

## Bounds on Some Variants of Clique Cover Numbers

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### Abstract

A clique covering of  $G$  is defined as a family of cliques of  $G$  such that every edge of  $G$  lies in at least one of the cliques. The weight of a clique covering is defined as the sum of the number of vertices of the cliques. The sigma clique cover number (resp. sigma clique partition number) of graph  $G$ , denoted by  $scc(G)$  (resp.  $scp(G)$ ), is defined as the smallest integer  $k$  for which there exists a clique covering (resp. clique partition) for  $G$  of weight  $k$ . In this paper, among some results we prove an upper bound on  $scc$ . Also, we provide a new lower bound on  $scp$  that improves a result of Erdős as a corollary. Then, we explore  $scc$  and  $scp$  for complete multipartite graphs as well as the product of graphs.

**Keywords:** Clique covering, Clique partition, Sigma clique covering, Sigma clique partition

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## 1 introduction

Throughout the paper, all graphs are simple and undirected. By a *clique* of a graph  $G$ , we mean a subset of mutually adjacent vertices of  $G$  as well as its corresponding complete subgraph. The *size* of a clique is the number of its vertices.

A *clique covering* of  $G$  is defined as a family of cliques of  $G$  such that every edge of  $G$  lies in at least one of the cliques comprising this family. The minimum size of a clique covering of  $G$  is called *clique cover number* of  $G$  and is denoted by  $cc(G)$ .

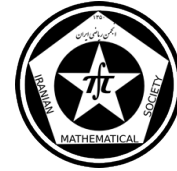
A clique covering in which each edge belongs to exactly one clique, is called a *clique partition*. The minimum size of a clique partition of  $G$  is called *clique partition number* of  $G$  and is denoted by  $cp(G)$ .

Chung et al. in [2] and independently Tuza in [10] defined the concept of *weight* for a clique covering. Let  $\mathcal{C}$  be a clique covering for graph  $G$ . The weight of  $\mathcal{C}$  is defined as  $\sum_{C \in \mathcal{C}} |V(C)|$ .

The *sigma clique cover number* of  $G$ , denoted by  $scc(G)$ , is defined as the minimum integer  $k$  for which there exists a clique covering  $\mathcal{C}$  for  $G$  of weight  $k$ . In fact,

$$scc(G) = \min_{\mathcal{C}} \sum_{C \in \mathcal{C}} |C|,$$

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where the minimum is taken over all clique coverings of  $G$ .

Analogously, one can define *sigma clique partition number* of  $G$ , denoted by  $\text{scp}(G)$ . As a general upper bound, in [1, 6, 7] it was proved that for every graph  $G$  on  $n$  vertices,  $\text{scc}(G) \leq \text{scp}(G) \leq n^2/2$ .

Clique covering parameters have close relation to other combinatorial concepts such as *set representations*, *line hypergraph* and *pairwise balanced designed*. For a survey of the classical results on the clique coverings see [8, 9].

## 2 General Bounds

### 2.1 Upper Bound for $\text{scc}$

Let  $G$  be a graph on  $n$  vertices. The only known general upper bound on  $\text{scc}(G)$  is  $n^2/2$  [1, 7, 6]. In the following theorem, using the probabilistic methods, we establish an upper bound for  $\text{scc}(G)$ .

**Theorem 2.1.** *If  $G$  is a graph on  $n$  vertices with no isolated vertex and  $\Delta(\overline{G}) = d - 1$ , then*

$$\text{scc}(G) \leq (e^2 + 1)nd \left\lceil \ln \left( \frac{n-1}{d-1} \right) \right\rceil.$$

**Sketch of proof.** Let  $0 < p < 1$  be a fixed number and let  $S$  be a random subset of  $V(G)$  defined by choosing every vertex  $u$  independently with probability  $p$ . For every vertex  $u \in S$ , if there exists a non-neighbour of  $u$  in  $S$ , then remove  $u$  from  $S$ . The resulting set is a clique of  $G$ . Repeat this procedure  $t$  times, independently, to get  $t$  cliques  $C_1, C_2, \dots, C_t$  of  $G$ .

Let  $F$  be the set of all the edges which are not covered by the cliques  $C_1, \dots, C_t$ . The cliques  $C_1, \dots, C_t$  along with all edges in  $F$  comprise a clique covering of  $G$ . Hence,

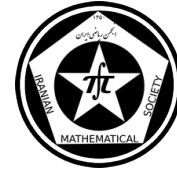
$$\begin{aligned} \text{scc}(G) &\leq \mathbf{E} \left( \sum_{i=1}^t |C_i| + 2|F| \right) \\ &\leq npt + 2 \binom{n}{2} e^{-tp^2(1-p)^{2(d-1)}}. \end{aligned}$$

Finally, set  $p := 1/d$  and  $t := \lceil e^2 d^2 \ln(\frac{n-1}{d-1}) \rceil > 0$  to get the desired corollary.  $\square$

### 2.2 Lower Bound for $\text{scp}$

**Theorem 2.2.** *Let  $U$  and  $V$  be a partition of vertices of  $G$  into the two sets. If  $G$  has  $t$  edges between parts  $U$  and  $V$ , then  $\text{scp}(G) \geq 2(t - (p + q))$ , in which  $p$  and  $q$  are number of edges of  $G$  with both ends in  $U$  and  $V$ , respectively. Moreover, equality holds if and only if there exists a clique partition of edges of  $G$ , say  $\mathcal{C}$ , such that for each  $C_i \in \mathcal{C}$ ,  $|C_i \cap U| = |C_i \cap V|$ .*

**Remark 2.3.** Without loss of generality assume that  $p \leq q$ . Erdős et al. in [5] proved that  $\text{cp}(G) \geq t - 2p - q$ . On the other hand, by Theorem 2 (ii) in [4],  $\text{cp}(G) \geq \text{scp}^2(G)/(2m + \text{scp}(G))$ , where  $m$  is the number of edges of  $G$ . Since  $x^2/(2m + x)$  is increasing for  $x > 0$ , Theorem 2.2 concludes that  $\text{cp}(G) \geq (t - (p + q))^2/t$  which improves Erdős bound if and only if  $t \leq (p + q)^2/q$ .



### 3 Clique Covering of Special Graphs

In this section, our focus is on determining  $scc$  and  $scp$  for some well-known families of graphs. First, we consider the Turan graphs because of their importance in covering problems. Then, by determining the value of  $scc$  and  $scp$  for *Cartesian product* of graphs, we give a tight lower bound for  $scp$  of *tensor product* of complete graphs and study its asymptotic behaviour.

The complement of the union of complete graphs is the  $s$ -partite complete graph  $K_{t_1, t_2, \dots, t_s}$ , whose parts are of size  $t_1, t_2, \dots, t_s$ , respectively. If each part has the same size,  $t_1 = t_2 = \dots = t_s = t > 1$ , then we denote the graph by  $K_s(t)$ .

**Theorem 3.1.** *Let  $N(t)$  be the maximum number of mutually orthogonal Latin squares of order  $t$ . If  $N(t) \geq s - 2$ , then  $scc(K_s(t)) = scp(K_s(t)) = st^2$ .*

**Theorem 3.2.** *If  $G \square H$  is the Cartesian product of  $G$  and  $H$ , then*

$$\begin{aligned} scc(G \square H) &= n(G) scc(H) + n(H) scc(G) \\ scp(G \square H) &= n(G) scp(H) + n(H) scp(G). \end{aligned}$$

For the tensor product of complete graphs,  $K_n \times K_n$ , we have the following theorem.

**Theorem 3.3.**  $scp(K_n \times K_n) \geq n^3 - n^2$ . *If  $n$  is a prime power, then equality holds.*

**Sketch of proof.** By Theorem 2.5 in [3], for a graph  $G$  on  $n$  vertices, if  $\max\{\omega(G), \omega(\overline{G})\} \leq \lfloor \sqrt{n} \rfloor$ , then  $scp(G) + scp(\overline{G}) \geq n(\sqrt{n} + 1)$ .

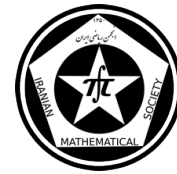
First note that complement of  $K_n \times K_n$  is  $K_n \square K_n$ . Since  $\omega(K_n \times K_n) = \omega(K_n \square K_n) = n$ , we conclude that  $scp(K_n \times K_n) \geq n^2(n + 1) - scp(K_n \square K_n)$ . Thus, the lower bound is proved by Theorem 3.2.

Now, let  $n$  be a prime power. Thus, there exist  $(n - 2)$  idempotent MOLS( $n$ ) and equivalently an  $(n, n)$ -orthogonal array. Consider each row of the  $(n, n)$ -orthogonal array as a clique except the row  $in + (i + 1)$ , for  $0 \leq i \leq n - 1$ . These  $n^2 - n$  cliques of size  $n$ , form a clique partition for the edges of  $K_n \times K_n$ .  $\square$

**Theorem 3.4.** *For large enough  $n$ ,  $scp(K_n \times K_n) \sim n^3$ .*

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