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Bounds on some variants of clique cover numbers

Bounds on Some Variants of Clique Cover Numbers

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Abstract

A clique covering of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques. The weight of a clique covering is defined as the sum of the number of vertices of the cliques. The sigma clique cover number (resp. sigma clique partition number) of graph G, denoted by scc(G) (resp. scp(G)), is defined as the smallest integer k for which there exists a clique covering (resp. clique partition) for G of weight k. In this paper, among some results we prove an upper bound on scc. Also, we provide a new lower bound on scp that improves a result of Erdős as a corollary. Then, we explore scc and scp for complete multipartite graphs as well as the product of graphs.

Keywords: Clique covering, Clique partition, Sigma clique covering, Sigma clique partition

Mathematics Subject Classification [2010]: 05C70,05C62,05D05

1 introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph G, we mean a subset of mutually adjacent vertices of G as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices.

A clique covering of G is defined as a family of cliques of G such that every edge of G lies in at least one of the cliques comprising this family. The minimum size of a clique covering of G is called *clique cover number* of G and is denoted by cc(G).

A clique covering in which each edge belongs to exactly one clique, is called a *clique* partition. The minimum size of a clique partition of G is called *clique partition number* of G and is denoted by cp(G).

Chung et al. in [2] and independently Tuza in [10] defined the concept of *weight* for a clique covering. Let \mathcal{C} be a clique covering for graph G. The weight of \mathcal{C} is defined as $\sum_{C \in \mathcal{C}} |V(C)|$.

The sigma clique cover number of G, denoted by scc(G), is defined as the minimum integer k for which there exists a clique covering C for G of weight k. In fact,

$$\operatorname{scc}(G) = \min_{\mathcal{C}} \sum_{C \in \mathcal{C}} |C|,$$

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where the minimum is taken over all clique coverings of G.

Analogously, one can define sigma clique partition number of G, denoted by scp(G). As a general upper bound, in [1, 6, 7] it was proved that for every graph G on n vertices, $scc(G) \leq scp(G) \leq n^2/2$.

Clique covering parameters have close relation to other combinatorial concepts such as *set representations*, *line hypergraph* and *pairwise balanced designed*. For a survey of the classical results on the clique coverings see [8, 9].

2 General Bounds

2.1 Upper Bound for scc

Let G be a graph on n vertices. The only known general upper bound on scc(G) is $n^2/2$ [1, 7, 6]. In the following theorem, using the probabilistic methods, we stablish an upper bound for scc(G).

Theorem 2.1. If G is a graph on n vertices with no isolated vertex and $\Delta(\overline{G}) = d - 1$, then

$$\operatorname{scc}(G) \le (e^2 + 1)nd \left\lceil \ln\left(\frac{n-1}{d-1}\right) \right\rceil.$$

Sketch of proof. Let 0 be a fixed number and let <math>S be a random subset of V(G) defined by choosing every vertex u independently with probability p. For every vertex $u \in S$, if there exists a non-neighbour of u in S, then remove u from S. The resulting set is a clique of G. Repeat this procedure t times, independently, to get t cliques C_1, C_2, \ldots, C_t of G.

Let F be the set of all the edges which are not covered by the cliques C_1, \ldots, C_t . The cliques C_1, \ldots, C_t along with all edges in F comprise a clique covering of G. Hence,

$$\operatorname{scc}(G) \leq \mathbf{E}\left(\sum_{i=1}^{t} |C_i| + 2|F|\right)$$
$$\leq npt + 2\binom{n}{2}e^{-tp^2(1-p)^{2(d-1)}}.$$

Finally, set p := 1/d and $t := \left[e^2 d^2 \ln\left(\frac{n-1}{d-1}\right)\right] > 0$ to get the desired corollary.

2.2 Lower Bound for scp

Theorem 2.2. Let U and V be a partition of vertices of G into the two sets. If G has t edges between parts U and V, then $scp(G) \ge 2(t - (p + q))$, in which p and q are number of edges of G with both ends in U and V, respectively. Moreover, equality holds if and only if there exists a clique partition of edges of G, say C, such that for each $C_i \in C$, $|C_i \cap U| = |C_i \cap V|$.

Remark 2.3. Without loss of generality assume that $p \leq q$. Erdős et al. in [5] proved that $cp(G) \geq t - 2p - q$. On the other hand, by Theorem 2 (ii) in [4], $cp(G) \geq scp^2(G)/(2m + scp(G))$, where *m* is the number of edges of *G*. Since $x^2/(2m + x)$ is increasing for x > 0, Theorem 2.2 concludes that $cp(G) \geq (t - (p + q))^2/t$ which improves Erdős bound if and only if $t \leq (p + q)^2/q$.

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3 Clique Covering of Special Graphs

In this section, our focus is on determining scc and scp for some well-known families of graphs. First, we consider the Turan graphs because of their importance in covering problems. Then, by determining the value of scc and scp for *Cartesian product* of graphs, we give a tight lower bound for scp of *tensor product* of complete graphs and study its asymptotic behaviour.

The complement of the union of complete graphs is the s-partite complete graph K_{t_1,t_2,\ldots,t_s} , whose parts are of size t_1, t_2, \ldots, t_s , respectively. If each part has the same size, $t_1 = t_2 = \cdots = t_s = t > 1$, then we denote the graph by $K_s(t)$.

Theorem 3.1. Let N(t) be the maximum number of mutually orthogonal Latin squares of order t. If $N(t) \ge s - 2$, then $scc(K_s(t)) = scp(K_s(t)) = st^2$.

Theorem 3.2. If $G \Box H$ is the Cartesian product of G and H, then

$$scc(G\Box H) = n(G)scc(H) + n(H)scc(G)$$
$$scp(G\Box H) = n(G)scp(H) + n(H)scp(G).$$

For the tensor product of complete graphs, $K_n \times K_n$, we have the following theorem.

Theorem 3.3. $scp(K_n \times K_n) \ge n^3 - n^2$. If n is a prime power, then equality holds.

Sketch of proof. By Theorem 2.5 in [3], for a graph G on n vertices, if $\max\{\omega(G), \omega(\overline{G})\} \leq \lfloor \sqrt{n} \rfloor$, then $\operatorname{scp}(G) + \operatorname{scp}(\overline{G}) \geq n(\sqrt{n} + 1)$.

First note that complement of $K_n \times K_n$ is $K_n \Box K_n$. Since $\omega(K_n \times K_n) = \omega(K_n \Box K_n) = n$, we conclude that $scp(K_n \times K_n) \ge n^2(n+1) - scp(K_n \Box K_n)$. Thus, the lower bound is proved by Theorem 3.2.

Now, let n be a prime power. Thus, there exist (n-2) idempotent MOLS(n) and equvalently an (n, n)-orthogonal array. Consider each row of the (n, n)-orthogonal array as a clique except the row in + (i + 1), for $0 \le i \le n - 1$. These $n^2 - n$ cliques of size n, form a clique partition for the edges of $K_n \times K_n$.

Theorem 3.4. For large enough n, $scp(K_n \times K_n) \sim n^3$.

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