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Bounds on Some Variants of Clique Cover Numbers

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Abstract

A clique covering of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques. The weight of a clique covering is defined as the sum of the number of vertices of the cliques. The sigma clique cover number (resp. sigma clique partition number) of graph G, denoted by $\mathrm{scc}(G)$ (resp. $\mathrm{scp}(G)$), is defined as the smallest integer k for which there exists a clique covering (resp. clique partition) for G of weight k. In this paper, among some results we prove an upper bound on scc. Also, we provide a new lower bound on scp that improves a result of Erdős as a corollary. Then, we explore scc and scp for complete multipartite graphs as well as the product of graphs.

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1 introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph G, we mean a subset of mutually adjacent vertices of G as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices.

A *clique covering* of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques comprising this family. The minimum size of a clique covering of G is called *clique cover number* of G and is denoted by $\mathrm{cc}(G)$.

A clique covering in which each edge belongs to exactly one clique, is called a *clique partition*. The minimum size of a clique partition of G is called *clique partition number* of G and is denoted by $\mathrm{cp}(G)$.

Chung et al. in [2] and independently Tuza in [10] defined the concept of *weight* for a clique covering. Let $\mathcal{C}$ be a clique covering for graph G. The weight of $\mathcal{C}$ is defined as $\sum_{C\in\mathcal{C}} |V(C)|$.

The *sigma clique cover number* of G, denoted by $\mathrm{scc}(G)$, is defined as the minimum integer k for which there exists a clique covering $\mathcal{C}$ for G of weight k. In fact,

$$\mathrm{scc}(G) = \min_{\mathcal{C}} \sum_{C\in\mathcal{C}} |C|,$$

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where the minimum is taken over all clique coverings of G.

Analogously, one can define *sigma clique partition number* of G, denoted by $\mathrm{scp}(G)$. As a general upper bound, in [1, 6, 7] it was proved that for every graph G on n vertices, $\mathrm{scc}(G) \leq \mathrm{scp}(G) \leq n^2/2$.

Clique covering parameters have close relation to other combinatorial concepts such as *set representations*, *line hypergraph* and *pairwise balanced designed*. For a survey of the classical results on the clique coverings see [8, 9].

2 General Bounds

2.1 Upper Bound for scc

Let G be a graph on n vertices. The only known general upper bound on $\mathrm{scc}(G)$ is $n^2/2$ [1, 7, 6]. In the following theorem, using the probabilistic methods, we stablish an upper bound for $\mathrm{scc}(G)$.

Theorem 2.1. *If G is a graph on n vertices with no isolated vertex and $\Delta(\overline{G}) = d-1$, then*

$$\mathrm{scc}(G) \leq (e^2+1)nd\left\lceil \ln\left(\frac{n-1}{d-1}\right)\right\rceil.$$

Sketch of proof. Let $0 < p < 1$ be a fixed number and let S be a random subset of $V(G)$ defined by choosing every vertex u independently with probability p. For every vertex $u \in S$, if there exists a non-neighbour of u in S, then remove u from S. The resulting set is a clique of G. Repeat this procedure t times, independently, to get t cliques $C_1, C_2, \ldots, C_t$ of G.

Let F be the set of all the edges which are not covered by the cliques $C_1, \ldots, C_t$. The cliques $C_1, \ldots, C_t$ along with all edges in F comprise a clique covering of G. Hence,

$$\begin{aligned}\mathrm{scc}(G) &\leq \mathbf{E}\left(\sum_{i=1}^{t}|C_i| + 2|F|\right)\\ &\leq npt + 2\binom{n}{2}e^{-tp^2(1-p)^{2(d-1)}}.\end{aligned}$$

Finally, set $p := 1/d$ and $t := \lceil e^2d^2\ln(\frac{n-1}{d-1})\rceil > 0$ to get the desired corollary. □

2.2 Lower Bound for scp

Theorem 2.2. *Let U and V be a partition of vertices of G into the two sets. If G has t edges between parts U and V, then $\mathrm{scp}(G) \geq 2(t-(p+q))$, in which p and q are number of edges of G with both ends in U and V, respectively. Moreover, equality holds if and only if there exists a clique partition of edges of G, say $\mathcal{C}$, such that for each $C_i \in \mathcal{C}$, $|C_i \cap U| = |C_i \cap V|$.*

Remark 2.3. Without loss of generality assume that $p \leq q$. Erdős et al. in [5] proved that $cp(G) \geq t-2p-q$. On the other hand, by Theorem 2 (ii) in [4], $\mathrm{cp}(G) \geq \mathrm{scp}^2(G)/(2m+\mathrm{scp}(G))$, where m is the number of edges of G. Since $x^2/(2m+x)$ is increasing for $x > 0$, Theorem 2.2 concludes that $\mathrm{cp}(G) \geq (t-(p+q))^2/t$ which improves Erdős bound if and only if $t \leq (p+q)^2/q$.

3 Clique Covering of Special Graphs

In this section, our focus is on determining scc and scp for some well-known families of graphs. First, we consider the Turan graphs because of their importance in covering problems. Then, by determining the value of scc and scp for *Cartesian product* of graphs, we give a tight lower bound for scp of *tensor product* of complete graphs and study its asymptotic behaviour.

The complement of the union of complete graphs is the s-partite complete graph $K_{t_1,t_2,\ldots,t_s}$, whose parts are of size $t_1, t_2, \ldots, t_s$, respectively. If each part has the same size, $t_1 = t_2 = \cdots = t_s = t > 1$, then we denote the graph by $K_s(t)$.

Theorem 3.1. *Let $N(t)$ be the maximum number of mutually orthogonal Latin squares of order t. If $N(t) \geq s-2$, then $\text{scc}(K_s(t)) = \text{scp}(K_s(t)) = st^2$.*

Theorem 3.2. *If $G\Box H$ is the Cartesian product of G and H, then*

$$\text{scc}(G\Box H) = n(G)\,\text{scc}(H) + n(H)\,\text{scc}(G)$$
$$\text{scp}(G\Box H) = n(G)\,\text{scp}(H) + n(H)\,\text{scp}(G).$$

For the tensor product of complete graphs, $K_n \times K_n$, we have the following theorem.

Theorem 3.3. $\text{scp}(K_n \times K_n) \geq n^3 - n^2$. *If n is a prime power, then equality holds.*

Sketch of proof. By Theorem 2.5 in [3], for a graph G on n vertices, if $\max\{\omega(G), \omega(\overline{G})\} \leq \lfloor\sqrt{n}\rfloor$, then $\text{scp}(G) + \text{scp}(\overline{G}) \geq n(\sqrt{n}+1)$.

First note that complement of $K_n \times K_n$ is $K_n\Box K_n$. Since $\omega(K_n \times K_n) = \omega(K_n\Box K_n) = n$, we conclude that $\text{scp}(K_n \times K_n) \geq n^2(n+1) - scp(K_n\Box K_n)$. Thus, the lower bound is proved by Theorem 3.2.

Now, let n be a prime power. Thus, there exist $(n-2)$ idempotent MOLS(n) and equvallently an (n,n)-orthogonal array. Consider each row of the (n,n)-orthogonal array as a clique except the row $in + (i+1)$, for $0 \leq i \leq n-1$. These $n^2 - n$ cliques of size n, form a clique partition for the edges of $K_n \times K_n$. □

Theorem 3.4. *For large enough n,* $\text{scp}(K_n \times K_n) \sim n^3$.

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