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### **On** C3-Like Finsler Metrics

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ABSTRACT. In this paper, we study the class of of C3-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

Keywords: Finsler metric, C3-like metric, semi-C-reducible metric.

#### **2000** Mathematics subject classification: <u>53C60</u>, 53C25.

## 1. INTRODUCTION

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians [3][5][14][16]. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg, and etc [6][7]. In [5], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hojo proves that the converse is true too [2]. A Randers metric  $F = \alpha + \beta$  is just a Riemannian metric  $\alpha$  perturbated by a one form  $\beta$ , which has important applications both in mathematics and physics [15].

Let us remark some important curvatures in Finsler geometry. Let (M, F) be a Finsler manifold. The second derivatives of  $\frac{1}{2}F_x^2$  at  $y \in T_x M_0$  is an inner product  $\mathbf{g}_y$  on  $T_x M$ . The third order derivatives of  $\frac{1}{2}F_x^2$  at  $y \in T_x M_0$  is a symmetric trilinear forms  $\mathbf{C}_y$  on  $T_x M$ . We call  $\mathbf{g}_y$  and  $\mathbf{C}_y$  the fundamental

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form and the Cartan torsion, respectively. The rate of change of  $\mathbf{C}_y$  along geodesics is the Landsberg curvature  $\mathbf{L}_y$  on  $T_x M$  for any  $y \in T_x M_0$ . F is said to be Landsbergian if  $\mathbf{L} = 0$ .

In [11], Prasad-Singh introduced a new class of Finsler spaces named by C3-like spaces which contains the class of semi-C-reducible spaces, as special case (see [8], [9], [10]). A Finsler metric F is called C3-like if its Cartan tensor is given by

(1) 
$$C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\},\$$

where  $a_i = a_i(x, y)$  and  $b_i = b_i(x, y)$  are homogeneous scalar functions on TMof degree -1 and 1, respectively. We have some special cases as follows: (i) if  $a_i = 0$ , then we have  $C_{ijk} = \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}$ , contracting it with  $g^{ij}$ implies that  $b_i = 1/(3C^2)I_i$ . Then F is a C2-like metric; (ii) if  $b_i = 0$ , then we have  $C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\}$ , contracting it with  $g^{ij}$  implies that  $a_i = 1/(n+1)I_i$ . Then F is a C-reducible metric; (iii) if  $a_i = p/(n+1)I_i$ and  $b_i = q/(3C^2)I_i$ , where p = p(x, y) and q = q(x, y) are scalar functions on TM, then F is a semi-C-reducible metric. It is remarkable that, in [3] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian  $(\alpha, \beta)$ -metric on a manifold M of dimension  $n \ge 3$ is semi-C-reducible. Therefore the study of the class of C3-like Finsler spaces will enhance our understanding of the geometric meaning of  $(\alpha, \beta)$ -metrics.

In this paper, we study C3-like metrics and find a condition on C3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent. More precisely, we prove the following.

**Theorem 1.1.** Let (M, F) be a C3-like Finsler manifold. Suppose that  $b_i = b_i(x, y)$  is constant along Finslerian geodesics. Then F is a weakly Landsberg metric if and only if it is a Landsberg metric.

There are many connections in Finsler geometry [12][13]. In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by "  $\parallel$ " and ", " respectively.

# 2. Preliminaries

Let M be a n-dimensional  $C^{\infty}$  manifold. Denote by  $T_x M$  the tangent space at  $x \in M$ , and by  $TM = \bigcup_{x \in M} T_x M$  the tangent bundle of M.

A Finsler metric on M is a function  $F: TM \to [0, \infty)$  which has the following properties:

(i) F is  $C^{\infty}$  on  $TM_0 := TM \setminus \{0\};$ 

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,

(iii) for each  $y \in T_x M$ , the following quadratic form  $\mathbf{g}_y$  on  $T_x M$  is positive definite,

$$\mathbf{g}_{y}(u,v) := \frac{1}{2} \left[ F^{2}(y + su + tv) \right]|_{s,t=0}, \quad u,v \in T_{x}M.$$

Let  $x \in M$  and  $F_x := F|_{T_xM}$ . To measure the non-Euclidean feature of  $F_x$ , define  $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$  by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ \mathbf{g}_{y+tw}(u, v) \right] |_{t=0}, \quad u, v, w \in T_x M.$$

The family  $\mathbf{C} := {\mathbf{C}_y}_{y \in TM_0}$  is called the Cartan torsion. It is well known that  $\mathbf{C=0}$  if and only if F is Riemannian. For  $y \in T_xM_0$ , define mean Cartan torsion  $\mathbf{I}_y$  by  $\mathbf{I}_y(u) := I_i(y)u^i$ , where  $I_i := g^{jk}C_{ijk}$  and  $u = u^i \frac{\partial}{\partial x^i}|_x$ . By Diecke Theorem, F is Riemannian if and only if  $\mathbf{I}_y = 0$ .

For  $y \in T_x M_0$ , define the Matsumoto torsion  $\mathbf{M}_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by  $\mathbf{M}_y(u, v, w) := M_{ijk}(y) u^i v^j w^k$  where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \},\$$

and  $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$  is the angular metric. A Finsler metric F is said to be C-reducible if  $\mathbf{M}_y = 0$ . This quantity is introduced by Matsumoto [5]. Matsumoto proves that every Randers metric satisfies that  $\mathbf{M}_y = 0$ . A Randers metric  $F = \alpha + \beta$  on a manifold M is just a Riemannian metric  $\alpha = \sqrt{a_{ij}y^iy^j}$  perturbated by a one form  $\beta = b_i(x)y^i$  on M such that  $\|\beta\|_{\alpha} < 1$ . Later on, Matsumoto-Hōjō proves that the converse is true too.

**Lemma 2.1.** ([2]) A Finsler metric F on a manifold of dimension  $n \ge 3$  is a Randers metric if and only if  $\mathbf{M}_y = 0, \forall y \in TM_0$ .

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n} \{ h_{ij}I_k + h_{jk}I_i + h_{ki}I_j \} + \frac{q}{C^2} I_i I_j I_k,$$

where p = p(x, y) and q = q(x, y) are scalar function on TM and  $C^2 = I^i I_i$ . Multiplying the definition of semi-C-reducibility with  $g^{jk}$  shows that p and qmust satisfy p + q = 1. If p = 0, then F is called C2-like metric. In [3], Matsumoto and Shibata proved that every  $(\alpha, \beta)$ -metric is semi-C-reducible. Let us remark that an  $(\alpha, \beta)$ -metric is a Finsler metric on M defined by  $F := \alpha\phi(s)$ , where  $s = \beta/\alpha, \phi = \phi(s)$  is a  $C^{\infty}$  function on the  $(-b_0, b_0)$  with certain regularity,  $\alpha$  is a Riemannian metric and  $\beta$  is a 1-form on M [4].

**Theorem 2.2.** ([3][4]) Let  $F = \phi(\frac{\beta}{\alpha})\alpha$  be a non-Riemannian  $(\alpha, \beta)$ -metric on a manifold M of dimension  $n \ge 3$ . Then F is semi-C-reducible.

The horizontal covariant derivatives of **C** along geodesics give rise to the Landsberg curvature  $\mathbf{L}_y: T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$  defined by

$$\mathbf{L}_{y}(u, v, w) := L_{ijk}(y)u^{i}v^{j}w^{k},$$

where  $L_{ijk} := C_{ijk|s} y^s$ ,  $u = u^i \frac{\partial}{\partial x^i}|_x$ ,  $v = v^i \frac{\partial}{\partial x^i}|_x$  and  $w = w^i \frac{\partial}{\partial x^i}|_x$ . The family  $\mathbf{L} := {\mathbf{L}_y}_{y \in TM_0}$  is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if  $\mathbf{L} = 0$ .

### 3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1.

### **Proof of Theorem 1.1**: *F* is *C*3-like metric

(2) 
$$C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\}$$

where  $a_i = a_i(x, y)$  and  $b_i = b_i(x, y)$  are scalar functions on TM. Multiplying (2) with  $g^{ij}$  implies that

(3) 
$$a_i = \frac{1}{n+1} \{ (1 - 2I^m b_m) I_i - C^2 b_i \},$$

where  $C^2 = I^m I_m$ . By plugging (3) in (2), we get

$$C_{ijk} = \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \} - \frac{2I^m b_m}{n+1} \{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \}$$

$$(4) \qquad - \frac{C^2}{n+1} \{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \} + \{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \},$$

or equivalently

$$(4) - \frac{C^2}{n+1} \{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \} + \{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \},$$
  
or equivalently  
$$M_{ijk} = -\frac{2I^m b_m}{n+1} \{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \} - \frac{C^2}{n+1} \{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \}$$
  
(5) 
$$+ \{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \}.$$

By taking a horizontal derivation of (5), we have

$$\widetilde{M}_{ijk} = -\frac{2}{n+1} (J^m b_m + I^m b'_m) \{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\} - \frac{2I^m b_m}{n+1} \{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\} - \frac{C^2}{n+1} \{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\} - \frac{1}{n+1} (J^m I_m + I^m J_m) \{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\} + \{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\} (6) + \{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\},$$

where  $b'_i = b_{i|s} y^s$  and

$$\widetilde{M}_{ijk} = L_{ijk} - \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \}$$

Let F be a weakly Landsberg metric. Since  $b_i$  is constant along geodesics, i.e.,  $b'_i = 0$ , then (6) reduces to following

(7) 
$$L_{ijk} = \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \} = 0.$$

This means that F is a Landsberg metric.

Corollary 3.1. Let (M, F) be a weakly Landsberg C3-like Finsler manifold. Suppose that q = q(x, y) is constant along Finslerian geodesics. Then F is a Landsberg metric.

*Proof.* Since F is weakly Landsberg, then (6) reduces to following

(8) 
$$L_{ijk} = -\frac{C^2}{n+1} \{ b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij} \} + \{ b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j \}.$$

It is obvious that if q = q(x, y) is constant along Finslerian geodesics, i.e., q' = 0 then F is a Landsberg metric.

**Corollary 3.2.** Let (M, F) be a semi-C-reducible Finsler manifold. Suppose that q = q(x, y) is constant along Finslerian geodesics. Then F is a weakly Landsberg metric if and only if it is a Landsberg metric.

*Proof.* According to Theorem 1.1, a weakly Landsberg semi-C-reducible metric is a Landsberg metric if and only if the following holds

$$0 = b'_{i} = \frac{q'}{3C^{2}}I_{i} + \frac{q}{3C^{2}}J_{i} - \frac{q}{3C^{4}}(I^{m}J_{m} + J^{m}I_{m})I_{i}$$
$$= \frac{q'}{3C^{2}}I_{i}$$

Thus  $b'_i = 0$  if and only if q' = 0.

(9)

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