, and so the category **Act-S** is complete (has all products and equalizers). In fact, limits in this category are computed as in the category **Set** of sets and equipped with a natural action. In particular, the terminal object of **Act-S** is the singleton $\{0\}$, with the obvious $S$-action. Also, for $S$-acts $A, B$, their cartesian product $A \times B$ with the $S$-action defined by $(a,b)s = (as, bs)$ is the *product* of $A$ and $B$ in **Act-S**.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

in **Act-S** is the subact $P = \{(c,a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps. Notice that for the case where $g$ is an inclusion, $P$ can be taken as $f^{-1}(C)$.

All colimits in **Act-S** exist and are calculated as in **Set** with the natural action of $S$ on them. In particular, $\emptyset$ with the empty action of $S$ on it, is the initial object of **Act-S**. Also, the *coproduct* of $S$-acts $A, B$ is their disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ with the obvious action, and coproduct injections are defined naturally.

The pushout of a given diagram

$$\begin{array}{ccc} & A & \stackrel{g}{\to} & C \\ f & \downarrow & & \\ & B & & \end{array}$$

in **Act-S** is the factor act $Q = (B \sqcup C)/\theta$ where $\theta$ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $q_1 = \pi u_C : C \to (B \sqcup C)/\theta,\ q_2 = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Act-S** are constructed analogously.

Recall that for a family $\{A_i : i \in I\}$ of $S$-acts, each with a unique fixed element 0, the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number of indices.

Let **I** be a small category and $\mathcal{A} : \mathbf{I} \to$ **Act-S** be a diagram in **Act-S** determining the acts $A_\alpha$, for $\alpha \in I = Obj\mathbf{I}$, and $S$-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor\mathbf{I}$. Recall that the limit of this diagram is $\varprojlim_\alpha A_\alpha := \bigcap_{\alpha \in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha \in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta} p_\alpha(a) = p_\beta(a)\}$ and $p_\alpha, p_\beta$ are the $\alpha, \beta$th projection maps of the product. The limit $S$-maps are $q_\alpha : \varprojlim_\alpha A_\alpha \to A_\alpha$. Also the limit has the universal property which is, if $\{f_\alpha : A \to A_\alpha\}$ is a family of morphisms such that $g_{\alpha\beta} f_\alpha(a) = f_\beta(a)$, then there is a morphism $f : A \to \varprojlim_\alpha A_\alpha$ such that $q_\alpha f = f_\alpha$.

Remind that a directed system of $S$-acts and $S$-maps is a family $(B_\alpha)_{\alpha \in I}$ of $S$-acts indexed by an updirected set $I$ endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha \leq \beta \in I}$ of $S$-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, also $g_{\alpha\alpha} = id$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha \in I}, (g_{\alpha\beta})_{\alpha \leq \beta \in I})$ in **Act-S** is given as $\varinjlim_\alpha B_\alpha = \coprod_\alpha B_\alpha / \rho$ where the congruence $\rho$ is given by $b_\alpha \rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$, in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha$ of $S$-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for

$\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \varinjlim_\alpha B_\alpha$ is the natural $S$-map. Also directed colimit has a dual universal property of limit.

## 2. $C^{sd}$-Closure operator

In this section, we introduce and briefly study a closure operator, so called $C^{sd}$-*Closure operator*. Let us denote the finite subset $T$ of $S$ by $T \subset\!\!\cdot\, S$. First recall the following definition of $C^{sd}$-closure operator.

**Definition 2.1.** A family $C^{sd} = (C^{sd}_B)_{B\in \mathbf{Act-S}}$, with $C^{sd}_B : sub(B) \to Sub(B)$, is defined as

$C^{sd}_B(A) = \{b \in B : bS \subseteq A \text{ and } \forall T \subset\!\!\cdot\, S,\ \exists a_T \in A,\ a_T t = bt (t \in T)\}$.

It is easy to show that $C^{sd}$ is a closure operator on **Act-S** in the sense of [5]. This means that $C^{sd}_B(A)$ is a subact of $B$ and,

(i) $A \subseteq C^{sd}_B(A)$,

(ii) $A_1 \subseteq A_2 \subseteq B$ implies $C^{sd}_B(A_1) \subseteq C^{sd}_B(A_2)$,

(iii) for every homomorphism $f : B \to D$ and each subact $A$ of $B$, $f(C^{sd}_B(A)) \subseteq C^{sd}_D(f(A))$.

We just prove (iii). Let $f : B \to C$ be a homomorphism and $b \in C^{sd}_B(A)$. For every $s \in S, f(b)s = f(bs) \in f(A)$, then $f(b)S \subseteq f(A)$. If $T$ is a finite subset of $S$, there is an element $a_T \in A$ such that $a_T t = bt(t \in T)$. Hence $f(a_T)t = f(b)t$, which means that $f(b) \in C^{sd}_B(f(A))$.

Notice that in the case where $S$ is a monoid, $C^{sd}_B(A) = A$ for every $A \subseteq B$. So, it is more interesting to consider the closure operator $C^{sd}$ only for semigroups, or for semigroup part $S$ of monoids of the form $T = S^1$.

Dikranjan and Tholen in [5] state some properties of a closure operator in general. Here we are going to investigate those for the closure operator $C^{sd}$ satisfy or not.

**Definition 2.2.** The closure operator $C^{sd}$ is said to be:

(1) idempotent, if for $A \subseteq B$, $C^{sd}_B(A) = C^{sd}_B(C^{sd}_B(A))$.

(2) hereditary, if for $A_1 \subseteq A_2 \subseteq B$, $C^{sd}_{A_2}(A_1) = C^{sd}_B(A_1) \cap A_2$.

(3) weakly hereditary, if for every $A \subseteq B$, $C^{sd}_{C^{sd}_B(A)}(A) = C^{sd}_B(A)$.

(4) grounded, if $C^{sd}_B(\emptyset) = \emptyset$.

(5) additive, if for subacts $A, C$ of $B$, $C^{sd}_B(A \bigcup C) = C^{sd}_B(A) \bigcup C^{sd}_B(C)$.

(6) productive, if for every family of subacts $A_i$ of $B_i$, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C^{sd}_B(A) = \prod_i C^{sd}_{B_i}(A_i)$.

(7) fully additive, if for $A_i \subseteq B$, $C^{sd}_B(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} C^{sd}_B(A_i)$.

(8) discrete, if $C^{sd}_B(A) = A$ for every $S$-act $B$ and $A \subseteq B$.

(9) trivial, if $C^{sd}_B(A) = B$ for every $B$ and $A \subseteq B$.

(10) minimal, if for $C \subseteq A \subseteq B$ one has $C^{sd}_B(A) = A \cup C^{sd}_B(C)$.

**Theorem 2.3.** *The closure operator $C^{sd}$ is hereditary, weakly hereditary, grounded and productive.*

*Proof.* It is easy to check that the closure operator $C^{sd}$ is hereditary, weakly hereditary and grounded. We just prove productivity. Let $b \in C^{sd}_B(A)$, $b = \{b_i\}$. For every $s \in S, bs \in A$, then for each $i \in I$, $b_i s \in A_i$. Let $T$ be a finite subset of $S$. So there exists $\{a_i\} \in \prod A_i$ such that $\{a_i\}t = \{b_i\}t (t \in T)$. Thus for every $i \in I, a_i t = b_i t$ and hence for each $i \in I, b_i \in C^{sd}_{B_i}(A_i)$. It deduces that $b \in \prod C^{sd}_{B_i}(A_i)$. The converse is done in a similar way. □

**Theorem 2.4.** *The closure operator $C^{sd}$ is idempotent.*

*Proof.* By definition of the closure operator for each $A \subseteq B$ we see that $C^{sd}_B(A) \subseteq C^{sd}_B(C^{sd}_B(A))$. *Conversly*, let $b \in C^{sd}_B(C^{sd}_B(A))$. For each $t \in S$, there exists $a_1 \in C^{sd}_B(A)$ such that $bt = a_1 t$, and since $a_1 \in C^{sd}_B(A)$, there exists $a \in A$ such that $a_1 t = at$, which implies $bt \in A$. Thus $bS \subseteq A$. Now let $T$ be a finite subset of $S$. There is an element $a_T \in C^{sd}_B(A)$ such that $a_T t = bt (t \in T)$. Since $a_T \in C^{sd}_B(A)$, there exists an element $a'_T \in A$ such that $a'_T t = a_T t$ and hence $a'_T t = bt$. Therefore $b \in C^{sd}_B(A)$. □

**Theorem 2.5.** *Let $A$ and $C$ be two subacts of $B$ such that $A \bigcap C = \emptyset$. Then $C^{sd}_B(A \bigcup C) = C^{sd}_B(A) \bigcup C^{sd}_B(C)$.*

*Proof.* By definition of the closure operator, $C_B^{sd}(A)\bigcup C_B^{sd}(C) \subseteq C_B^{sd}(A\bigcup C)$. Consider $x \in C_B^{sd}(A\bigcup C)$. So $xS \subseteq A\bigcup C$. Let $xS\bigcap A \neq \emptyset$ and $xS\bigcap C \neq \emptyset$. Then there exist elements $t_1, t_2 \in S$ such that $xt_1 \in A \setminus C$ and $xt_2 \in C \setminus A$. Set $T_1 = \{t_1, t_2\}$. There is an element $y \in A \cup C$ such that $xt_1 = yt_1$ and $xt_2 = yt_2$. If $y \in A$, then $xt_2 \in A$ and if $y \in C$, then $xt_1 \in C$ which are contradictions. Suppose that $xS \cap C = \emptyset$, so $xS \subseteq A$. For every finite subset $T \subseteq S$, there exists $x_T \in A \cup C$ such that $xt = x_T t$. It is clear that $x_T \notin C$, thus $x_T \in A$ and hence $x \in C_B^{sd}(A)$. □

Now we show that some of the properties of closure operator do not satisfy in general. But first recall another closure operator $C^d$ defined by

$$C_B^d(A) = \{b \in B \mid bS \subseteq A\}.$$

A subact $A$ is, by definition, $s$-dense in $B$ if $C_B^d(A) = B$.

**Lemma 2.6.** *The closure operator $C^{sd}$ is not necessary fully additive.*

*Proof.* Let $S = (\mathbb{N}, min)$ be a semigroup, $B = \mathbb{N}^\infty$ and $A = \mathbb{N}$. Set $A_n = \{m \in \mathbb{N} \mid m \leq n\}$ for each $n \in \mathbb{N}$. It is easy to check that $C_{\mathbb{N}^\infty}^{sd}(A_n) = A_n$ and hence $\bigcup C_{\mathbb{N}^\infty}^{sd}(A_n) = \bigcup(A_n) = \mathbb{N}$, but $C_{\mathbb{N}^\infty}^{sd}(\cup A_n) = C_{\mathbb{N}^\infty}^{sd}(\mathbb{N}) = \mathbb{N}^\infty$. □

**Lemma 2.7.** (i) *Let $\{A_i \mid i \in I\}$ be a family of subacts of $A$. If $C_A^{sd}(\bigcap A_i) = C_A^d(\bigcap A_i)$, then $C_A^{sd}(\bigcap A_i) = \bigcap C_A^{sd}(A_i)$.*

(ii) *If for every $i \in I$, $C_A^{sd}(A_i) = C_A^d(A_i)$ and $C_A^{sd}(\bigcap A_i) = \bigcap C_A^{sd}(A_i)$, then $C_A^{sd}(\bigcap A_i) = C_A^d(\bigcap A_i)$.*

*Proof.* (i) By the hypothesis we see that $\bigcap C_A^{sd}(A_i) \subseteq \bigcap C_A^d(A_i) = C_A^d(\bigcap A_i) = C_A^{sd}(\bigcap A_i)$. Thus $C_A^{sd}(\bigcap A_i) = \bigcap C_A^{sd}(A_i)$.

(ii) $C_A^d(\bigcap A_i) = \bigcap C_A^d(A_i) = \bigcap C_A^{sd}(A_i) = C_A^{sd}(\bigcap A_i)$. □

**Lemma 2.8.** *For every semigroup $S$, the closure operator $C^{sd}$ is not discrete nor trivial and minimal.*

*Proof.* Let $0 \in A$ be a fixed element of a nonempty $S$-act $A$. Adjoin two elements $\theta, \omega$ to A with actions $\omega s = \omega$ and $\theta s = 0$. Consider

$B = A \cup \{\theta, \omega\}$. It is clear that $C_B^{sd}(A) = A \cup \{\theta\}$. This shows that $C^{sd}$ is neither discrete nor trivial. Also, it is not minimal. Because, adjoining two elements $\theta, \omega$ to a nonempty $S$-act $C$ with actions $\omega s = \theta$ and $\theta s = \theta$, and taking $A = C \cup \{\theta\}$, $B = C \cup \{\theta, \omega\}$, we get $C \subset A \subset B$, and $C_B^{sd}(A) = B$ while $C_B^{sd}(C) = C$. □

**Theorem 2.9.** (i) *The closure $C^{sd}$ is discrete if and only if $S$ has a left identity element.*

(ii) *The closure $C^{sd}$ is trivial if and only if $S$ is the empty set.*

*Proof.* (i) Let $C^{sd}$ be a discrete closure operator and $S$ do not have a left identity. Consider $t_0 \in S$ and adjoin an element $x$ to $S$ defined by $xs = t_0 s$ for each $s \in S$. It is clear that $C_{S^x}^{sd}(S) = S^x$ and by the hypothesis we have $C_{S^x}^{sd}(S) = S$. So $S^x = S$ which is a contradiction.

(ii) Let the closure $C^{sd}$ be trivial and $S \neq \emptyset$. If $B$ is an $S$-act such that all of its elements are fixed and $A$ is a subact of $B$, then $C_B^{sd}(A) = A \neq B$ which is a contradiction. Thus $S$ is the empty set.

*Conversly*, let $S = \emptyset$. Then it is clear that $C_B^{sd}(A) = B$. □

## 3. Categorical properties of st-s-dense monomorphisms

In this section we investigate the categorical and algebraic properties of the class $\mathcal{M}_{sd}$ of st-$s$-dense monomorphisms in the following three subsections.

### 3.1. Composition Property.

In this subsection we investigate some properties of the class $\mathcal{M}_{sd}$ of strongly-$s$-dense monomorphisms which are mostly related to the composition of st-$s$-dense monomorphisms. These properties and the ones given in the next two subsections are what normally used to study injectivity with respect to a class of monomorphisms (see [1]). The class $\mathcal{M}_{sd}$ is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.

**Definition 3.1.** An $S$-act $A$ is *strongly-s-dense (or simply st-s-dense)* subact of $B$, if for every $b \in B, bS \subseteq A$ and for every finite subset $T$ of $S$ there is an element $a_T \in A$ such that $a_T t = bt (t \in T)$. In other word $C_B^{sd}(A) = B$. A monomorphism $A \xrightarrow{f} B$ is an st-$s$-dense monomorphism, if $f(A)$ is an st-$s$-dense subact of $B$.

**Lemma 3.2.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two monomorphisms. The monomorphism $gf$ is an st-s-dense monomorphism if and only if $f$ and $g$ are st-s-dense monomorphisms too.*

*Proof.* Suppose that $gf$ is an st-$s$-dense monomorphism. It is clear that $f$ and $g$ are $s$-dense monomorphisms. Now let $b \in B$, $c \in C$ and $T$ be a finite subset of $S$. Since $gf$ is st-$s$-dense, then there exist $a_{T_1}$ and $a_{T_2}$ in $A$ such that $a_{T_1} t = bt$ and $a_{T_2} t = ct$ for each $t \in T$. Thus $f$ and $g$ are st-$s$-dense monomorphisms.

Conversely, assume each $f$ and $g$ is st-$s$-dense monomorphism. Let $T$ be a finite subset of $S$ and $c \in C$. By the hypothesis there exist $b \in B$ and $a \in A$ such that for every $t \in T$, $ct = bt$ and $bt = at$. If $T$ is a one element subset of $S$, the above equations show that $A$ is $s$-dense in $C$. Therefore $A$ is st-$s$-dense in $C$. □

**Definition 3.3.** The semigroup $S$ locally has left identity element if every finitely generated right ideal of $S$ has a left identity element in $S$.

In the following lemma we have the characterization of a semigroup $S$ over which all $s$-dense extensions are st-$s$-dense. First recall that every st-$s$-dense monomorphism is an $s$-dense monomorphism.

**Lemma 3.4.** *For a semigroup $S$, the following are equivalent:*

(i) *Every s-dense extension is st-s-dense extension.*

(ii) *The semigroup $S$ is st-s-dense in $S^1$.*

(iii) *The semigroup $S$ locally has left identity element.*

*Proof.* (i)⇒(ii) is obvious.

(ii)⇒(iii) Let $I = \bigcup_{i=1}^{n} t_i S^1$ be a finitely generated right ideal of $S$. Since $S$ is st-$s$-dense in $S^1$, then there exists $s_0 \in S$ such that $s_0 t_i = 1t_i = t_i (1 \leq i \leq n)$. So $S$ locally has left identity.

(iii)⇒(i) Let $B$ be an $s$-dense extension of $A$, $T$ be a finite subset of $S$ and $b \in B$. Then $A \cup bS^1 = A \cup \{b\}$ and since $S$ locally has left identity element, there exists $t_0 \in S$ such that $t_0 t = t (t \in T)$. Thus $(bt_0)t = bt$ and so $A$ is st-$s$-dense in $B$. □

3.2. **Limits of st-$s$-dense monomorphisms.**

In this subsection we will investigate the behaviour of st-$s$-dense monomorphisms with respect to limits. First recall that, we say the class $\mathcal{M}_{sd}$ is closed under products (coproduct, direct sum), if for every family of st-$s$-dense monomorphisms $\{f_i : A_i \to B_i\}$, $\prod f_i : \prod A_i \to \prod B_i (\coprod f_i, \oplus f_i)$ is st-$s$-dense monomorphism. The proof of the following is straightforward.

**Proposition 3.5.** (i) *The class $\mathcal{M}_{sd}$ is closed under products.*

(ii) *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a family of st-s-dense monomorphisms. Then their product homomorphism $h : A \to \prod_{\alpha \in I} B_\alpha$ is also an st-s-dense monomorphism.*

**Proposition 3.6.** *The class $\mathcal{M}_{sd}$ is closed under direct sums.*

**Theorem 3.7.** *In the category* **Act-S**, *the following are equivalent:*

(i) *pullbacks transfer st-s-dense monomorphisms.*

(ii) *The semigroup S locally has left identity element.*

*Proof.* (i)⇒(ii) By Lemma 3.4, it is enough to show that $S$ is st-$s$-dense in $S^1$. Let $E$ be an injective $S$-act and 0 be a fixed element of $E$ which exists by [4]. Adjoin an element $\theta$ to $E$ by action $\theta s = 0$ for all $s \in S$. Then, $E$ is clearly st-$s$-dense in $E^\theta = E \cup \{\theta\}$. Taking a homomorphism $f : S^1 \longrightarrow E^\theta$ given by $f(s) = \theta s$ $(s \in S^1)$, one gets the pullback diagram:

$$\begin{array}{ccc} S & \stackrel{\tau}{\longrightarrow} & S^1 \\ f\downarrow & & \downarrow f \\ E & \stackrel{\tau'}{\longrightarrow} & E^\theta \end{array}$$

where $\tau$, $\tau'$ are inclusion maps. By the hypothesis, Since $\tau'$ is st-$s$-dense; so is $\tau$.

(ii)$\Rightarrow$(i) Consider the pullback diagram: $\begin{array}{ccc} P & \stackrel{q}{\longrightarrow} & B \\ p\downarrow & & \downarrow g \\ A & \stackrel{f}{\longrightarrow} & C \end{array}$ where $P = \{(a,b)|f(a) = g(b)\}$ and $f$ is an st-$s$-dense monomorphism. We show that $q$ is a monomorphism. Let $q(a_1, b_1) = q(a_2, b_2)$. So $b_1 = b_2$ and $fp(a_1, b_1) = fp(a_2, b_2)$ which implies $a_1 = a_2$ by using that $f$ is a monomorphism. Thus $q$ is a monomorphism. Now one should show that $q$ is an $s$-dense monomorphism. Let $b \in B$ and $s \in S$. Since $f$ is $s$-dense, $g(bs) = g(b)s \in f(A)$. So there exist elements $a_s \in A(s \in S)$, such that $g(bs) = f(a_s)$. Hence $(a_s, bs) \in P$ and $bs = q(a_s, bs)$ which implies that $q$ is $s$-dense. Now by using Lemma 3.4 the proof is complete. □

### 3.3. Colimits of st-$s$-dense monomorphisms.

This subsection is devoted to the study of the behaviour of st-$s$-dense monomorphisms with respect to colimits.

**Proposition 3.8.** *$\mathcal{M}_{sd}$ is closed under coproducts.*

*Proof.* Consider the diagram

$$\begin{array}{ccccc} & A_i & \stackrel{f_i}{\rightarrow} & B_i & \\ u_i & \downarrow & & \downarrow & u'_i \\ & \coprod_{i\in I} A_i & \stackrel{f}{\rightarrow} & \coprod_{i\in I} B_i & \end{array}$$

in which $\{f_i : A_i \to B_i : i \in I\}$ is a family of st-$s$-dense monomorphisms. We want to show that $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ is an st-$s$-dense monomorphism. It is not difficult to show that $f$ is an $s$-dense monomorphism. Now let $b \in \coprod_{i\in I} B_i$, and $T$ be a finite subset of $S$. Then there exists

$i \in I$ and $b_i \in B_i$ such that $b = u'_i(b_i)$. Since $f_i$ is st-$s$-dense, there exists $a_i \in A_i$ such that $f_i(a_i)t = b_i t$ for $t \in T$. Thus $fu_i(a_i)t = u'_i f_i(a_i)t = bt$ which means that $f$ is an st-$s$-dense monomorphism. □

**Theorem 3.9.** *Pushouts transfer st-s-dense monomorphisms. That is, for the following pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

*in* **Act-S**, *If $f$ is st-s-dense, then $h$ is st-s-dense too.*

*Proof.* Recall that $Q = (B \sqcup C)/\theta$ where $\theta = \rho(H)$ and $H$ consists of all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C$, $u_C : C \to B \sqcup C$ are coproduct injections. And $h = \pi u_C : C \to (B \sqcup C)/\theta$, $h' = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. By [6], pushout transfers $s$-dense monomorphism. So $h$ is an $s$-dense monomorphism.

Let $T$ be a finite subset of $S$ and $q \in Q$ be a solution for $\Sigma = \{xt_1 = h(c_1), xt_2 = h(c_2), ..., xt_m = h(c_m)\}$. Two cases may be occur: (i) there exists $c \in C$ such that $q = [u_C(c)]_{\rho(H)}$.Then for all $1 \leq i \leq m$, $h(c)t_i = h(c_i)$. (ii) There exists $b \in B$ such that $q = [u_B(b)]_{\rho(H)}$. For every $1 \leq i \leq m$, $[u_B(bt_i)]_{\rho(H)} = h(c_i) = [u_C(c_i)]_{\rho(H)}$, and so there exist $a_{i1}, a_{i2}, ...a_{in} \in A$, $s_{i1}, s_{i2}, ..., s_{in} \in S^1$ such that $u_B(b)t_1 = u_B f(a_{i1})s_{i1}$, $u_C g(a_{i1})s_{i1} = u_C g(a_{i2})s_{i2}, u_B f(a_{i2})s_{i2} = u_B f(a_{i3})s_{i3}$, ..., $u_B f(a_{i(n-1)})s_{i(n-1)} = u_B f(a_{in})s_{in}$, $u_C g(a_{in})s_{in} = u_C(c_i)$. Since $f$ is a monomorphism, $a_{i2}s_{i2} = a_{i3}s_{i3}, a_{i4}s_{i4} = a_{i5}s_{i5}, \cdots, a_{i(n-1)}s_{i(n-1)} = a_{in}s_{in}$, and hence $u_C g(a_{i1}s_{i1}) = u_C g(a_{i2}s_{i2}) = u_C g(a_{i3}s_{i3}) = \cdots = u_C g(a_{in}s_{in}) = u_C(c_i)$. Thus $g(a_{i1}s_{i1}) = c_i$. Now, for all $1 \leq i \leq m$, $bt_i = f(a_{i1}s_{i1}) \in f(A)$, and hence $\Sigma_1 = \{xt_1 = f(a_{11}s_{11}), \cdots xt_m = f(a_{m1}s_{m1})\}$ has a solution $b \in B$. So $\Sigma_1$ has a solution $f(a)$ for some $a \in A$. Then for every $1 \leq i \leq m$, $hg(a)t_i = h'f(a)t_i = h'f(a_{i1}s_{i1}) = hg(a_{i1}s_{i1}) = h(c_i)$ which yields $hg(a)$ is a solution for $\Sigma$. □

**Theorem 3.10.** *The category* **Act-S** *has $\mathcal{M}_{sd}$-directed colimits.*

*Proof.* Let $g_{\alpha\beta} : B_\alpha \to B_\beta$ $(\alpha \leq \beta)$ be a directed system of homomorphisms and $h : A \to \underrightarrow{lim}_\alpha B_\alpha$ be a directed colimit in **Act-S** of st-$s$-dense monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$, with the colimit maps $g_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$. Since $h = \underrightarrow{lim}_\alpha h_\alpha = g_\alpha h_\alpha$ for each $\alpha \in I$, then $h$ is an $s$-dense monomorphism because of each $h_\alpha$. Now we show that $h$ is st-$s$-dense. Let $b \in \underrightarrow{lim}_\alpha B_\alpha$ and $T$ be a finite subset of $S$. Since $b \in \underrightarrow{lim}_\alpha B_\alpha$, there exists $x_\alpha \in B_\alpha$ such that $b = [x_\alpha]_\rho$ and since $h_\alpha$ is st-$s$-dense, there exists an element $a_T \in A$ with $h_\alpha(a_T)t = x_\alpha t$ for all $t \in T$. Then $bt = [x_\alpha]_\rho t = g_\alpha(x_\alpha)t = g_\alpha(x_\alpha t) = g_\alpha h_\alpha(a_T)t = h(a_T)t$. □

We say that *multiple pushouts* transfer st-$s$-dense monomorphisms if in multiple pushout $(P, A_\alpha \overset{h_\alpha}{\to} P)$ of a family of st-$s$-dense monomorphisms $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$, every $h_\alpha$, $\alpha \in I$, is an st-$s$-dense monomorphism. In multiple pushout diagram for every $\alpha, \beta \in I$, $h_\alpha f_\alpha = h_\beta f_\beta$, which calls diagonal map.

**Theorem 3.11.** *Multiple pushouts transfer st-s-dense monomorphisms.*

*Proof.* Let $(P, A_\alpha \overset{h_\alpha}{\to} P)$ be a multiple pushout of the family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of st-$s$-dense monomorphisms. We know that $P = \coprod A_\alpha/\rho(H)$ where $H = \{(f_\alpha(a), f_\beta(a)) \mid a \in A, \alpha, \beta \in I\}$ (we have taken the image of each element of $A_\alpha$ under coproduct morphisms equal to itself). Let $h_\alpha(a) = h_\alpha(a')$, $a, a' \in A_\alpha$. So there exist $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A, s_1, s_2, ..., s_n \in S^1$ where for $i = 1, ..., n, (p_i, q_i) \in H \cup H^{-1}$ and such that $a = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, ..., q_n s_n = a'$. Then, $a = f_\alpha(a_1)s_1$ and there exists $\beta \in I$ such that $f_\beta(a_1)s_1 = f_\beta(a_2)s_2$. Since $f_\beta$ is a monomorphism, $a_1 s_1 = a_2 s_2$. Continuing this process, we get that $a_1 s_1 = a_2 s_2 = ... = a_n s_n$, and therefore $a = a'$. Now let $q \in P$ and $s \in S$. There exist $\beta \in I$ and $p \in A_\beta$ such that $q = h_\beta(p)$. Since $f_\beta$ is $s$-dense then $ps = f_\beta(a_s)$ and so $qs = h_\beta(ps) = h_\beta(f_\beta(a_s)) = h_\alpha(f_\alpha(a_s))$. Thus $h_\alpha$ is $s$-dense. Let $x \in P$, $x = [p]_{\rho(H)}$ and $T$ be a finite subset of $S$. If $p \in A_\alpha$, the result is true. If $p \in A_\beta$, $\beta \neq \alpha$, then for every

$t \in T$, $[pt]_{\rho(H)} = [f_\beta(a_{_T})t]_{\rho(H)} = h_\beta f_\beta(a_{_T}t) = h_\alpha f_\alpha(a_{_T}t)$ and thus $h_\alpha$ is an st-$s$-dense monomorphism. □

**Corollary 3.12.** *In every multiple pushout diagram of st-s-dense monomorphisms the diagonal map is an st-s-dense monomorphism.*

*Proof.* Apply Lemma 3.2 and Theorem 3.11. □

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