Regulating Control of the Angular Velocity of a Rigid Body with Two Torque Actuators

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***Abstract*—This paper presents two different kinds of regulating control strategies for the angular velocity of a rigid body model with two torque actuators. The first strategy presents piece-wise constant, states dependent feedback control laws. The method is based on the construction of a cost function V (not a Lyapunov function), which is sum of two semi-positive definite functions V_1 and V_2. The semi-positive definite function V_1 is dependent on the first m state variables which can be steered along the given vector fields and V_2 is dependent on the remaining $n-m$ state variables which can be steered along the missing Lie brackets. The values of the functions V_1 and V_2 allow in determining a desired direction of system motion and permit to construct a sequence of controls such that the sum of these functions decreases in an average sense. The second strategy presents a time-varying feedback law based on the model reference approach, where the trajectory of the extended system is chosen as the model reference trajectory. The controllers are designed in such a way that after each time period T, the trajectory of the rigid body model intersects the trajectory of the model reference which can be made asymptotically stable. The proposed feedback law is as a composition of a standard stabilizing feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop control problem stated for an abstract equation on a Lie group.**

***Index Terms*—Feedback regulating control, systems with drift, Lie bracket extension, Lie groups, logarithmic coordinates.**

I. Introduction

There has been much interest over the past decade in the problem of stabilization of rigid body with only two torque inputs and has been studied by many researchers [1]-[4]. Many strategies have been proposed for the stabilization and asymptotic stabilization problems. The asymptotic stabilization problem for the case of a single control aligned with an axis having components along all three principal axes was studied in [5], [6]. If there are two torque inputs along two principle axes and the uncontrolled principal axis is not an axis of symmetry then the system can be asymptotically stabilized by using a variety of design schemes. By finding a Lyapunov function, Brockett [7], proved that the null solution of the angular velocity equations is asymptotically stabilizable by two control torques aligned with two principal axes if the uncontrolled axis is not an axis of symmetry. Later, Aeyels [8] applied center manifold theory to reduce the problem to one of lower dimension and thereby obtained another locally stabilizing feedback control law. Byrnes *et al.* [9] used the general methodology of non-linear zero dynamics to derive globally stabilizing feedback control law for the system. Averaging theory was also used to control the under actuated mechanical systems [3], [10], [11]. Astolfi presented the output feedback stabilization of the angular velocity of a rigid body [1].

In this article we present two different kinds of regulating control strategies for the angular velocity of a rigid body model with two torque actuators. The first approach is based on the construction of a cost function which is a sum of two semi-positive definite functions $V_1(x)$ and $V_2(x)$, where $V_1(x)$ is corresponds to of the first m state variables which can be steered along the given vector fields and $V_2(x)$ is dependent on the remaining $n-m$ state variables which can be steered along the missing Lie brackets. The values of these functions allow in determining a desired direction of system motion and permit to construct a sequence of controls such that the sum of these functions decreases in an average sense. The individual functions are hence not restricted to decrease monotonically but their oscillations are limited and coordinated in a way to guarantee convergence. The task of the control is to decay the non-differentiable cost function along the controlled system trajectories only asymptotically.

In the second method a model reference approach is used. The trajectory of the extended system for rigid body model is chosen as the model reference trajectory. Since the extended system has equal number of inputs and state variables, i.e. $m = n$, therefore can be made asymptotically stable by choosing an arbitrary Lyapunov function. This state feedback is then combined with a periodic continuation of a parameterized solution to an open loop steering problem for the comparison of flows of the original and extended systems. Since the controllability Lie algebra associated with this system is locally nilpotent, the latter can be recast as an open loop control problem for a finite set of the logarithmic coordinates of flows [12]. In combination with the time invariant state feedback for the extended system, the solution to this open loop problem delivers a time varying control which provides for periodic intersection of the trajectories of the controlled extended system and the original system. For steering the original system, the extended system trajectory serves as a reference. The time-invariant feedback for the extended system dictates the speed of convergence of the system trajectory to the desired terminal point, the open loop solution serves the averaging purpose in that it ensures that the average motion of the original system is that of the controlled extended system. The construction proposed here demonstrates that synthesis of time varying feedback stabilizers for rigid body with two control torques can be viewed as a procedure of combining state feedback laws

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for a Lie bracket extension of the system with a solution of an open loop trajectory interception control problem.

II. A Dynamical Model of a Rigid Body

Consider a rigid body, which is controlled by means of two torque inputs, applied about two principal axes. Let ω_1, ω_2 and ω_3 be the angular velocity components with respect to the principle axes, and J_1, J_2, J_3 denote the respective principal moments of inertia. For simplicity, assume that the two torque inputs are about the first two principal axes. Then the Euler's equations of motion of the rigid body are given by [2]:

$$\begin{aligned} \dot\omega_1 &= J_{23}\,\omega_2\omega_3 + u_1 \\ \dot\omega_2 &= J_{31}\,\omega_3\omega_1 + u_2 \\ \dot\omega_3 &= J_{12}\,\omega_1\omega_2 \end{aligned} \tag{1}$$

where, $J_{23} = (J_2 - J_3)/J_1$, $J_{31} = (J_3 - J_1)/J_2$ and $J_{12} = (J_1 - J_2)/J_3$. For the null solution of the system (1), if the uncontrolled principal axis is an axis of symmetry, i.e., $J_{12} = 0$, or $J_1 = J_2$, then the system cannot be asymptotically stabilized. If the uncontrolled principal axis is not an axis of symmetry, i.e., $J_{12} \neq 0$, or $J_1 \neq J_2$, then the system can be globally asymptotically stabilized [2]. Moreover, the inertia moments of rigid body satisfy the inequalities $0 < J_1 < J_2 + J_3$ which in particular, imply $-1 < J_{ik} < 1$. For simplicity, let $x = [x_1\ x_2\ x_3]^T \underline{\underline{\Delta}} [\omega_1\ \omega_2\ \omega_3]^T$ then (1) can be written in the following standard form:

$$\dot x = g_0(x) + g_1(x)u_1 + g_2(x)u_2 \tag{2}$$

where

$$g_0(x) = \begin{bmatrix} J_{23}\,x_2x_3 \\ J_{31}\,x_3x_1 \\ J_{12}\,x_1x_2 \end{bmatrix},\quad g_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \text{and}\ g_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The kinematics model (2) has the following important properties:

- (P1) The vector fields $g_0(x)$, $g_1(x)$ and $g_2(x)$ are real analytic and complete and, additionally, $g_0(0) = 0$.
- (P2) System (2) satisfies the LARC (Lie algebra rank condition) for accessibility [13], namely that $L(g_0, g_1, g_2)$, the Lie algebra of vector fields generated by $g_0(x), g_1(x)$ & $g_2(x)$, spans R^3 at each point $x \in R^3$ that is

$$\operatorname{span}\{g_1, g_2, g_5\}(x) = R^3, \qquad \forall\ x \in R^3 \tag{3}$$

- (P3) All brackets involved in (3) are good brackets which is necessary condition for accessibility condition (for detail see [14]).
- (P4) The controllability Lie algebra $L(g_0, g_1, g_2)$ is "locally" nilpotent (see [15]).

To verify property P2, it is sufficient to calculate the following Lie brackets of $g_0(x), g_1(x)$ & $g_2(x)$:

$$g_3(x) \underline{\underline{\Delta}} [g_0(x), g_1(x)] = \begin{bmatrix} 0 \\ -J_{31}\,x_3 \\ -J_{12}\,x_2 \end{bmatrix}$$

$$g_4(x) \underline{\underline{\Delta}} [g_0(x), g_2(x)] = \begin{bmatrix} -J_{23}\,x_3 \\ 0 \\ -J_{12}\,x_1 \end{bmatrix}$$

$$g_5(x) \underline{\underline{\Delta}} [g_2(x), [g_0(x), g_1(x)]]$$

$$= [g_1(x), [g_0(x), g_2(x)]] = \begin{bmatrix} 0 \\ 0 \\ -J_{12} \end{bmatrix}$$

which satisfy the LARC condition:

$$span\{g_1, g_2, [g_1, [g_0, g_2]]\}(x) = R^3 \quad \text{and}$$
$$span\{g_1, g_2, [g_2, [g_0, g_1]]\}(x) = R^3$$

Since all the brackets of order greater than two vanish at the origin, therefore system is locally nilpotent of order 2.

III. A Control Problem

(SP): Given a desired set point $x_{des} \in \Re^3$, construct a feedback strategy in terms of the controls $u_i : \Re^3 \to \Re, i = 1, 2$ such that the desired set point x_{des} is an attractive set for (2), so that there exists an $\varepsilon > 0$, such that $x(t; t_0, x_0) \to x_{des}$, as $t \to \infty$ for any initial condition $(t_0, x_0) \in \Re^+ \times B(x_{des}; \varepsilon)$.

Without the loss of generality, it is assumed that $x_{des} = 0$, which can be achieved by a suitable translation of the coordinate system.

IV. First Approach (Discontinuous Regulating Control Law)

A. Basic Approach to Feedback Control Synthesis

It is clear that for system (2) there does not exist any Lyapunov function V for which the set $S \overset{def}{=} \{x \in \Re^3 : L_{g_0}V(x) = L_{g_i}V(x) = 0, i = 1, 2\} = \{0\}$. which disables the construction of control laws $u_i(x), i = 1, 2$, which render $\mathrm{d}V(x)/\mathrm{d}t < 0$ along the trajectories of the controlled system. A different approach is therefore suggested which relies on the construction of two functions $V_i(x), i = 1, 2$, whose behavior along the trajectories of the controlled system is not limited to $\mathrm{d}V_i(x)/\mathrm{d}t < 0, i = 1, 2$. While allowing one of the functions $V_i(x)$ to increase it is possible to construct feedback controls $u_i(x)$, $i = 1, 2$, in such a way that the sum $V(x) \overset{def}{=} V_1(x) + V_2(x)$ decreases on average.

B. Construction of the Cost Function and Feedback Strategy

For the construction of the functions $V_1(x)$ and $V_2(x)$ consider the following two groups of vector fields and missing Lie brackets:

$$G_1(x) \overset{def}{=} \{g_1(x), g_2(x)\} \ \text{and}\ G_2(x) \overset{def}{=} \{g_5(x)\}.$$

We introduce the following semi-positive definite functions:

$$V_1(x)=\frac{1}{2}x^T G_1(0)G_1^T(0)x=\frac{1}{2}x^T\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}x$$

$$=\frac{1}{2}x^T\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}x=\frac{1}{2}(x_1^2+x_2^2)$$

$$V_2(x)=\frac{1}{2}z^T G_2(0)G_2^T(0)x=\frac{1}{2}x^T\begin{bmatrix}0\\0\\-J_{12}\end{bmatrix}\begin{bmatrix}0&0&-J_{12}\end{bmatrix}x.$$

$$=\frac{1}{2}x^T\begin{bmatrix}0&0&0\\0&0&0\\0&0&(J_{12})^2\end{bmatrix}x=(J_{12})^2(x_3^2)$$

$$V(x)\overset{def}{=}V_1(x)+V_2(x)=\frac{1}{2}(x_1^2+x_2^2+(J_{12})^2x_3^2)\cdot$$

The solution to the steering problem of system (2) can be obtained by partitioning the state space $\Re^3$ in to two subspaces i.e. $\Re^3=S\oplus\{\Re^3\setminus S\}$, where

$$S\overset{def}{=}\{x\in\Re^3: L_{g_0}V(x)=L_{g_i}V(x)=0,\, i=1,2,$$
$$Lg_5V(x)\neq 0\}\ =\{x\in\Re^3: x_1=x_2=0,\ x_3\neq 0\}$$

$$\Re^3\setminus S\overset{def}{=}\{x\in\Re^3:,\ L_{g_5}V(x)=0\}\ =\{x\in\Re^3: x_3=0\}\cdot$$

Now consider the case when $x\in\Re^3\setminus S$, then the following control

$$u_1=-Lg_1V(x)-\frac{(J_{23}+J_{31}+J_{12})}{2J_{23}}g_1^T(x)g_0(x)$$
$$=-x_1-\frac{(J_{23}+J_{31}+J_{12})}{2}x_2x_3$$

$$u_2=-Lg_2V(x)-\frac{(J_{23}+J_{31}+J_{12})}{2J_{31}}g_2^T(x)g_0(x)$$
$$=-x_2-\frac{(J_{23}+J_{31}+J_{12})}{2}x_1x_3$$

steer the system from any initial state $x_0=\begin{bmatrix}x_1 & x_2 & 0\end{bmatrix}^T$ to the desired state $x_{des}=\begin{bmatrix}0&0&0\end{bmatrix}^T$ because:

$$\frac{d}{dt}V(x)=x^T\dot{x}=x^T\{g_0(x)+g_1(x)u_1+g_2(x)u_2\}$$
$$=(J_{23}+J_{31}+J_{12})x_1x_2x_3+x_1u_1+x_2u_2$$
$$=(J_{23}+J_{31}+J_{12})x_1x_2x_3+x_1\left(-\frac{(J_{23}+J_{31}+J_{12})}{2}x_2x_3-x_1\right)+$$
$$x_2\left(-\frac{(J_{23}+J_{31}+J_{12})}{2}x_1x_3-x_2\right)$$
$$=-x_1^2-x_2^2<0\ \text{ if } x=[x_1\ x_2\ 0]^T.$$

If $x\in S$ then the above strategy is failed due to the fact that $dV(x)/dt=0$ where as $x\neq 0$. Further decrease in $V(x)$ is not possible by the above control law.

The drift vector $g_0(p)=0$ and the functions $V_1(p)=0$ & $V_2(p)\neq 0$ if $p\in S$. Now to make $V_2(p)=0$ we have to steer the system away from S which is equivalent to generate the system motion along the Lie bracket direction

$$g_5(x)\overset{def}{=}[g_2(x),[g_0(x),g_1(x)]]=[g_1(x),[g_0(x),g_2(x)]]\cdot$$

By using the controls $u_1=-1,\ \ u_2=1$, the system motion can be generated along this Lie bracket.

The controls: $u_1=-1,\ \ u_2=1$ will change x_1 & x_2 (and hence $V_1(p)$) from zero to a nonzero value, therefore the system will be steered away from S and these controls will also activate the drift vector by making $g_0(p)\neq 0$.

It is then clear that the controls $u_1=-1, u_2=1$ results in the increase of $V_1(p)$ along the control trajectory. It is logical to assume that controls $u_1=-1, u_2=1$ are employed until the value of $V_1(x)$ becomes comparable with the value of $V(p)$ at a point p at which S was last traversed, i.e. until $V_1(x)\geq\alpha V(p),\ \ \alpha>1$. At this point, the drift can be taken into advantage by resetting the controls to: $u_1=u_2=0$ until $L_{g_5}V(x)=0$ i.e. $x_3=0$, which causes a decrease in $V_2(x)$. After $V_2(x)$ reaches to zero, $V_1(x)$ is restored to its previous value (zero) by reversing the controls $u_1=1, u_2=-1$. Repeating the above results in a decrease in V. The above translates into the following algorithmic feedback strategy:

C. Steering Feedback Strategy

Data: $\alpha\geq 1$.

(1) Steer the system towards the surface S by employing the control:

$$u_1=-Lg_1V(x)-\frac{(J_{23}+J_{31}+J_{12})}{2J_{23}}g_1^T(x)g_0(x)$$
$$=-x_1-\frac{(J_{23}+J_{31}+J_{12})}{2}x_2x_3$$

$$u_2=-Lg_2V(x)-\frac{(J_{23}+J_{31}+J_{12})}{2J_{31}}g_2^T(x)g_0(x)$$
$$=-x_2-\frac{(J_{23}+J_{31}+J_{12})}{2}x_1x_3$$

until $L_{g_i}V(x)=0,\quad i=1,2$ i.e. $x_1=x_2=0$.

This step will generate motion along the vector fields $g_i, i=1,2$, and gives $V_1=0$ and $V_2\neq 0$.

(2) Define $p\overset{def}{=}x(t)$ in which t is the exit of step 1 (when the surface S is traversed). If $p=0$, then stop, else if $p\neq 0$, then

(2a) Employ the controls

$u_1=-1,\ u_2=1$ until $V_1(x)\geq\alpha V(p)$

This step will increase V_1 and gives $g_0(p)\neq 0$.

(2b) Until $Lg_5V(x)=0$ i.e. $x_3=0$ employ the controls

$u_i=0,\ i=1,2$

In this step, motion is due to the drift and $V_2\to 0$.

(2c) Employ the controls $u_1=1,\ u_2=-1$ until $V_1(x)=0$, i.e. $x_1=x_2=0$.

The steps (2a)-(2c) will generate the system motion along the Lie bracket $g_5(x)=[g_1,[g_0,g_2]](x)=[g_2,[g_0,g_1]]\}(x)$.

The strategy employs uniformly bounded, piecewise constant controls, and can even lead to dead-beat control. The strategy is based on simple principles; the values of the semi-definite functions provide an on line convergence verification test. The strategy is applied to a model of rigid body with two torque actuators and the simulation results are shown in Figs 1-3.

In the first simulation we choose $x(0)=\begin{bmatrix}1 & 0.8 & 0.5\end{bmatrix}^T$, the $J_{23}=0.4,\ J_{31}=0.8,\ J_{12}=0.5$ results are shown in Figs. 1(a)-1(c).

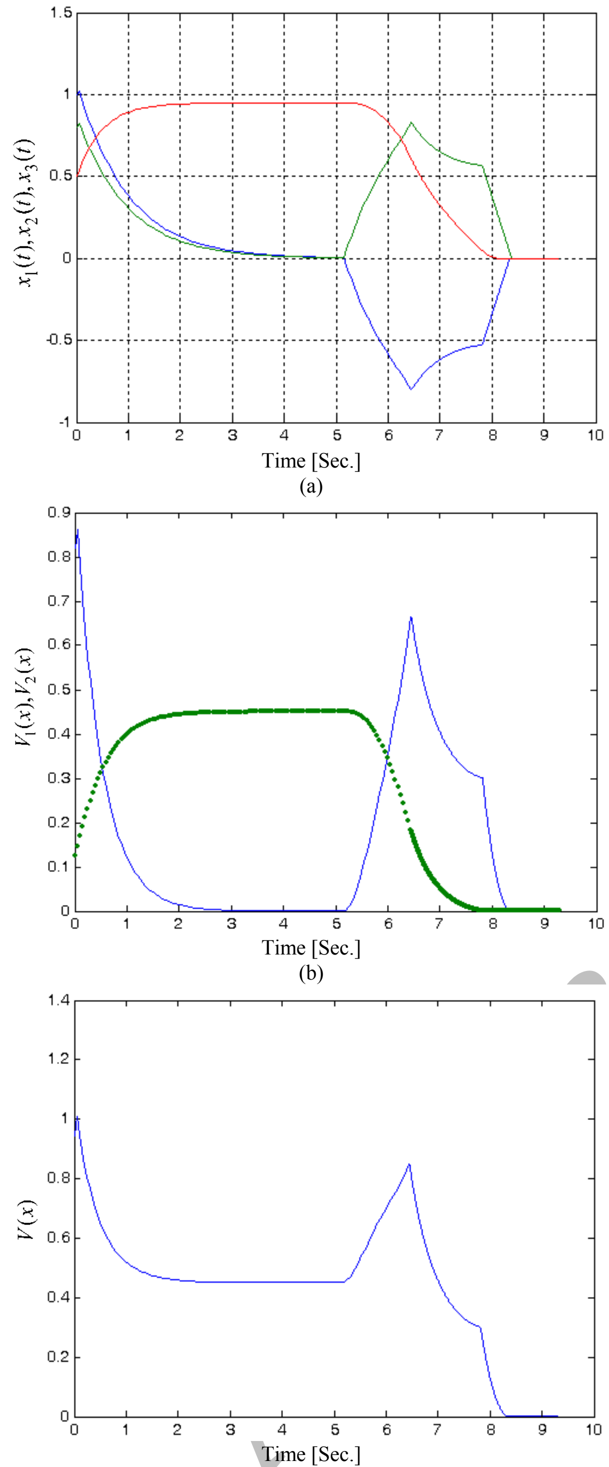

Fig. 1. Plots of the controlled state trajectories versus time, (a) $x_1(t), x_2(t), x_3(t)$, (b) $V_1(x), V_2(x)$, and (c) $V(x)$.

In the second simulation, $x(0) = [1 \quad 0.8 \quad 0.5]^T$, $J_{23} = 0.5$, $J_{31} = 0.6$, $J_{12} = 0.8$ the results are shown in Figs. 2(a)-2(c).

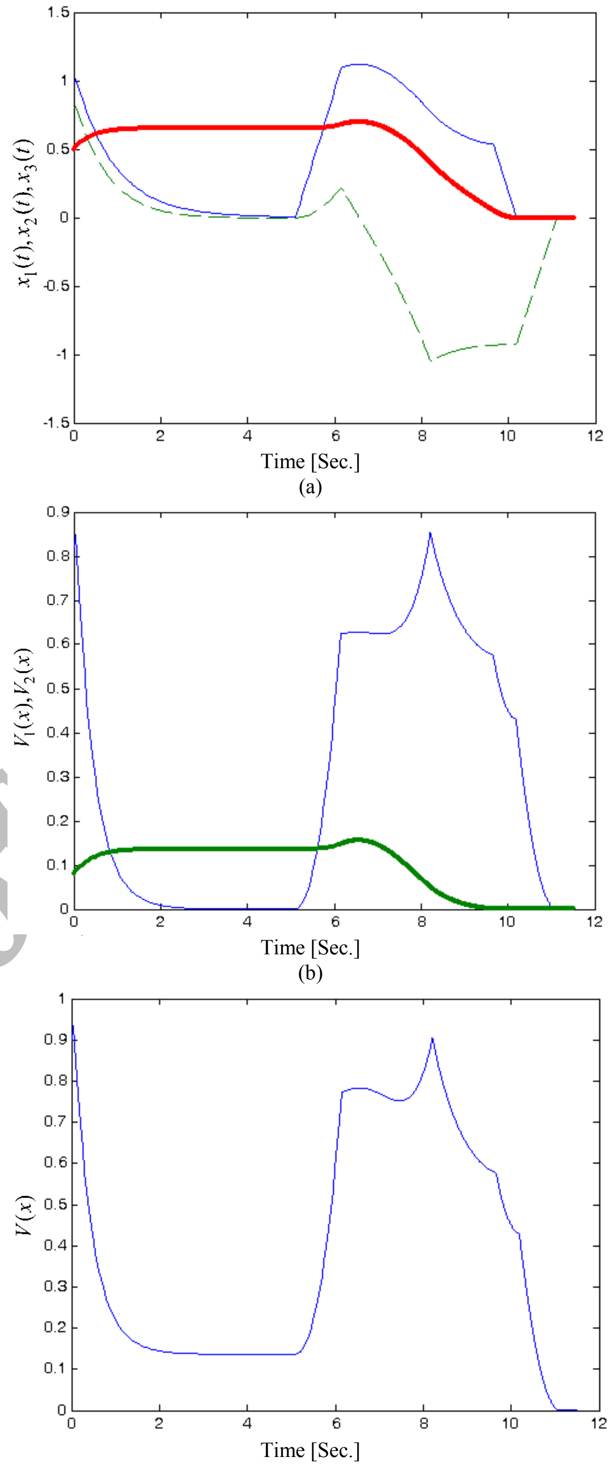

Fig. 2. Plots of the controlled state trajectories versus time, (a) $x_1(t), x_2(t), x_3(t)$, (b) $V_1(x), V_2(x)$, and (c) $V(x)$.

V. Second Approach (Time-Varying Control Law)

A. *Extended System*

The construction of the stabilizing feedback, presented in the this section, employs as its base a Lie bracket extension for the original system (2) as defined in Lafferriere *et al*. [14], and Liu [16]. This extension is a new system whose right hand side is a linear combination of the vector fields which locally span the state space. The “coefficients” of this linear combination are regarded as “extended” controls. The extended system can be written as:

$$\dot{x} = g_0(x) + g_1(x)v_1 + g_2(x)v_2 + g_5(x)v_5 \qquad (4)$$

Henceforth, equations (2) and (4) are referred to as the “original system”, and the “extended system”, respectively. The importance of the extended system for the purpose of control synthesis lies in the fact that, unlike the original system, it permits instantaneous motion in the “missing" Lie bracket direction $g_5 = [g_1,[g_0, g_2]] = [g_2,[g_0, g_1]]$.

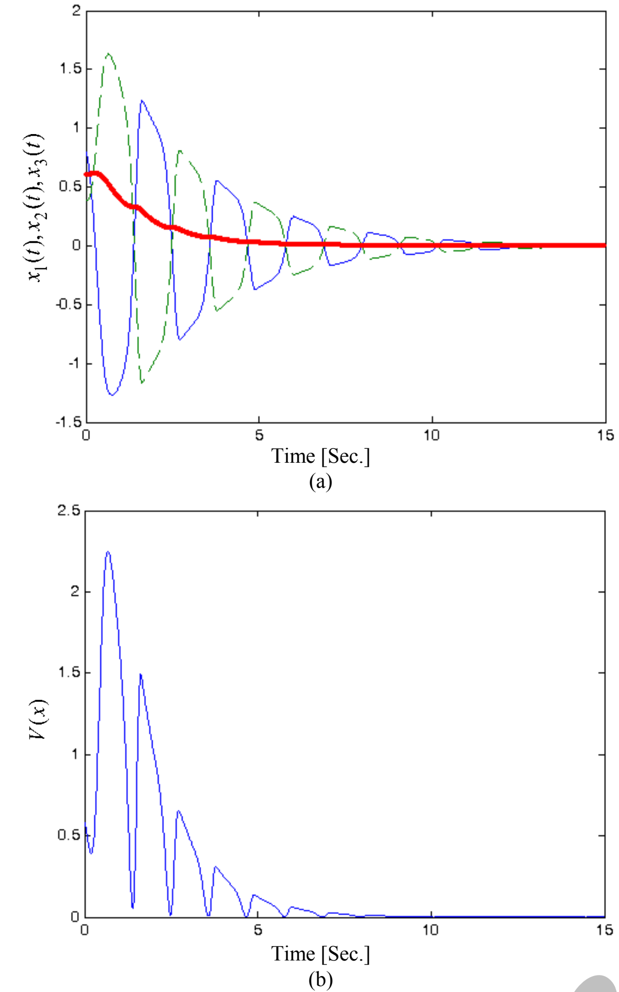

Fig. 3. Plots of (a) the controlled state trajectories versus time, $x_1(t), x_2(t), x_3(t)$, (b) Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$.

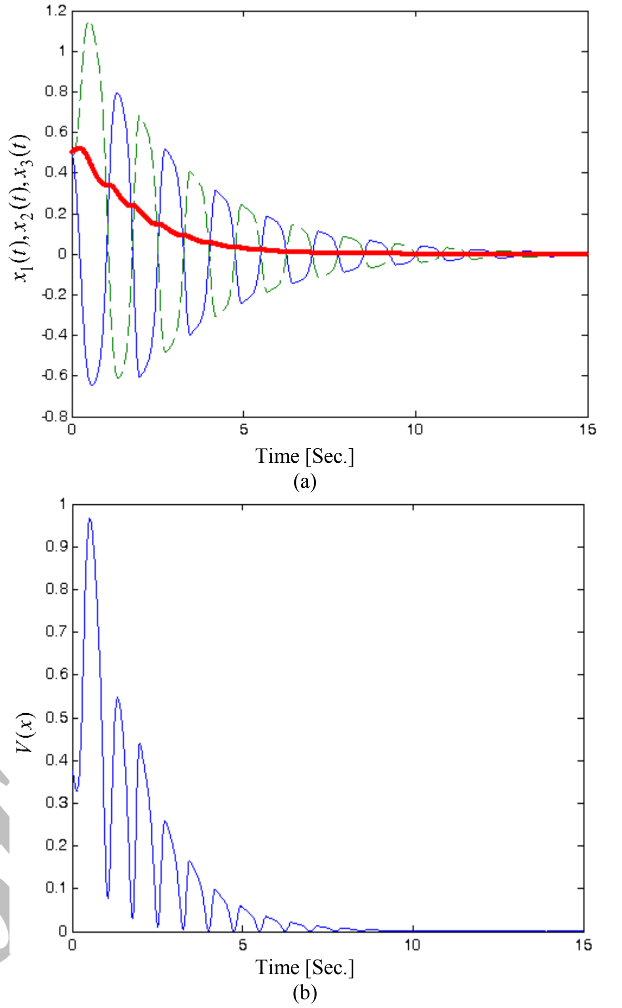

Fig. 4. Plots of (a) the controlled state trajectories versus time, $x_1(t), x_2(t), x_3(t)$, (b) Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{3} x_i^2(t)$.

B. Stabilization of the Extended System

The extended system (4) can be made globally asymptotically stable if we define the following control inputs:

$$v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \\ v_5(x) \end{bmatrix} = \{G(x)\}^{-1}(-x - g_0(x))$$
$$= \begin{bmatrix} -x_1 - J_{23}x_2x_3 \\ -x_2 - J_{13}x_1x_3 \\ -\frac{1}{J_{12}}(-x_3 - J_{12}x_1x_2) \end{bmatrix} \quad (5)$$

where $G(x) = [g_1(x), g_2(x), g_5(x)]$. The existence of $\{G(x)\}^{-1}$ is guaranteed by the LARC condition.

Lemma:

The extended system (4) can be made asymptotically stable by using the feedback control as given in (5).

Proof: By considering a Lyapunov function $V(x) = 0.5x^T Q\, x$, where Q is some symmetric and positive definite matrix, it follows that, along the controlled extended system trajectories,

$$\frac{d}{dt}V(x) = x^T Q\,(g_0(x) + G(x)\{G(x)\}^{-1}(-x - g_0(x))$$
$$= -x^T Q\, x = -2\,V(x) < 0, \quad \forall\, x \in R^3 \setminus \{0\}$$

Confirming the asymptotic stability of (4) with feedback controls (5).

The discretization of the above control in time, with sufficiently high sampling frequency $1/T$, does not prejudice stabilization in that if the feedback control (5) is substituted by the discretized control:

$$v_i^T(x(t) \overset{def}{=} v_i(x(nT)), \quad t \in [nT, (n+1)T),$$
$$n = 0, 1, 2, \ldots \quad i = 1, 2, 5$$

This leads to a parameterized extended system:

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_5(x)a_5 \quad (6)$$

where $a_i = v_i^T(x(t)),\ i = 1, 2, 5$, (which are constant over each interval $[nT, (n+1)T)$. For a sufficiently small T, the discretization of the extended controls preserves their stabilizing properties.

Lemma:

Suppose that the extended system (4) is asymptotical stable by using the feedback controls (5). Then, for any compact region $R \subset M$ which contains the origin, there exists a constant $T > 0$ such that the discretized controlled extended system (6) is asymptotical stable with region of attraction R [15].

C. The Trajectory Interception Problem

{TIP:} Find control functions $m_i(a,t),\ i = 1, 2$, in the class of functions which are continuous in $a \overset{def}{=} [a_1, a_2, a_5]$, and piece-wise continuous and locally bounded in t, such that for any initial condition $x(0) = x_0$ the trajectory $x^a(t; x_0, 0)$ of the extended, parameterized system (6) intersects the trajectory $x^m(t; x_0, 0)$ of the system (2) with controls m_i, $i = 1, 2$ i.e. the trajectory of the system

$$\dot{x} = g_0(x) + g_1(x)m_1(a,t) + g_2(x)m_2(a,t) \quad (7)$$

intercept with the trajectory of

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_5(x)a_5$$

precisely at time T, so that

$$x^a(T; x_0, 0) = x^m(T; x_0, 0) \quad (8)$$

Theorem:

Suppose that a solution to the TIP problem can be found. Then, there exists an admissible time horizon $T_{\max}$ and a neighborhood of the origin R such that for any $T < T_{\max}$, the time-varying feedback controls [15]:

$$u_i(t) \overset{def}{=} m_i(v^T(x), t), \quad i = 1, 2, \quad v^T \overset{def}{=} [v_1^T, v_2^T, v_5^T]$$

are asymptotically stabilizing the system (2) with the region of attraction R.

D. The TIP in Logarithmic Coordinates of Flows

To solve the TIP; we employ the formalism of [14] by considering a formal equation for the evolution of flows for the system (6):

$$\dot{U}(t) = U(t)\left(\sum_{i=0}^{5} g_i\, w_i\right), \ w_0 = 1, w_3 = w_4 = 0, U(0) = I \quad (9)$$

and its solution can be expressed locally as:

$$U(t) = \prod_{i=0}^{5} e^{\gamma_i(t) g_i} \quad (10)$$

Where the functions $\gamma_i, i = 0, 1, ..., 5$ are the logarithmic coordinates for this flow and can be computed approximately as follows:

Firstly, (10) is substituted into (9) which yields:

$$\begin{aligned} &g_0 w_0 + g_1 w_1 + ... + g_5 w_5 = \dot{\gamma}_0 g_0 + \dot{\gamma}_1 (e^{\gamma_0 Adg_0}) g_1 \\ &+ \dot{\gamma}_2 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 + ... + \\ &\dot{\gamma}_5 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5 \end{aligned} \quad (11)$$

where $(e^{AdX})Y \overset{def}{=} e^X Y e^{-X}$ and $(AdX)Y \overset{def}{=} [X, Y]$.

Employing the Campbell-Baker-Hausdorff formula:

$$(e^{AdX})Y = e^X Y e^{-X} = Y + [X, Y] + [X, [X, Y]]/2! + ...$$

and ignoring all other Lie brackets which are not involved in LARC equation (3). This gives

$$\begin{aligned} &(e^{\gamma_0 Adg_0}) g_1 = e^{\gamma_0 g_0} g_1 e^{-\gamma_0 g_0} \\ &= g_1 + (\gamma_0/1!)[g_0, g_1] + (\gamma_0^2/2!)[g_0, [g_0, g_1]] + ... \\ &\approx g_1 + \gamma_0 g_3 \end{aligned} \quad (12)$$

Similarly

$$\begin{aligned} &(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 = e^{\gamma_0 Adg_0} (e^{\gamma_1 Adg_1} g_2) \\ &= e^{\gamma_0 Adg_0}(g_2) \approx g_2 + \gamma_0 [g_0, g_2] \\ &= g_2 + \gamma_0 g_4 \end{aligned} \quad (13)$$

$$\begin{aligned} (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2}) g_3 &= e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (e^{\gamma_2 Adg_2} g_3) \\ &\approx e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3 + \gamma_2 [g_2, g_3]) \\ &= e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3 + \gamma_2 g_5) \\ &\approx e^{\gamma_0 Adg_0} (g_3 + \gamma_2 g_5) \approx (g_3 + \gamma_2 g_5) \end{aligned} \quad (14)$$

In a similar way we can obtain

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3}) g_4 \approx g_4 + \gamma_1 g_5 \quad (15)$$

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5 \approx g_5 \quad (16)$$

Substituting (12)-(15) into (11) and comparing the coefficients of $g_i, i = 0, 1, ..., 5$ yields the following approximate equations for the evolution of the logarithmic coordinates $\gamma_i, i = 0, 1, ..., 5$:

$$\begin{aligned} &\dot{\gamma}_0 = 1 \\ &\dot{\gamma}_1 = w_1 \\ &\dot{\gamma}_2 = w_2 \\ &\dot{\gamma}_3 = -\gamma_0 w_1 \\ &\dot{\gamma}_4 = -\gamma_0 w_2 \\ &\dot{\gamma}_5 = -\gamma_0 \gamma_2 w_1 + \gamma_0 \gamma_1 w_2 + w_5 \\ &\text{with} \quad \gamma_i(0) = 0, \quad i = 0, 1, 2, ..., 5 \end{aligned} \quad (17)$$

The TIP problem can thus be recast in the logarithmic coordinates as follows:

[TIP in LC:] On a given time horizon $T > 0$, find control functions $m_i(a,t)$, $i = 1, 2$, in the class of functions which are continuous in $a \overset{def}{=} [a_1, a_2, a_5]$, and piece-wise continuous, and locally bounded in t, such that the trajectory $t \mapsto \gamma^a(t)$ of

$$\dot{\gamma} = M(\gamma)\, a, \qquad \gamma(0) = 0 \quad (18)$$

intersects the trajectory $t \mapsto \gamma^m(t)$ of

$$\dot{\gamma} = M(\gamma)\, m(a,t), \qquad \gamma(0) = 0 \quad (19)$$

in which $m(a,t) \overset{def}{=} [m_1(a,t), m_2(a,t), 0]$ at time T, so that

$$\gamma^a(T) = \gamma^m(T) \quad (20)$$

The TIP in logarithmic coordinates now takes the form of a trajectory interception problem for the following two control systems

$$\begin{aligned} CS1: \quad &\dot{\gamma}_0 = 1 \\ &\dot{\gamma}_1 = m_1 \\ &\dot{\gamma}_2 = m_2 \\ &\dot{\gamma}_3 = -\gamma_0 m_1 \\ &\dot{\gamma}_4 = -\gamma_0 m_2 \\ &\dot{\gamma}_5 = -\gamma_0 \gamma_2 m_1 + \gamma_0 \gamma_1 m_2 \\ CS2: \quad &\dot{\gamma}_0 = 1 \\ &\dot{\gamma}_1 = a_1 \\ &\dot{\gamma}_2 = a_2 \\ &\dot{\gamma}_3 = -\gamma_0 a_1 \\ &\dot{\gamma}_4 = -\gamma_0 a_2 \\ &\dot{\gamma}_5 = -\gamma_0 \gamma_2 a_1 + \gamma_0 \gamma_1 a_2 + a_5 \end{aligned} \quad (21)$$

with initial conditions *with* $\gamma_i(0) = 0, \quad i = 0, 1, 2, ..., 5$.

A solution to TIP is calculated by approximating the flow of $\dot{x} = g_0 + [g_1, [g_0, g_2]]$ by the flow of

$\dot{y} = c\, g_1 \sin(2\pi t/T) + c\, g_2 \sin(2\pi t/T)$, therefore we adopt the following parameterization of $m_i, \quad i = 1, 2$:

$$m_1 = c_1 + c_5 \sin\frac{2\pi\ t}{T},\ m_2 = c_2 + c_5 \sin\frac{2\pi\ t}{T} \qquad (22)$$

where $c_i, \quad i = 1, 2, 5$ are unknown functions of the control parameters $a = [a_1, a_2, a_5]$ and T, were found:

$$c_1 = a_1,\ c_2 = a_2,\ c_5 = ((4a_1 + 4a_2)\pi\ T^2 + \sqrt{d})/(2T^2)$$

where

$$d = (32\, a_5\, \pi^2\, T^2 + (4\, a_1 + 4\, a_2)^2\, \pi^2\, T^4)$$

The time varying stabilizing controls for the model (2), are thus given by

$$\begin{aligned} u_1 &= c_1(v^T(x)) + c_5(v^T(x)) \sin\frac{2\pi t}{T} \\ u_2 &= c_2(v^T(x)) + c_5(v^T(x)) \sin\frac{2\pi t}{T} \end{aligned} \quad . \qquad (23)$$

The simulation results employing the above controls are depicted in Figs. 3 and 4. In the first simulation we choose $x(0) = [0.5 \quad 0.5 \quad 0.5]^T$, $J_{23} = 0.4,\ J_{31} = 0.8,\ J_{12} = 0.5$, $T = 1.5$ the results are shown in Figs. 3(a) and 3(b).

In the second simulation we choose $x(0) = [0.8 \quad 0.4 \quad 0.6]^T$, $J_{23} = 0.8,\ J_{31} = 0.6,\ J_{12} = 0.2$, $T = 2.2$ the results are shown in Figs. 4(a) and 4(b).

VI. Conclusion

Two different strategies are presented for the regulating control of the angular velocity of a rigid body model with two torque actuators. The first strategy gives piece-wise constant, states dependent feedback control laws based on the construction of a cost function V which is the sum of two semi-positive definite functions V_1 and V_2. Where, V_1 is the function of the first two state variables which can be steered along the given vector fields while V_2 is the function of the third state variable which can be steered along the missing Lie bracket. The second strategy presents a time-varying feedback law based on the model reference approach, where the trajectory of the extended system is chosen as the model reference trajectory. By using an arbitrary Lyapunov function the controllers are designed in such a way that after each time period T, the trajectory of the rigid body model intersects the trajectory of the model reference which is asymptotically stable.

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