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ON bD-SETS AND ASSOCIATED SEPARATION AXIOMS†

A. KESKIN* AND T. NOIRI

Abstract. Here, first we introduce and investigate *bD-sets* by using the notion of b-open sets to obtain some weak separation axioms. Second, we introduce the notion of *gb-closed sets* and then investigate some relations of between b-closed and gb-closed sets. We also give a characterization of b-$T_{1/2}$ spaces via gb-closed sets. We introduce two new weak homeomorphisms which are important keys between General Topology and Algebra. Using the notion of m_X-*structures*, we give a characterization theorem of m_X-T_2 spaces. Finally, we give some examples related to the digital line.

1. Introduction

It is known that open sets play a very important role in General Topology. In [38], Tong introduced the notion of D-sets by using open sets and used the notion to define some separation axioms. Later, the modifications of these notions for α-open (resp. semi-open, preopen, δ-semi-open) sets are introduced and some of their properties investigated in [6] (resp.[4], [20], [7]) and [20], respectively. The notion of b-open sets were introduced by Andrijević [3]. The set was named as γ-open and sp-open by El-Atik [17] and Doncthev and Przemski [13], respectively.

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∗Corresponding author

The notion of b-open sets is stronger than the notion of β-open sets and is weaker than the notions of semi-open sets and preopen sets. Since then, these notions has been extensively investigated in the literature (see [32], [33], [36], [14], [11], [37], [8], [9], [15]).

Here, first we introduce the notion of *bD-sets* as the difference sets of b-open sets. Second, we introduce the notion of *gb-closed sets* and investigate some relations between b-closed and gb-closed sets. We also give a characterization of b-$T_{1/2}$ spaces via gb-closed sets. Then, we investigate some preservation theorems. We must state that we introduce new two weak homeomorphisms. It is well-known that the notion of homeomorphisms is an important key between General Topology and Algebra. By using the notion of m_X-*structures*, we give a characterization theorem of m_X-T_2 spaces. Finally, we give some examples. Some applications of our results may relate to the digital line.

2. **Preliminaries**

Through out the paper, by (X, τ) and (Y, φ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subset X$. Then A is called *b-open* [3] if $A \subset Cl(Int(A)) \cup Int(Cl(A))$, where $Cl(A)$ and $Int(A)$ denote the closure and interior of A in (X, τ), respectively. The complement A^c of a b-open set A is called *b-closed* [3] and the *b-closure* of a set A, denoted by $bCl(A)$, is defined by the intersection of all b-closed sets containing A. The *b-interior* of a set A, denoted by $bInt(A)$, is the union of all b-open sets contained in A. The symbols $bCl(A)$ and $bInt(A)$ were first used by Andrijević [3]. The family of all b-open (resp. b-closed) sets in (X, τ) will be denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$) as in [3]. The family of all b-open sets containing x of X will be denoted by $BO(X, x)$ as in [37]. It was shown that [3, Proposition 2.3(a)] the union of any family of b-open sets is a b-open set.

We recall some definitions used in the sequel.

Definition 2.1. A subset A of a space (X, τ) is said to be
(a) α-open [34] if $A \subset Int(Cl(Int(A)))$,
(b) semi-open [27] if $A \subset Cl(Int(A))$,
(c) preopen [30] if $A \subset Int(Cl(A))$,
(d) β-open [1] if $A \subset Cl(Int(Cl(A)))$.

Through out the paper, the family of all α-open (resp. semi-open, preopen) sets in a topological space (X, τ) is denoted by $\alpha(X)$ (resp. $SO(X,\tau)$, $PO(X,\tau)$).

Definition 2.2. A subset S of a topological space X is called
(a) a D-set [38] if there are $U, V \in \tau$ such that $U \neq X$ and $S = U \backslash V$,
(b) an αD-set [6] if there are $U, V \in \alpha(X)$ such that $U \neq X$ and $S = U \backslash V$,
(c) a sD-set [4] if there are $U, V \in SO(X,\tau)$ such that $U \neq X$ and $S = U \backslash V$,
(d) a pD-set [21] if there are $U, V \in PO(X,\tau)$ such that $U \neq X$ and $S = U \backslash V$.

Observe that every open (resp. α-open, semi-open, preopen) set U different from X is a D-set (resp. an αD-set, a sD-set, a pD-set) if $S = U$ and $V = \varnothing$. Furthermore, since every open set is α-open, then every α-open set is semi-open and preopen. We have the following properties.

Proposition 2.3. *(a) Every D-set is an αD-set,*
(b) every αD-set is an sD-set, and
(c) every αD-set is a pD-set.

In [6], Caldas et al. showed that the converses of (b) and (c) need not be true, in general. One can see related examples [4, Example 3.1] and [4 , Example 3.2]. Since the notions of semi-open sets and preopen sets are independent, then one can easily obtain that the notions of sD-sets and pD-sets are independent of each other.

3. bD-sets and associated separation axioms

Definition 3.1. A subset S of a topological space X is called a bD-set if there are $U, V \in BO(X,\tau)$ such that $U \neq X$ and $S = U \setminus V$.

It is true that every b-open set U different from X is a bD-set if $S = U$ and $V = \varnothing$. So, we can observe the following.

Remark 3.2. Every proper b-open set is a *bD-set*. But, the converse is false as the next example shows.

Example 3.3. Let X={a,b,c,d} and τ={X,$\varnothing$,{a},{a,d},{a,b,d},{a,c,d}}. Then, {b} is a *bD-set* but it is not a b-open. In really, since BO(X,τ)={X,$\varnothing$,{a},{a,b},{a,c},{a,d},{a,b,c},{a,b,d},{a,c,d}}, then U={a,b}$\neq X$ and V={a,c} are b-open sets in X. For U and V, since $S = U\backslash V$={a,b}\{a,c}={b}, then we have S={b} is a bD-set but it is not b-open.

We have diagram I below.

$$\begin{array}{ccccccc}
 & & \text{preopen} & \longrightarrow & b\text{-open} & & \\
 & \nearrow & | & & \nearrow & | & \\
\text{open}\longrightarrow & \alpha\text{-open} & |\longrightarrow & \text{semi-open} & | & & \\
| & | & \downarrow & | & \downarrow & & \\
| & | & pD\text{-set}\longrightarrow & | & bD\text{-set} & & \\
\downarrow & \downarrow & \nearrow & \downarrow & \nearrow & & \\
D\text{-set}\longrightarrow & \alpha D\text{-set}\longrightarrow & sD\text{-set} & & & &
\end{array}$$

Diagram I

Definition 3.4. A topological space (X, τ) is called b-D$_0$ [10] (resp. b-D$_1$ [10]) if for any pair of distinct points x and y of X there exists a bD-set of X containing x but not y or (resp. and) a bD-set of X containing y but not x.

Definition 3.5. A topological space (X, τ) is called b-D$_2$ [10] if for any pair of distinct points x and y of X there exist disjoint bD-sets G and E of X containing x and y, respectively.

Definition 3.6. A space X is called b-T$_0$ [10] if for every pair of distinct points x and y of X, there exists a b-open set of X containing x but not y or a b-open set of X containing y but not x.

We recall that a topological space (X, τ) is called b-T_1 ([2], [8], [10]) if for each pair of distinct points x and y of X, there exist b-open sets U and V containinig x and y, respectively, such that $y \notin U$ and $x \notin V$. Additionally, in [37], Park introduced the notion of b-T_2 spaces as follows: A topological space (X, τ) is called b-T_2 if for any pair of distinct points x and y in X, there exist $U \in BO(X, x)$ and $V \in BO(X, y)$ such that $U \cap V = \varnothing$.

The following remark and theorem are due to [10].

Remark 3.7. (i) For a topological space (X, τ), the following properties hold:
(i-1) (Caldas and Jafari [10]) (a) If (X, τ) is b-T_i, then it is b-D_i, i=0,1,2.
(b) If (X, τ) is b-D_i, then it is b-D_{i-1}, i=1,2.
(c) If (X, τ) is b-T_i, then it is b-T_{i-1}, i=1,2.
(i-2) (Caldas and Jafari [10]) (a) If (X, τ) is b-D_0 if and only if it is b-T_0.
(b) If (X, τ) is b-D_1 if and only if it is b-D_2.

(ii) In [10], the authors proved that every topological space is b-T_0.

(iii) Using Remark 3.7 (i-1)(a) or (i-2)(a) above, every topological space is b-D_0. The Sierpinski space is not b-D_1.

Definition 3.8. A subset A of a topological space (X, τ) is called a generalized b-closed (briefly gb-closed) set if $bCl(A) \subset U$, whenever $A \subset U$ and U is *b-open* in (X, τ).

The following notion is due to [2].

A topological space (X, τ) is called a B-$T_{1/2}$ space if each singleton is either b-open or b-closed. The authors proved that "every topological space is B-$T_{1/2}$" [2]. Here, we define the concept of "b-$T_{1/2}$-spaces".

Definition 3.9. A topological space (X, τ) is called b-$T_{1/2}$ if every gb-closed set is b-closed.

It is obvious that every b-closed is gb-closed (Definition 3.7). Recall that a topological space (X, τ) is called:

a) *b-symmetric* [15] if for each x and y in X, $x \in bCl(\{y\})$ implies $y \in bCl(\{x\})$;

b) b-R_0 [15] if its every b-open set contains the b-closure of each singleton.

Theorem 3.10. *For a topological space (X, τ) , the following properties hold:*

(i) *(Abd El-Monsef, El-Atik and Sharkasy [2]) Let x be a point of (X, τ). Then, $\{x\}$ is b-open or b-closed.*

(ii) *A space (X, τ) is b-$T_{1/2}$ if and only if each singleton is b-open or b-closed in (X, τ).*

(iii) *Every topological space is a b-$T_{1/2}$-space, i.e., a subset A is gb-closed in (X, τ) if and only if A is b-closed.*

(iv) *For a space (X, τ), the following properties are equivalent:*
(1) (X, τ) is b-symmetric; (2) (X, τ) is b-T_1 ;(3) (X, τ) is b-R_0 .

(v) *For each pair of distinct points x, y of X, $bCl(\{x\}) \neq bCl(\{y\})$.*

Proof. (i) This is obtained in [2; the proof of Lemma 2.3], but here we will give an alternative proof. By [22; Lemma 2], for every point x of any topological space (X, τ), $\{x\}$ is preopen or nowhere dense (i.e., $Int(Cl(\{x\})) = \varnothing$) and so $\{x\}$ is preopen or semi-closed. Therefore, $\{x\}$ is b-open or b-closed.

(ii) $Necessity$: Let $x \in X$. When $\{x\} \notin BC(X, \tau)$, $X\backslash\{x\} \notin BO(X, \tau)$, then for any b-open set U satisfying a property $X\backslash\{x\} \subset U$, we have $U = X$ only and so $bCl(X\backslash\{x\}) \subset U$. This shows that $X\backslash\{x\}$ is gb-closed and, by assumption, the singleton $\{x\}$ is b-open.

$Sufficiency$: Let A be a gb-closed set of (X, τ). In order to prove $bCl(A) = A$, let $x \in bCl(A)$. When $\{x\}$ is b-open, $\{x\} \cap A \neq \varnothing$ and so $x \in A$. When $\{x\}$ is b-closed, $X\backslash\{x\} \in BO(X, \tau)$. For this case, suppose that $x \notin A$. Since $A \subset X\backslash\{x\}$ and A is a gb-closed, we have that $x \in bCl(A) \subset X\backslash\{x\}$ and hence $x \in X\backslash\{x\}$. This contradiction shows that $x \in A$ for a point satisfying $x \in bCl(A)$ and $A \in BC(X, \tau)$. Therefore, every gb-closed set is b-closed in (X, τ).

(iii) It follows from (i) and (ii) that every topological space is b-$T_{1/2}$.

(iv) (1) $\Longrightarrow$ (2). Let $x \in X$. We claim that $bCl(\{x\}) \subset \{x\}$. Let $y \in bCl(\{x\})$. Then, by (i), $x \in bCl(\{y\})$ holds. If $\{x\}$ is b-open, then $\{x\} \cap \{y\} \neq \varnothing$ and so $y \in \{x\}$. If $\{x\}$ is b-closed, $y \in bCl(\{x\}) = \{x\}$ and so $y \in \{x\}$. By using (i), the claim is proved. Therefore, (X, τ) is b-T_1 .
(2) $\Longrightarrow$ (3). Let $G \in BO(X, \tau)$. For a point $x \in G$, $bCl(\{x\}) = \{x\} \subset G$. Thus, (X, τ) is b-R_0 .
(3) $\Longrightarrow$ (1). It is similar to [15].

(v) Suppose that there exists a pair of distinct points x and y such that $bCl(\{x\}) = bCl(\{y\})$. Then, by using (i), $\{x\}$ is b-open or b-closed.

If $\{x\}$is b-open, then $\{x\} \cap \{y\} \neq \varnothing$, because $x \in bCl(\{y\})$. Thus, we have $x = y$.

If $\{x\}$is b-closed, $\{x\} = bCl(\{x\}) = bCl(\{y\})$ and so $\{x\} = \{y\}$. For both cases, we have contradiction. □

For a subset A of a topological space (X, τ) and a family m_X of subsets of (X, τ) satisfying properties $\varnothing, X \in m_X$, the following subset $\Lambda_m(A)$ is defined in [11]: $\Lambda_m(A) = \cap\{U \mid A \subset U, U \in m_X\}$. Such a family m_X is called an *m_X-structure* on X [35]. For $m_X = \tau$ (resp. $SO(X,\tau), PO(X,\tau), BO(X,\tau)$), the set $\Lambda_m(A)$ is denoted by $\Lambda(A)$ [28] (resp. $\Lambda_s(A)$ [5], $\Lambda_p(A)$ [19], $\Lambda_b(A)$ [14]).

Corollary 3.11. *Let A be a subset of a topological space (X, τ).*

(i) $\Lambda_b(A) \subset \Lambda_s(A) \cap \Lambda_p(A)$ *and* $\Lambda_s(A) \cup \Lambda_p(A) \subset \Lambda(A)$ *hold.*

(ii) *(a) Assume that $BO(X,\tau)$ is a topology of X. If $\Lambda_b(\{x\}) \neq X$ for a point $x \in X$, then $\{x\}$ is a bD-set of (X, τ).*
(b) If a singleton $\{x\}$ is a bD-set of (X, τ), then $\Lambda_b(\{x\}) \neq X$.

(iii) *If $\Lambda(\{x\}) \neq X$ for a point $x \in X$, then $\{x\}$ is a bD-set of (X, τ).*

(iv) *For a topological space (X, τ) with at least two points, (X, τ) is a b-D_1-space if and only if $\Lambda_b(\{x\}) \neq X$ holds for every point $x \in X$.*

(v) *Let X be a set with at least two points. If there exists a point $x \in X$ such that $\Lambda_b(\{x\}) = X$, then (X, τ) is not b-D_1 (thus, it is not b-D_2).*

Proof. (i) According to [3], since $\tau \subset SO(X,\tau) \cap PO(X,\tau)$ and $SO(X,\tau) \cup PO(X,\tau) \subset BO(X,\tau)$, then we have $\Lambda_b(A) \subset \Lambda_s(A)$, $\Lambda_b(A) \subset \Lambda_p(A)$, $\Lambda_s(A) \subset \Lambda(A)$ and $\Lambda_p(A) \subset \Lambda(A)$. This shows that we have the required implications.

(ii) (a) Since $\Lambda_b(\{x\}) \neq X$ for a point $x \in X$, then there exists a subset $U \in BO(X,\tau)$ such that $\{x\} \subset U$ and $U \neq X$. Using Theorem 3.10(i) for the point x, then $\{x\}$ is b-open or b-closed in (X, τ). When the singleton $\{x\}$ is b-open, $\{x\}$ is a bD-set of (X, τ). When the singleton $\{x\}$ is b-closed, then $(\{x\})^c$ is b-open in (X, τ). Put $U_1 = U$ and $U_2 = U \cap (\{x\})^c$. Then, $\{x\} = U_1 \backslash U_2$, $U_1 \in BO(X,\tau)$ and $U_1 \neq X$. It follows from the hypothesis that $U_2 \in BO(X,\tau)$ and so $\{x\}$ is a bD-set.
(b) Since $\{x\}$ is a bD-set of (X, τ), then there exist two subsets $U_1 \in BO(X,\tau)$ and $U_2 \in BO(X,\tau)$ such that $\{x\} = U_1 \backslash U_2$, $\{x\} \subset U_1$ and $U_1 \neq X$. Thus, we have that $\Lambda_b(\{x\}) \subset U_1 \neq X$ and so $\Lambda_b(\{x\}) \neq X$.

(iii) Since $\Lambda(\{x\}) \neq X$, then there exists a subset $U \in \tau$ such that $\{x\} \subset U$ and $U \neq X$. Using Theorem 3.10(i) for the point x, $\{x\}$ is b-open or b-closed in (X, τ). When the singleton $\{x\}$ is b-open, then $\{x\}$ is a bD-set of (X, τ). When the singleton $\{x\}$ is b-closed, then $(\{x\})^c$ is b-open in (X, τ). Put $U_1 = U$ and $U_2 = U \cap (\{x\})^c$. By [3, Proposition 2.3(b)], the set U_2 is b-open. Therefore, $\{x\} = U_1 \backslash U_2$ and $\{x\}$ is a bD-set, because $U_1 \in BO(X, \tau)$ and $U_1 \neq X$.

(iv) Necessity: Let $x \in X$. For a point $y \neq x$, there exists a bD-set U such that $x \in U$ and $y \notin U$. Say $U = U_1 \backslash U_2$, where $U_i \in BO(X, \tau)$ for each $i \in \{1, 2\}$ and $U_1 \neq X$. Thus, for the point x, we have a b-open set U_1 such that $\{x\} \subset U_1$ and $U_1 \neq X$. Hence, $\Lambda_b(\{x\}) \neq X$.
Sufficiency: Let x and y be a pair of distinct points of X. We prove that there exist bD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Theorem 3.10(i), we can take the subsets A and B for the following four cases for two points x and y.

Case1. $\{x\}$ is b-open and $\{y\}$ is b-closed in (X, τ). Since $\Lambda_b(\{y\}) \neq X$, then there exists a b-open set V such that $y \in V$ and $V \neq X$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = V$, then $\{y\}^c$, V is a b-open set with $V \neq X$ and $\{y\}^c$ is b-open, and B is a required bD-set containing y such that $x \notin B$. Obviously, A is a required bD-set containing x such that $y \notin A$.

Case 2. $\{x\}$ is b-closed and $\{y\}$ is b-open in (X, τ). The proof is similar to Case 1.

Case 3. $\{x\}$ and $\{y\}$ are b-open in (X, τ). Put $A = \{x\}$ and $B = \{y\}$.

Case 4. $\{x\}$ and $\{y\}$ are b-closed in (X, τ). Put $A = \{y\}^c$ and $B = \{x\}^c$.

For each case above, the subsets A and B are the required bD-sets. Therefore, (X, τ) is a b-D_1-space.

(v) By (iv) and Remark 3.7 (i-2)(b), (v) is obtained. □

Remark 3.12. (i) The converse of Corollary 3.11(iii) is not true, in general. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then, $\Lambda_b(\{c\}) = \{c\} \neq X$ and the singleton $\{c\} = \{b, c\} \backslash \{b\}$is a *bD-set*; $\Lambda(\{c\}) = X$ holds.

(ii) It follows from Corollary 3.11 (i) that for a point $x \in X$, $\Lambda_b(\{x\}) \neq X$ if $\Lambda(\{x\}) \neq X$; $\Lambda(\{x\}) = X$ if $\Lambda_b(\{x\}) = X$.

4. **Preservation theorems**

Here, first we recall some definitions. Then, we will give several preservation theorems.

Definition 4.1. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be

(a) α-continuous [31] if $f^{-1}(V)$ is α-open in (X, τ), for every open set V of (Y, φ),

(b) α-open [31] if $f(U)$ is α-open in $(Y,\ \varphi)$, for every open set U of (X, τ),

(c) γ-*irresolute* [14] if $f^{-1}(V)$ is γ-open in X, for every γ-open set V of Y,

(d) γ-*continuous* [17] if $f^{-1}(V)$ is γ-*open* in (X, τ), for every open set V of (Y, φ).

We note that since the notion of b-open sets and the notion of γ-open sets are the same, then here we will use the term of b-irresolute (resp. b-continuous) functions instead of γ-irresolute (resp. γ-continuous) functions. In [10; Definition 6] the authors used the term of b-continuous functions instead of γ-irresolute functions.

Theorem 4.2. *If $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a b-continuous (resp. b-irresolute) surjective function and S is a D-set (resp. bD-set) of (Y, φ), then $f^{-1}(S)$ is a bD-set of (X, τ).*

Proof. Let $S = O_1 \backslash O_2$ be a D-set (resp. bD-set) of $(Y,\ \varphi)$, where $O_i \in \varphi$ (resp. $O_i \in BO(Y,\ \varphi)$), for each $i \in \{1, 2\}$ and $O_1 \neq Y$. We have that $f^{-1}(O_i) \in BO(X, \tau)$, for each $i \in \{1, 2\}$ and $f^{-1}(O_1) \neq X$. Hence, $f^{-1}(S) = f^{-1}(O_1) \cap (X \backslash f^{-1}(O_2))$. Therefore, $f^{-1}(S)$ is a bD-*set*. □

Theorem 4.3. *If (Y, φ) is a D_1 space (resp. b-D_1 space) and $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a b-continuous (resp. b-irresolute) bijective function, then (X, τ) is a b-D_1 space.*

Proof. Suppose that Y is a D_1 space (resp. b-D_1 space). Let x and y be any pair of distinct points in X. Since f is injective and Y is D_1 (resp. b-D_1), then there exist D-sets (resp. bD-sets) S_x and S_y of Y containing

$f(x)$ and $f(y)$, respectively, such that $f(x) \notin S_y$ and $f(y) \notin S_x$. By Theorem 4.2, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are *bD-sets* in X containing x and y, respectively, such that $x \notin f^{-1}(S_y)$ and $y \notin f^{-1}(S_x)$. This implies that X is a b-D_1 space. □

Theorem 4.4. *A topological space* (X, τ) *is b-*D_1 *if for each pair of distinct points* x, $y \in X$, *there exists a b-continuous (resp. b-irresolute) surjective function* $f : (X,\tau) \longrightarrow (Y,\varphi)$, *where* (Y,φ) *is a* D_1 *space (resp. b-*D_1 *space) such that* $f(x)$ *and* $f(y)$ *are distinct.*

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists a b-continuous (resp. b-irresolute) surjective function f of a space (X,τ) onto a D_1 space (resp. b-D_1 space) (Y, φ) such that $f(x) \neq f(y)$. It follows from Theorem 4.2 of [37] (resp. Remark 3.7(i-2)(b)) that $D_1 = D_2$ (resp. b-D_1 $= b$-D_2). Hence, there exist disjoint D-sets (resp. bD-sets) S_x and S_y in Y such that $f(x) \in S_x$ and $f(y) \in S_y$. Since f is b-continuous (resp. b-irresolute) and surjective, by Theorem 4.2, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are disjoint bD-sets in X containing x and y, respectively. So, the space (X, τ) is b-D_1 . □

The following notion is due to Hatir and Noiri [20].

A filterbase $\mathbf{B}$ is called D-convergent to a point $x \in X$ if for any D-set A containing x, there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset A$.

Definition 4.5. Let (X,τ) be a topological space. A filter base $\mathbf{B}$ is called *bD-convergent* to a point $x \in X$, if for any *bD-set* A containing x, there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset A$.

Theorem 4.6. *If a function* $f : (X,\tau) \longrightarrow (Y,\varphi)$ *is b-continuous (resp.b-irresolute) and surjective, then for each point* $x \in X$ *and each filterbase* $\mathbf{B}$ *on* (X,τ), *bD-converging to x, the filterbase* $f(\mathbf{B})$ *is D-convergent (resp. bD-convergent) to* $f(x)$.

Proof. Let $x \in X$ and $\mathbf{B}$ be any filterbase bD-converging to x. Since f is a b-continuous (resp. b-irresolute) surjection, by Theorem 4.2, for each D-set (resp. bD-set) $V \subset Y$ containing $f(x)$, $f^{-1}(V) \subset X$ is a bD-set containing x. Since $\mathbf{B}$ is bD-converging to x, then there exists $B_1 \in \mathbf{B}$ such that $B_1 \subset f^{-1}(V)$ and hence $f(B_1) \subset V$. It follows that $f(\mathbf{B})$ is D-convergent (resp. bD-convergent) to $f(x)$. □

Recall that a topological space (X, τ) is said to be *D-compact* [20] if every cover of X by D-sets has a finite subcover.

Definition 4.7. A topological space (X, τ) is said to be *bD-compact* if every cover of X by *bD-sets* has a finite subcover.

Theorem 4.8. *Let a function* $f : (X, \tau) \longrightarrow (Y, \varphi)$ *be b-continuous (resp.b-irresolute) and surjective. If* (X, τ) *is bD-compact, then* (Y, φ) *is D-compact (resp. bD-compact).*

Proof. It is proved by using Theorem 4.2. □

Recall that a topological space (X, τ) is said to be D-connected [20] if (X, τ) cannot be expressed as the union of two disjoint nonempty D-sets.

Definition 4.9. A topological space (X, τ) is said to be bD-connected if (X, τ) cannot be expressed as the union of two disjoint nonempty *bD-sets*.

Theorem 4.10. *If* $f : (X, \tau) \longrightarrow (Y, \varphi)$ *is a b-continuous (resp. b-irresolute) surjection and* (X, τ) *is bD-connected, then* (Y, φ) *is D-connected (resp.bD-connected).*

Proof. It is proved by using Theorem 4.2. □

Remark 4.11. Theorems 4.2, 4.3, 4.4, 4.6, 4.8 and 4.10 are true for an α-continuous and α-open function f instead of a *b-irresolute* function f. For an α-continuous and α-open function f, the inverse image $f^{-1}(S)$ of each b-open set S is b-open (see El-Atik [17]).

It is well known that the notion of homeomorphisms is very important in General Topology. The following definition provides two new weak forms of homeomorphisms.

Definition 4.12. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called a br-homeomorphism (resp. b-homeomorphism) if f is a b-irresolute bijection (resp. b-continuous bijection) and $f^{-1} : (Y, \varphi) \longrightarrow (X, \tau)$ is a b-irresolute (resp. b-continuous).

Now, we can give the following definition by taking the space (X, τ), instead of the space (Y, φ).

Definition 4.13. For a topological space (X, τ), we define the following two collections of functions:

br-h$(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a *b-irresolute bijection*, $f^{-1} : (X, \tau) \longrightarrow (X, \tau)$ is *b-irresolute*$\}$;

b-h$(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a *b-continuous bijection*, $f^{-1} : (X, \tau) \longrightarrow (X, \tau)$ is *b-continuous*$\}$.

Theorem 4.14. *For a topological space (X, τ), the following properties hold:*

(i) $h(X, \tau) \subset br\text{-}h(X, \tau) \subset b\text{-}h(X, \tau)$, where $h(X, \tau) = \{f \mid f : (X, \tau) \longrightarrow (X, \tau)$ is a homeomorphism $\}$.

(ii) The collection $br\text{-}h(X, \tau)$ forms a group under the composition of functions.

(iii) The group $h(X, \tau)$ of all homeomorphisms on (X, τ) is a subgroup of $br\text{-}h(X, \tau)$.

Proof. *(i)* First we show that every homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$ is a *br*-homeomorphism. Indeed, for a subset $A \in BO(Y, \varphi)$, $f^{-1}(A) \subset f^{-1}(Cl(Int(A)) \cup Int(Cl(A))) = Cl(Int(f^{-1}(A))) \cup Int(Cl(f^{-1}(A)))$ and so $f^{-1}(A) \in BO(X, \tau)$. Thus, f is b-irresolute. In a similar way, it is shown that f^{-1} is b-irresolute. Hence, we have that $h(X, \tau) \subset br$-$h(X, \tau)$.

Finally, it is obvious that $br\text{-}h(X, \tau) \subset b\text{-}h(X, \tau)$, because every b-irresolute function is b-continuous.

(ii) If $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$ are *br*-homeomorphisms, then their composition $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is a *br*-homeomorphism. It is obvious that for a bijective *br*-homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$, $f^{-1} : (Y, \varphi) \longrightarrow (X, \tau)$ is also a *br*-homeomorphism and the identity $1 : (X, \tau) \longrightarrow (X, \tau)$ is a *br*-homeomorphism. A binary operation $\alpha : br\text{-}h(X, \tau) \times br\text{-}h(X, \tau) \longrightarrow br\text{-}h(X, \tau)$ is well defined by $\alpha(a, b) = b \circ a$, where $a, b \in br\text{-}h(X, \tau)$ and $b \circ a$ is the composition of a and b. By using the above properties, the set $br\text{-}h(X, \tau)$ forms a group under composition of functions.

(iii) For any $a, b \in h(X, \tau)$, we have $\alpha(a, b^{-1}) = b^{-1} \circ a \in h(X, \tau)$ and $1_X \in h(X, \tau) \neq \varnothing$. Thus, using *(i)* and *(ii)*, it is obvious that the group $h(X, \tau)$ is a subgroup of *br*-$h(X, \tau)$. □

For a topological space (X,τ), we can construct a new group *br*-$h(X, \tau)$ satisfying the property: if there exists a homeomorhism $(X, \tau) \cong (Y, \varphi)$, then there exists a group isomorphism *br*-$h(X, \tau) \cong$ *br*-$h(Y, \varphi)$.

Corollary 4.15. *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$ be two functions between topological spaces.*

(i) For a br-homeomorphism $f : (X, \tau) \longrightarrow (Y, \varphi)$, there exists an isomorphism, say $f_* :$ br-$h(X, \tau) \longrightarrow$ br-$h(Y, \varphi)$, defined by $f_*(a) = f \circ a \circ f^{-1}$, for any element $a \in$ br-$h(X, \tau)$.

(ii) For two br-homeomorphisms $f : (X, \tau) \longrightarrow (Y, \varphi)$ and $g : (Y, \varphi) \longrightarrow (Z, \eta)$, $(g \circ f)_* = g_* \circ f_* :$ br-$h(X, \tau) \longrightarrow$ br-$h(Z, \eta)$ holds.

(iii) For the identity function $1_X : (X, \tau) \longrightarrow (X, \tau)$, $(1_X)_* = 1 :$ br-$h(X, \tau) \longrightarrow$br-$h(X, \tau)$ holds, where 1 denotes the identity isomorphism.

Proof. Straightforward. □

Remark 4.16. (i) The following example shows that $h(X, \tau)$ is a proper subgroup of *br*-$h(X, \tau)$. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$. We note that $BO(X, \tau) = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$. It is shown that $h(X, \tau) = \{1_X\}$ and *br*-$h(X, \tau) = \{1_X, h_a\}$, where 1_X is the identity on (X, τ) and $h_a : (X, \tau) \longrightarrow (X, \tau)$ is a bijection defined by $h_a(a) = a$, $h_a(b) = c$ and $h_a(c) = b$.

(ii) The following example shows that *br*-$h(X, \tau)$ is a proper subset of *b*-$h(X, \tau)$. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a, b\}, X\}$. Then, $BO(X, \tau) = P(X) \backslash \{\{c\}\}$. There exists an element $h_b \in$ *b*-$h(X, \tau)$ such that$h_b \notin$ *br*-$h(X, \tau)$, where $h_b : (X, \tau) \longrightarrow (X, \tau)$ is a bijection defined by $h_b(b) = b$, $h_b(a) = c$ and $h_b(c) = a$.

(iii) The converse of Corollary 4.15(i) is not always true. Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\varphi = \{\varnothing, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a bijection between topological spaces defined by

$f(a) = b$, $f(b) = c$ and $f(c) = a$. Then, it is shown that $f_* : br\text{-}h(X, \tau) \longrightarrow br\text{-}h(Y, \varphi)$ is an isomorphism and the function f is not a *br-homeomorphism*. Indeed, $BO(X, \tau) = P(X) \backslash \{\{c\}\}$, $BO(Y, \varphi) = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, Y\}$, $br\text{-}h(X, \tau) = \{1_X, h_b\}$, $br\text{-}h(Y, \varphi) = \{1_Y, h_a\}$, where h_b (resp. h_a) is defined in (ii) (resp. (i)). Moreover, $f_*(h_b) = h_a$ holds and for a set $\{a\} \in BO(Y, \varphi)$, $f^{-1}(\{a\}) = \{c\} \notin BO(X, \tau)$ and so f is not a *br-homeomorphism*.

5. Some properties of b-T_2spaces

Since the notion of b-open sets and the notion of γ-open sets are the same, then in this paper we will use the term of b-open functions instead of γ-open functions. Recall that a function is called γ-open [14] if the image of every γ-open set is γ-open.

In the following theorems, for a non-empty topological space (Y, φ), we consider a family m_Y of subsets of (Y, φ) such that $m_Y \in \{SO(Y, \varphi), PO(Y, \varphi), BO(Y, \varphi)\}$. Namely, the family m_Y is only one element of $\{SO(Y, \varphi), PO(Y, \varphi), BO(Y, \varphi)\}$. We recall that (Y, m_Y) is called m_Y-T_2 [36] if for each pair of distinct points $x, y \in Y$, there exist $U, V \in m_Y$ containing x and y, respectively, such that $U \cap V = \varnothing$. A topological space (Y, φ) is called semi-T_2 [29] (resp. pre-T_2 [23], b-T_2 [37]) if (Y, m_Y) is m_Y-T_2, where $m_Y = SO(Y, \varphi)$ (resp. $PO(Y, \varphi)$, $BO(Y, \varphi)$). A function $f : (X, m_X) \longrightarrow (Y, m_Y)$ is called M-open [11], if for each set$A \in m_X$, $f(A) \in m_Y$. For topological spaces (X, τ), (Y, φ) with m_X-structure and m_Y-structure, respectively, here we call, a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ to be (m_X, m_Y)-*open* if $f : (X, m_X) \longrightarrow (Y, m_Y)$ is M-open in the sense of [11] given above.

Theorem 5.1. *Let R be an equivalance relation, $R \subset X \times X$, in a topological space (X, τ) and $(X/R, \Psi)$ an identification space. Let $(m_X, m_{X/R}) = (SO(X, \tau), SO(X/R, \Psi))$ (resp. $(PO(X, \tau), PO(X/R, \Psi))$, $(BO(X, \tau), BO(X/R, \Psi))$).*
Assume that:

(a) the identification function $\rho : (X, \tau) \longrightarrow (X/R, \Psi)$ is $(m_X, m_{X/R})$-open, and

(b) for each point $(x, y) \in (X \times X) \backslash R$, there exist subsets U_x, $U_y \in m_X$ such that $x \in U_x$, $y \in U_y$ and $U_x \times U_y \subset (X \times X) \backslash R$.

Then, $(X/R, m_{X/R})$ *is* $m_{X/R}$-T_2 .

Proof. Let $\rho(x)$ and $\rho(y)$ be distinct members of X/R. Since x and y are not equivalent, then $(x, y) \in (X \times X)\backslash R$. By assumption, there exists two subsets $U_x \in m_X$, $U_y \in m_X$ such that $x \in U_x$, $y \in U_y$ and $U_x \times U_y \subset (X \times X)\backslash R$. Then, we have that $U_x \cap U_y = \varnothing$, because $\{(z, z') \in X \times X \mid z = z'\} \subset R$. By the further assumption, $\rho(U_x)$ and $\rho(U_y)$ are the required subsets containing $\rho(x)$ and $\rho(y)$, respectively, i.e., $\rho(U_x), \rho(U_y) \in m_{X/R}$ and $\rho(U_x) \cap \rho(U_y) = \varnothing$. □

Theorem 5.2. *For a topological space* (X, τ) *and each family* $m_X \in \{SO(X, \tau), PO(X, \tau), BO(X, \tau)\}$*, the following properties are equivalent:*

(1) (X, m_X) *is* m_X-T_2 .

(2) For distinct points x *and* $y \in X$*, there exists a subset* $U \in m_X$ *such that* $x \in U$, $y \notin m_X$ -$Cl(U)$*, where* m_X -$Cl(U)$ *is defined by* $\cap\{F \mid U \subset F, X\backslash F \in m_X\}$.

(3) For each $x \in X$, $\cap\{m_X$ -$Cl(U) \mid U \in m_X , x \in U\} = \{x\}$.

(4) For each pair $(x, y) \in (X \times X)\backslash\Delta$*, there exist two subsets* $U_x, V_y \in m_X$ *such that* $x \in U_x$, $y \in V_y$ *and* $U_x \times V_y \subset (X \times X)\backslash\Delta$*, where* $\Delta = \{(x, x) \mid x \in X\}$.

Proof. (1) $\Longrightarrow$ (2). Let $x, y \in X$ with $x \neq y$. Then, there exist two subsets $U, V \in m_X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. It is obvious that $y \notin V^c$, m_X -$Cl(U) \subset m_X$ -$Cl(V^c) = V^c$ and therefore $y \notin m_X$ -$Cl(U)$.

(2) $\Longrightarrow$ (3). Assume that $y \notin \{x\}$. There exists a subset $U \in m_X$ such that $x \in U$ and $y \notin m_X$ -$Cl(U)$. So, we have that $y \notin \cap\{m_X$ -$Cl(U) \mid U \in m_X, x \in U\}$.

(3) $\Longrightarrow$ (4). Let $(x, y) \in (X \times X)\backslash\Delta$. Since $y \notin \cap\{m_X$ -$Cl(U) \mid U \in m_X, x \in U\}$, then there exists a subset $U \in m_X$ such that $x \in U$, $y \in (m_X$ -$Cl(U))^c$ and $(m_X$ -$Cl(U))^c \in m_X$ ([36, Lemma 3.2, Lemma 3.1(6), Remark 3.1(2)]). Set $U_x = U$ and $U_y = (m_X$ -$Cl(U))^c$. Then, it is shown that $x \in U_x$, $y \in U_y$ and U_x, $U_y \in m_X$. Besides, we have that $(U_x \times U_y) \cap \Delta = \varnothing$, because $U_x \cap U_y = \varnothing$. Therefore, we have $U_x \times U_y \subset (X \times X)\backslash\Delta$.

(4) $\Longrightarrow$ (1). Let $x \neq y$. Then $(x, y) \in (X \times X)\backslash\Delta$, and by (4) there exist two subsets U_x, $U_y \in m_X$ such that $(x, y) \in U_x \times U_y \subset (X \times X)\backslash\Delta$. Hence, we have that $(U_x \times U_y) \cap \Delta = \varnothing$, i.e., $U_x \cap U_y = \varnothing$. □

6. **Applications**

Here, we are able to apply Theorems 5.1 and 5.2 to investigate properties on the digital line using alternative construction [26; p. 908] of digital lines. In Example 6.2 (a)-(d), we use Theorem 5.1 to prove the b-T_2ness of the digital line. Morever, in Example 6.3, we use Theorem 5.2 to observe an alternative proof on the non-pre-T_2ness of the digital line. We recall the Khalimsky line or so called the digital line $(\mathbb{Z}, \kappa)$ is the set of the integers $\mathbb{Z}$ with the topology κ having $S = \{\{2m-1, 2m, 2m+1\} \mid m \in \mathbb{Z}\}$ as a subbase ([24],[25],[26]; eg.,[12],[18],[17]). In $(\mathbb{Z}, \kappa)$, for examples, each singleton $\{2m+1\}$ is open and each singleton $\{2m\}$ is closed, where $m \in \mathbb{Z}$. A subset U is open in $(\mathbb{Z}, \kappa)$ if and only if whenever $x \in U$ and x is an even integer, then $x-1$, $x+1 \in U$. A subset $\{2m-1, 2m, 2m+1\}$ is the smallest open set containing $2m$, where $m \in \mathbb{Z}$. It is shown directly that $(\mathbb{Z}, \kappa)$ is semi-T_2 ([17; Theorem 2.3]) and so it is b-T_2 . However, it is not pre-T_2 ([17; Theorem 4.8(ii)]) and so it is not T_2 , because $\kappa = PO(\mathbb{Z}, \kappa)$ holds([18; Theorem 2.1(i)(a)]).

Example 6.1. Let $(\mathbb{R}, \epsilon)$ be the Euclidan line and $q : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{Z}, \kappa)$ a function defined by $q(x) = 2n+1$ for every point x with $2n < x < 2n+2$, $q(2n) = 2n$, where $n \in \mathbb{Z}$([26; p. 908]). Let R be an equivalence relation in $(\mathbb{R}, \epsilon)$ defined by $R = (\cup\{V(2n, 2n+2) \times V(2n, 2n+2) \mid n \in \mathbb{Z}\}) \cup (\cup\{(2n, 2n) \mid n \in \mathbb{Z}\})$, where $V(2n, 2n+2) = \{t \in \mathbb{R} \mid 2n < t < 2n+2\}$. For points t and x of $\mathbb{R}$, t is equivalent to x if and only if $(t, x) \in R$. We denote the set of all equivalence classes by $\mathbb{R}/R = \{[t] \mid t \in \mathbb{R}\}$, where $[t] = \{x \in \mathbb{R} \mid (x, t) \in R\}$ is an equivalence class including t. Then, the projection $p : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{R}/R, \Psi)$ is well defined by $p(t) = [t]$, for every $t \in \mathbb{R}$; Ψ is the identification topology induced by the function p; a subset U_1 of $\mathbb{R}/R$ is open in $(\mathbb{R}/R, \Psi)$ (i.e., $U_1 \in \Psi$) if and only if $p^{-1}(U_1)$ is open in $(\mathbb{R}, \epsilon)$. It is shown that $p(t) = [q(t)]$, for every $t \in \mathbb{R}$

(a) The digital line $(\mathbb{Z}, \kappa)$ and $(\mathbb{R}/R, \Psi)$ are homeomorphic.

(b) For $m_{\mathbb{R}} = SO(\mathbb{R}, \epsilon)$ (resp. $(BO(\mathbb{R}, \epsilon))$ and R above, one of the assumptions in Theorem 5.1, i.e., (b), holds.

(c) The function $p : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{R}/R, \Psi)$ is $(SO(\mathbb{R}, \epsilon), SO(\mathbb{R}/R, \Psi))$-open, $(PO(\mathbb{R}, \epsilon), PO(\mathbb{R}/R, \Psi))$-open and $(BO(\mathbb{R}, \epsilon), BO(\mathbb{R}/R, \Psi))$-open.

(d) The digital line $(\mathbb{Z}, \kappa)$ is semi-T_2 and also b-T_2 .

Proof. (a) A continuous bijection $f : (\mathbb{Z}, \kappa) \longrightarrow (\mathbb{R}/R, \Psi)$ is well defined by $f(q(x)) = p(x)$. Then, $f \circ q = p$ and the inverse f^{-1} is continuous.

(b) In Theorem 5.1, let $(X, \tau) = (\mathbb{R}, \epsilon)$ and $R = (\cup\{V(2n, 2n+2) \times V(2n, 2n+2) \mid n \in \mathbb{Z}\}) \cup \{(2n, 2n) \mid n \in \mathbb{Z}\})$. We need the following notations: $V(2n, +\infty) = \{x \in \mathbb{R} \mid 2n < x\}$, $V[2n, +\infty) = \{x \in \mathbb{R} \mid 2n \leq x\}$, $V(-\infty, 2n) = \{x \in \mathbb{R} \mid x < 2n\}$ and $V(-\infty, 2n] = \{x \in \mathbb{R} \mid x \leq 2n\}$, where $n \in \mathbb{Z}$. It is shown that $\mathbb{R}^2 \setminus R = [\cup\{(V[2n, +\infty) \times V(-\infty, 2n)) \cup (V(2n, +\infty) \times V(-\infty, 2n]) \mid n \in \mathbb{Z}\}] \cup [\cup\{(V(-\infty, 2n] \times V(2n, +\infty)) \cup (V(-\infty, 2n) \times V[2n, +\infty)) \mid n \in \mathbb{Z}\}]$. Let$(x, y) \in \mathbb{R}^2 \setminus R$. Then, there exist subsets such that $(x, y) \in V[2n, +\infty) \times V(-\infty, 2n)$, $(x, y) \in V(2n, +\infty) \times V(-\infty, 2n]$, $(x, y) \in V(-\infty, 2n] \times V(2n, +\infty)$ or $(x, y) \in V(-\infty, 2n) \times V[2n, +\infty)$. Since $V[2n, +\infty)$, $V(-\infty, 2n)$, then $V(2n, +\infty)$ and $V(-\infty, 2n]$ are semi-open and also b-open in $(\mathbb{R}, \epsilon)$, the condition (b) of Theorem 5.1 holds for $m_{\mathbb{R}} = SO(\mathbb{R}, \epsilon)$ and also $m_{\mathbb{R}} = BO(\mathbb{R}, \epsilon)$.

(c) It is obvious that the function $q : (\mathbb{R}, \epsilon) \longrightarrow (\mathbb{Z}, \kappa)$ is open and continuous and $f \circ q = p$ holds. First, let $A \in SO(\mathbb{R}, \epsilon)$. Then, there exists an open subset U such that $U \subset A \subset Cl(U)$. Using f in the proof of (a) above, $f(q(U)) \in \Psi$ and $f(q(U)) \subset f(q(A)) \subset f(Cl(q(U))) = Cl(f(q(U)))$. Thus, we have that $p(U) \in \Psi$ and $p(U) \subset p(A) \subset Cl(p(U))$, i.e., $p(A) \in SO(\mathbb{R}/R, \Psi)$.

Second, let $B \in PO(\mathbb{R}, \epsilon)$. Then, there exists an open subset V such that $B \subset V \subset Cl(B)$. It is shown similarly as above that $p(V) \in \Psi$ and $p(B) \subset p(V) \subset Cl(p(B))$. Namely, $p(B) \in PO(\mathbb{R}/R, \Psi)$.

Finally, let $S \in BO(\mathbb{R}, \epsilon)$. It is well known that $S = sInt(S) \cup pInt(S)$ holds [3; Proposition 2.1]. Since $sInt(S) \in SO(\mathbb{R}, \epsilon)$ and $pInt(S) \in PO(\mathbb{R}, \epsilon)$, we have that $p(sInt(S)) \in SO(\mathbb{R}/R, \Psi)$ and $p(pInt(S)) \in PO(\mathbb{R}/R, \Psi)$.
Then, we have that $p(S) = p(sInt(S)) \cup p(pInt(S)) \subset Cl(Int(p(sInt(S)))) \cup Int(Cl(p(pInt(S)))) \subset Cl(Int(p(S))) \cup Int(Cl(p(S)))$. Namely, $p(S) \in BO(\mathbb{R}/R, \Psi)$.

(d) By (b) and (c), all assumptions of Theorem 5.1 are satisfied for $m_{\mathbb{R}} = SO(\mathbb{R}, \epsilon)$ (resp. $m_{\mathbb{R}} = BO(\mathbb{R}, \epsilon)$). Thus, $(\mathbb{R}/R, \Psi)$ is semi-T_2 (resp.
b-T_2). Since $f : (\mathbb{Z}, \kappa) \longrightarrow (\mathbb{R}/R, \Psi)$ is a homeomorphism, then we prove that
$(\mathbb{Z}, \kappa)$ is semi-T_2 (resp. b-T_2). □

Example 6.2. (a) The digital line is not pre-T_2 ([17; Theorem 4.8(ii)]). Using Theorem 5.2 (3), we have an alternative proof of the property above. Let $(X, \tau) = (\mathbb{Z}, \kappa)$ and $m_X = PO(\mathbb{Z}, \kappa)$ in Theorem 5.2. Then, the condition (3) of Theorem 5.2 is not satisfied. Indeed, let a point $x = 2n \in \mathbb{Z}$ for some integer n, and U_x be any preopen set $(\mathbb{Z}, \kappa)$ such that $x \in U_x$. By using [17; Lemma 3.3], it is shown that $\{2n-1, x, 2n+1\} \subset U_x$. Thus, we have that $\cap\{pCl(U) \mid U \in PO(\mathbb{Z}, \kappa),\ x \in U\} \supset pCl(\{2n-1, x, 2n+1\}) \supset \{2n-1, x, 2n+1\} \neq \{x\}$. Therefore, by Theorem 5.2, $(\mathbb{Z}, PO(\mathbb{Z}, \kappa))$ is not $PO(\mathbb{Z}, \kappa)$-T_2 . Namely, the digital line is not pre-T_2 .

(b) Using Theorem 5.2 (3), for $m_X = SO(\mathbb{Z}, \kappa)$, we have an alternative proof of (d) in Example 6.2 above. Let $x = 2n$ and $y = 2m + 1$ for some integers n and m. Then, $\{x, 2n+1\}, \{2n-1, x\}$ and $\{y\}$ are semi-open sets of $(\mathbb{Z}, \kappa)$ (e.g., [18]). Since $sCl(\{x, 2n+1\}) = \{x, 2n+1\}$, then $sCl(\{2n-1, x\}) = \{2n-1, x\}$, $\cap\{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa),\ x \in U\} \subset sCl(\{x, 2n+1\}) \cap sCl(\{2n-1, x\}) = \{x, 2n+1\} \cap \{2n-1, x\} = \{x\}$. Morever, $\cap\{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa), y \in U\} \subset sCl(\{y\}) = \{y\}$ holds. Therefore, we conclude that $\cap\{sCl(U) \mid U \in SO(\mathbb{Z}, \kappa),\ z \in U\} = \{z\}$ holds for any point $z \in \mathbb{Z}$. By Theorem 5.2, it is obtained that $(\mathbb{Z}, SO(\mathbb{Z}, \kappa))$ is $SO(\mathbb{Z}, \kappa)$-T_2 . Namely, the digital line is semi-T_2 and also it is b-T_2 .

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Aynur Keskin,

Selcuk University, Faculty of Sciences, Department of Mathematics, Campus, 42075, Konya/TURKEY.

Email: akeskin@selcuk.edu.tr

Takashi Noiri,

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken 869-5142, JAPAN.

Email: t.noiri@nifty.com