Iranian Journal of Mathematical Sciences and Informatics
Vol. 7, No. 1 (2012), pp 1-6

On $C3$-Like Finsler Metrics

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Abstract. In this paper, we study the class of of $C3$-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on $C3$-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

Keywords: Finsler metric, $C3$-like metric, semi-C-reducible metric.

2000 Mathematics subject classification: 53C60, 53C25.

1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians [3][5][14][16]. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg, and etc [6][7]. In [5], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hōjō proves that the converse is true too [2]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric α perturbated by a one form β, which has important applications both in mathematics and physics [15].

Let us remark some important curvatures in Finsler geometry. Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental

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Received 22 October 2010; Accepted 17 March 2011

form and the Cartan torsion, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$.

In [11], Prasad-Singh introduced a new class of Finsler spaces named by $C3$-like spaces which contains the class of semi-C-reducible spaces, as special case (see [8], [9], [10]). A Finsler metric F is called $C3$-like if its Cartan tensor is given by

$$(1) \qquad C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\},$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on TM of degree -1 and 1, respectively. We have some special cases as follows: (i) if $a_i = 0$, then we have $C_{ijk} = \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}$, contracting it with g^{ij} implies that $b_i = 1/(3C^2)I_i$. Then F is a $C2$-like metric; (ii) if $b_i = 0$, then we have $C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\}$, contracting it with g^{ij} implies that $a_i = 1/(n+1)I_i$. Then F is a C-reducible metric; (iii) if $a_i = p/(n+1)I_i$ and $b_i = q/(3C^2)I_i$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on TM, then F is a semi-C-reducible metric. It is remarkable that, in [3] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of $C3$-like Finsler spaces will enhance our understanding of the geometric meaning of (α, β)-metrics.

In this paper, we study $C3$-like metrics and find a condition on $C3$-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be a $C3$-like Finsler manifold. Suppose that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Then F is a weakly Landsberg metric if and only if it is a Landsberg metric.*

There are many connections in Finsler geometry [12][13]. In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_xM$ the tangent bundle of M.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C=0}$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i\frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([2]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM and $C^2 = I^iI_i$. Multiplying the definition of semi-C-reducibility with g^{jk} shows that p and q must satisfy $p + q = 1$. If $p = 0$, then F is called $C2$-like metric. In [3], Matsumoto and Shibata proved that every (α, β)-metric is semi-C-reducible. Let us remark that an (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M [4].

Theorem 2.2. ([3][4]) Let $F = \phi(\frac{\beta}{\alpha})\alpha$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1.

Proof of Theorem 1.1: F is $C3$-like metric

$$C_{ijk}=\{a_ih_{jk}+a_jh_{ki}+a_kh_{ij}\}+\{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\}, \tag{2}$$

where $a_i=a_i(x,y)$ and $b_i=b_i(x,y)$ are scalar functions on TM. Multiplying (2) with g^{ij} implies that

$$a_i=\frac{1}{n+1}\{(1-2I^mb_m)I_i-C^2b_i\}, \tag{3}$$

where $C^2=I^mI_m$. By plugging (3) in (2), we get

$$\begin{aligned} C_{ijk} =\ & \frac{1}{n+1}\{I_ih_{jk}+I_jh_{ki}+I_kh_{ij}\}-\frac{2I^mb_m}{n+1}\{I_ih_{jk}+I_jh_{ki}+I_kh_{ij}\} \\ -\ & \frac{C^2}{n+1}\{b_ih_{jk}+b_jh_{ki}+b_kh_{ij}\}+\{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\}, \end{aligned} \tag{4}$$

or equivalently

$$\begin{aligned} M_{ijk}=-\frac{2I^mb_m}{n+1}\{I_ih_{jk}+I_jh_{ki}+I_kh_{ij}\} -\ & \frac{C^2}{n+1}\{b_ih_{jk}+b_jh_{ki}+b_kh_{ij}\} \\ +\ & \{b_iI_jI_k+I_ib_jI_k+I_iI_jb_k\}. \end{aligned} \tag{5}$$

By taking a horizontal derivation of (5), we have

$$\begin{aligned} \widetilde{M}_{ijk}=-\ & \frac{2}{n+1}(J^mb_m+I^mb'_m)\{I_ih_{jk}+I_jh_{ki}+I_kh_{ij}\} \\ -\ & \frac{2I^mb_m}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}-\frac{C^2}{n+1}\{b'_ih_{jk}+b'_jh_{ki}+b'_kh_{ij}\} \\ -\ & \frac{1}{n+1}(J^mI_m+I^mJ_m)\{b_ih_{jk}+b_jh_{ki}+b_kh_{ij}\} \\ +\ & \{b_iJ_jI_k+b_iI_jJ_k+b_jJ_iI_k+b_jI_iJ_k+b_kJ_iI_j+b_kI_iJ_j\} \\ +\ & \{b'_iI_jI_k+b'_jI_iI_k+b'_kI_iI_j\}, \end{aligned} \tag{6}$$

where $b'_i=b_{i|s}y^s$ and

$$\widetilde{M}_{ijk}=L_{ijk}-\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}.$$

Let F be a weakly Landsberg metric. Since b_i is constant along geodesics, i.e., $b'_i=0$, then (6) reduces to following

$$L_{ijk}=\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}=0. \tag{7}$$

This means that F is a Landsberg metric. □

Corollary 3.1. *Let (M,F) be a weakly Landsberg $C3$-like Finsler manifold. Suppose that $q=q(x,y)$ is constant along Finslerian geodesics. Then F is a Landsberg metric.*

Proof. Since F is weakly Landsberg, then (6) reduces to following

$$L_{ijk} = -\frac{C^2}{n+1}\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\} + \{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\}. \tag{8}$$

It is obvious that if $q = q(x, y)$ is constant along Finslerian geodesics, i.e., $q' = 0$ then F is a Landsberg metric. □

Corollary 3.2. *Let (M, F) be a semi-C-reducible Finsler manifold. Suppose that $q = q(x, y)$ is constant along Finslerian geodesics. Then F is a weakly Landsberg metric if and only if it is a Landsberg metric.*

Proof. According to Theorem 1.1, a weakly Landsberg semi-C-reducible metric is a Landsberg metric if and only if the following holds

$$\begin{aligned} 0 = b'_i = & \frac{q'}{3C^2} I_i + \frac{q}{3C^2} J_i - \frac{q}{3C^4}(I^m J_m + J^m I_m) I_i \\ = & \frac{q'}{3C^2} I_i \end{aligned} \tag{9}$$

Thus $b'_i = 0$ if and only if $q' = 0$. □

Acknowledgments. The authors would like to express their gratitude to the referees for careful reading and helpful comments.

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