

Technical Note

A STUDY OF ELASTIC BUCKLING OF MASONRY COLUMNS

P.K. Bhandari^a, G.S Benipal^b, and M. Ahmed^{*c}

^aA.E., Zone-III, PWD (GOD), New Delhi, India

^bDepartment of, Civil Engineering, IIT Delhi, India

^cPBA Division, SB Group, Post Box –9887, Jeddah-21423, Kingdom of Saudi Arabia

Abstract

In this paper, the behavior of masonry columns under working load has been studied. The effect of cracking on compression area of column or variation of moment of inertia of column along its height has been considered. The analytical solution for the critical load and deflection of the column is presented. A numerical example is presented to illustrate the method. The proposed approach has provided more realistic behavior of the masonry columns and proved to give an optimized critical load. The implications of the proposed study on the Indian code have also been presented.

Keyword: Buckling; cracked masonry; eccentricity; slenderness ratio; critical load

1. Introduction

The vertical members of building structure i.e. columns are the most important structural members as the transfer of the load from the horizontal members to foundation is through the column. The failure of a column can cause failure of other members or failure of entire structure. The elastic stability or the sidewise buckling of such members is of great practical importance. The buckling and secondary effects, which reduce their ultimate strength, are more pronounced in slender columns as compared to that of a short column.

The behavior of masonry or concrete column is different with the steel column. In steel column, the area of cross section remains constant and slenderness ratio for specified length and cross section will remain constant. But in case of masonry column slenderness ratio ($1/r$ ratio) doesn't remain constant and vary along its height. In cracked section, the section below the neutral axis can be ignored for computation of moment of inertia and radius of gyration. Hence as the eccentricity increases, depth of neutral axis decreases with the result, moment of inertia and effective area of cross section also decreases. The decrease of radius of gyration leads to increases in slenderness ratio of the column and critical or Euler load for

* E-mail address of the corresponding author: mahmed@pbad.sbg.com.sa (M. Ahmed)

the column is decreases i.e. the strength to resist buckling is greatly affected.

$$n = f(e) = f(M/P)$$

where n is depth of neutral axis, M is moment and P is axial force.

The structural adequacy of masonry columns/walls depends upon a number of factors such as quality and strength of material and mortars, workmanship, effective length, eccentricity of loading etc. Eccentricity caused by an eccentric vertical load is maximum at the top of a member, that is, at the point of loading and it is assumed to reduce linearly to zero at the bottom of the member that is, just above the bottom lateral support, while eccentricity on account of slenderness of a member is zero at the two support and is maximum at the middle. As per Para 4.7 of Code of practice for Structural use of un-reinforced masonry [1,2], eccentricity of vertical loading at a particular function in a masonry columns/wall shall depends on factors, such as extent of bearing, magnitude of loads, still of slab or beam, fixity at the support and constructional details at junctions. The eccentricity of vertical loading on masonry elements increases its tendency to buckling and reduces its load carrying capacity. The effect is thus similar to that of slenderness of the members. Thus combined effect of slenderness and eccentricity is taken in to consideration in design calculation by the factor known as stress reduction factor.

Many researchers have attempted to study the stability of columns. Around 1757, Leonhard Euler made the pioneering study on slender columns using differential equations. Rankine in 1805 proposed an empirical formula for columns with cases ranging from very short to very long column. Recently an attempt was made [3] to study the behaviour of cracked masonry (No tension) and R.C. Column under working loads. The two existing fundametal theories for the determination of critical loads of four types of prismatic or non prismatic columns carrying compressive loads at the ends are as following [4]:

1. Leonhard Euler's differential equation.
2. Timoshenko's energy method.

Lateral stiffness of un-reinforced masonry circular columns under cracked conditions has been investigated by Gurel et al. [5]. They concluded that the lateral stiffness of a masonry column depends not only on the slenderness, material properties and vertical load ratio, but also on the displacement level and hence the extent of cracking. Rilippo Romano [6] presented an analytical approach to analyze the behavior of masonry member, with zero tensile strength. The theory is founded on the monomial non-linear gross stress-strain relationship i.e.

$$\sigma = E \varepsilon^n$$

where σ is the stress, E is the elastic modulus, ε = strain and n is constant ($n \leq 1$).

The analytical curves of the gross stress-strain relationship given by various researchers show the wide range of variability of exponent n , (n , varies from 0.54 to 0.93).

The object of this work is to study the behaviour of cracked masonry (No tension) under working loads. So far methods available on the subject scan to be inadequate in evaluation of critical load of cracked masonry accurately, particularly form the fact that none of them

takes in to account the variation of net area of compression of column or variation of moment of inertia of column along its height due to cracking. The analytical solution is presented for the deflection and critical load over which structural prismatic masonry collapse without reaching the compressive strength.

2. Analysis for Critical Load

The methods and concepts from elastic buckling theory have been applied for the analysis of cracked masonry and to investigate the onset of instability under working loads. In this study, the following assumptions are made.

1. Strain distribution across any section is linear.
2. The shear deformations are neglected
3. The neutral axis, about which the cross section rotates, lies within the cross section.

The masonry has no tensile strength

4. The shortening of a column under load is negligible.
5. The deflected shape of a column is symmetrical about its mid height.

Deflected shape of column is given in Figure 1. The variation of depth of neutral axis is shown in Figure 2. The centroidal axis of column will also shift along the height. The column is pinned at both ends and subjected to vertical load P at an eccentricity e_L at end. At ends $e = e_L$ is maximum, considering $n = d$

$$\sigma = \frac{P}{bd} - \frac{P e}{z} \quad (1)$$

For no tension column, The stress $\sigma = 0$,

By putting $z = \frac{bd^2}{6}$, we obtain

$$\frac{P}{bd} - \frac{P e}{\frac{bd^2}{6}} = 0$$

$$e = d/6$$

where $(b \times d)$ is the section of the column of length L , e is eccentricity of force P at a section x distance from end. Hence $e_{\max} = e_L = d/6$

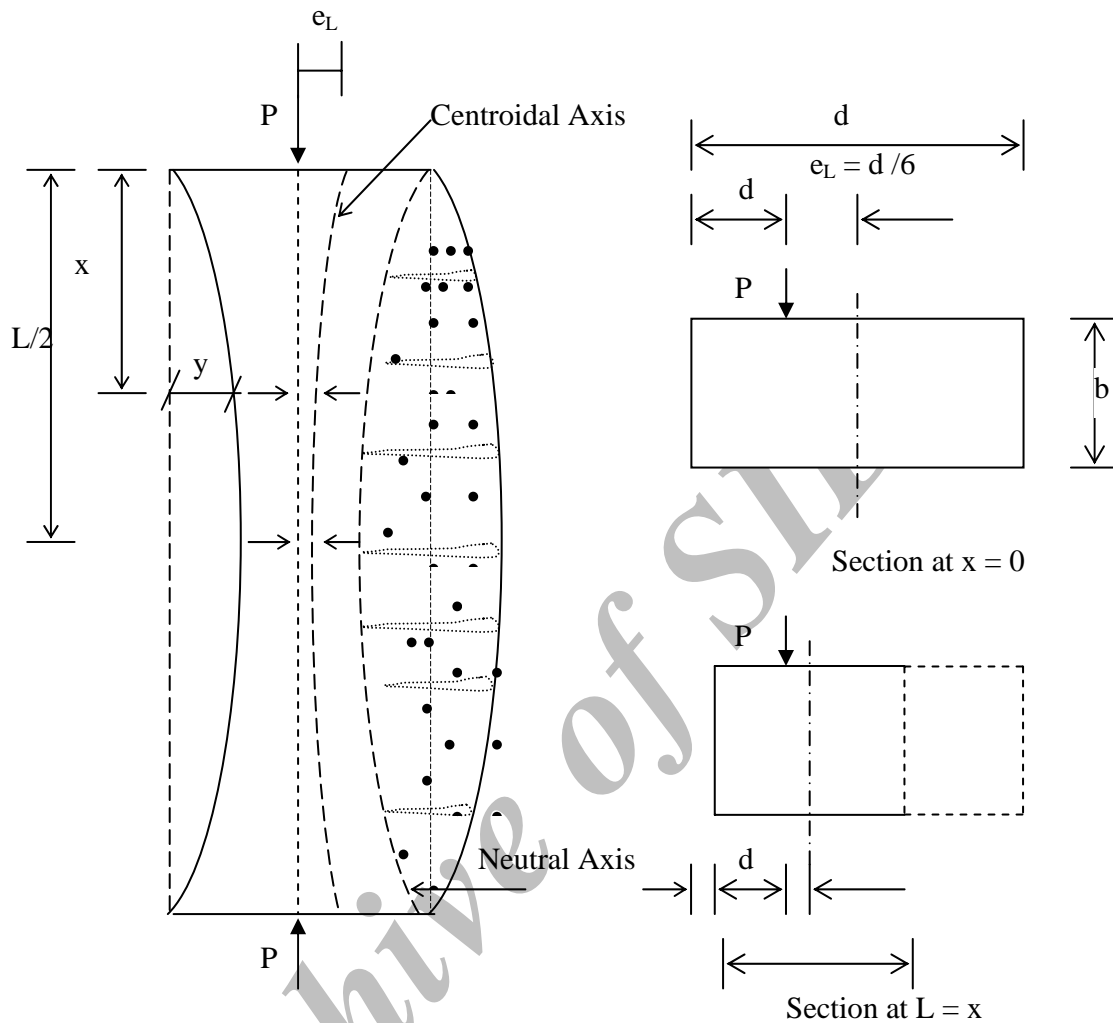


Figure 1. Variation of centroidal axis depth and neutral axis depth along the span of masonry column

The maximum stress in member occurs at $x=L/2$, since depth of neutral axis at this section will be minimum.

Considering $n < d$ and from Figure 1,

$$e = \frac{n}{2} + y - \frac{d}{3}$$

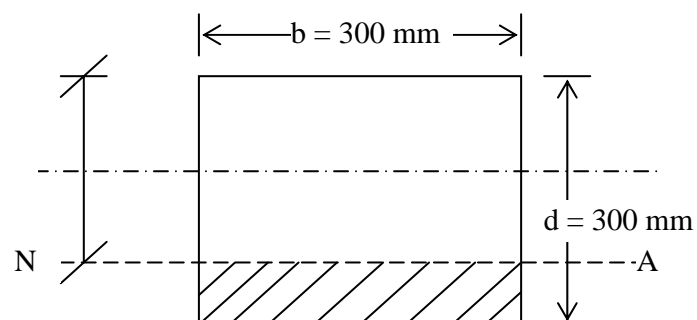
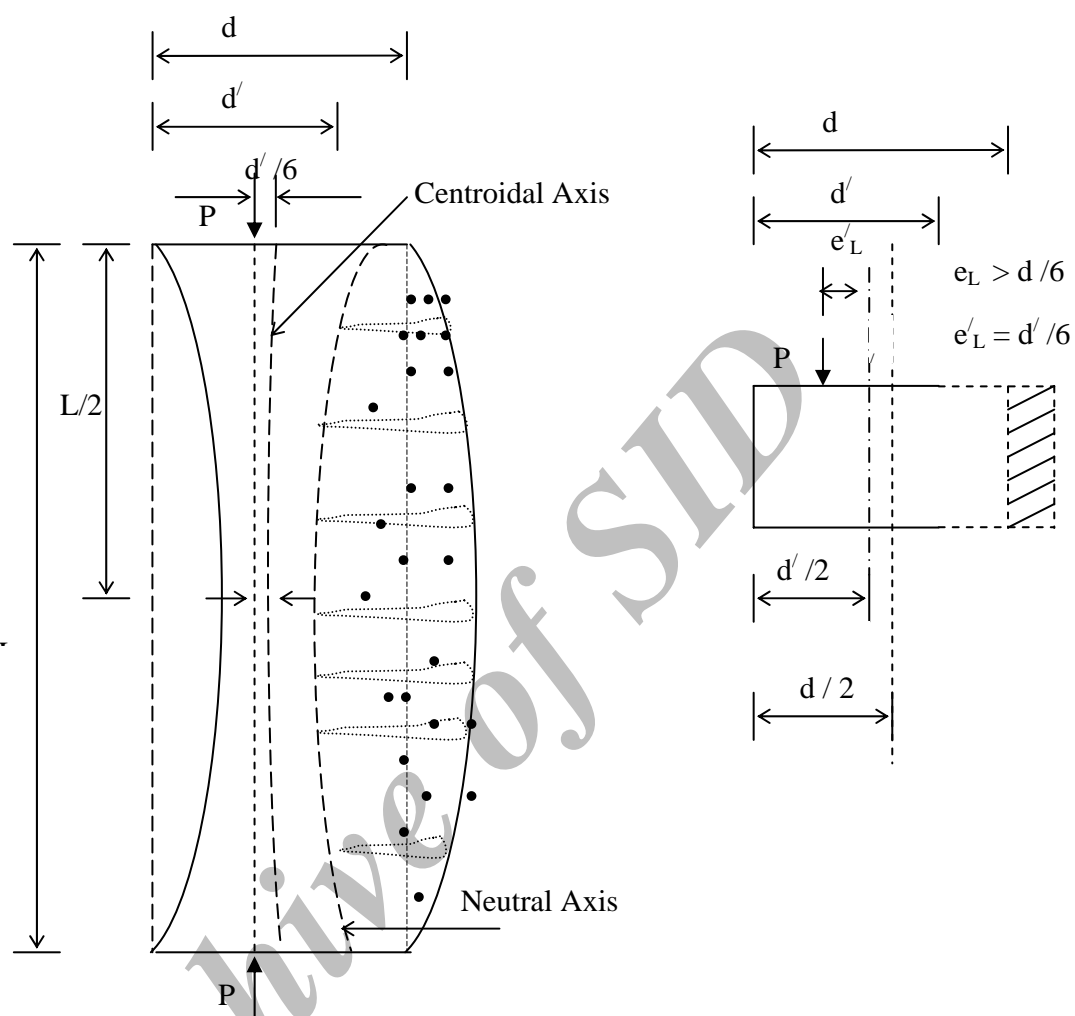


Figure 3. Masonry column section

where y is the lateral displacement from original position of maximum compressed fiber at a distance x from the end.

For no tension case, $e = \frac{n}{6}$, or $n = 6e$

$$\begin{aligned} e &= \frac{6e}{2} + y - \frac{d}{3} \\ n = 6e &= 6 \cdot \frac{d-3y}{6} = d-3y \end{aligned} \quad (2)$$

where n is the neutral axis of section at $L = x$

Now, the equilibrium equation for any section at a distance x from end can be written as

$$EI \frac{d^2 y}{dx^2} - P \cdot e = 0 \quad (3)$$

For cracked section, Moment of inertia, $I = \frac{bn^3}{12}$, and $n = d - 3y$

substituting the values of e and I in Eq. (3), results in

$$\begin{aligned} \frac{bn^3}{12} \cdot \frac{d^2 y}{dx^2} &= -\frac{P(d-3y)}{6} \\ \frac{d^2 y}{dx^2} &= \frac{-2P}{(d-3y)^2 Eb} \end{aligned} \quad (4)$$

Now from Eq. (2), we get

$$\begin{aligned} 6e &= d-3y \\ \frac{d^2 y}{dx^2} &= -2 \frac{d^2 e}{dx^2} \end{aligned}$$

By substituting $(d-3y) = 6e$ and $\frac{d^2 y}{dx^2} = -2 \frac{d^2 e}{dx^2}$ in Eq. (4), we get

$$\begin{aligned} \frac{d^2 e}{dx^2} &= \frac{\Delta}{e^2} \\ \Delta &= \frac{P}{36Eb} \end{aligned} \quad (5)$$

where

Multiplying both sides of Eq. (5) by $\frac{de}{dx}$ and integrating, we get

$$\int \frac{de}{dx} \cdot \frac{d^2e}{dx^2} \cdot dx = \int \frac{\Delta}{e^2} \frac{de}{dx} \cdot dx$$

$$\frac{de}{dx} = \pm \sqrt{2 \left(C - \frac{\Delta}{e} \right)}$$

where C is a constant of integration.

Considering the positive sign,

$$\frac{de}{dx} = \sqrt{2C} \cdot \sqrt{1 - \frac{\Delta}{Ce}} = \sqrt{2C} \cdot \sqrt{1 - \frac{B}{e}}$$

where $B = \frac{\Delta}{C}$

By integrating, we get

$$\frac{1}{\sqrt{2C}} \int \frac{de}{\sqrt{1 - \frac{B}{e}}} = \int dx$$

$$\frac{1}{\sqrt{2C}} \int \frac{\sqrt{e}}{\sqrt{e - B}} de = x - D$$

where D is another integration constant.

Substituting the value of B and solving the above equation, we get

$$x - D = \frac{1}{\sqrt{2C}} \left[\sqrt{e^2 - \frac{\Delta}{C}e} + \frac{\Delta}{C} \ln \left(\sqrt{e - \frac{\Delta}{C}} + \sqrt{e} \right) - \frac{\Delta}{2C} \ln \frac{\Delta}{C} \right] \quad (6)$$

The boundary conditions are given by

$$\text{at } x = \frac{L}{2}, \quad \frac{dy}{dx} = 0, \text{ (slope is zero at mid span), } e = e_0, \quad \frac{\Delta}{C} = e_0$$

$$\text{and at } x = L, \quad e = e_L = \frac{d}{6}$$

Now applying the boundary conditions in Eq. (6), we get

$$D = L - \sqrt{\frac{e_0}{2\Delta}} \left[\sqrt{\left(\frac{d}{6}\right)^2 - e_0 \cdot \frac{d}{6}} + e_0 \ln \left(\sqrt{\frac{d}{6} - e_0} + \sqrt{\frac{d}{6}} \right) - \frac{e_0}{2} \ln e_0 \right]$$

Again substituting the value of C&D in Eq. (6), we get

$$x = L - \sqrt{\frac{e_0}{2\Delta}} \left[\sqrt{\left(\frac{d}{6}\right)^2 - \frac{d}{6} \cdot e_0} + e_0 \ln \left(\sqrt{\frac{d}{6} - e_0} + \sqrt{\frac{d}{6}} \right) - \sqrt{e^2 - ee_0} - e_0 \ln(\sqrt{e - e_0} + \sqrt{e}) \right] \quad (7)$$

The critical section shall be at $x = \frac{L}{2}$ and at $x = \frac{L}{2}$, $e = e_0$,

$$\Delta = \frac{2e_0}{L^2} \left[\sqrt{\left(\frac{d}{6}\right)^2 - \frac{d}{6} \cdot e_0} + e_0 \ln \left(\sqrt{\frac{d}{6} - e_0} + \sqrt{\frac{d}{6}} \right) - \frac{e_0}{2} \ln e_0 \right]^2$$

By substituting the value Δ as $\Delta = \frac{P}{36Eb}$, we obtain

$$P = P_{eq} \cdot \frac{4m}{\pi^2} \left[\sqrt{1-m} + m \ln(\sqrt{1-m} + 1) - \frac{m}{2} \ln m \right]^2 \quad (8)$$

where $P_{eq} = \frac{\pi^2 Ebd^3}{12L^2}$

Here, P_{eq} is the critical load of a prismatic column having length L and m is ratio of minimum and maximum eccentricity, which is given by the relation

$$m = \frac{e_0}{d/6} = \frac{e_0}{e_L}$$

For maximum value of P , i.e. P_{cr} , $\frac{P}{P_{eq}}$ should be maximum.

For $\frac{P}{P_{eq}}$ maximum, $\frac{d\left(\frac{P}{P_{eq}}\right)}{dm} = 0$

$$\frac{d}{dm} \cdot \frac{4m}{\pi^2} \left[\sqrt{1-m} + m \ln(\sqrt{1-m} + 1) - \frac{m}{2} \ln m \right]^2 = 0$$

$$\frac{4}{\pi^2} \left[\sqrt{1-m} + m \ln(\sqrt{1-m} + 1) - \frac{m}{2} \ln m \right] \left[\frac{1-3m}{\sqrt{1-m}} + 3m \ln \left(\frac{\sqrt{1-m} + 1}{\sqrt{m}} \right) \right] = 0$$

$$\left[\frac{1-3m}{\sqrt{1-m}} + 3m \ln \left(\frac{\sqrt{1-m} + 1}{\sqrt{m}} \right) \right] = 0$$

Solving above equations iteratively, we can find for rectangular section
 $m = 0.61165$

$$e_0 = 0.61165 \times \frac{d}{6} = 0.10194d$$

Substituting value of m in Eq. (8), we get

$$P = 0.2837 P_{eq}$$

At $x = L/2$, the P will be the P_{cr} . We can write above equation

$$P_{cr} = 0.2837 \times \frac{\pi^2 EI}{L^2}$$

$$\sigma_{cr} = 0.2837 \times \frac{\pi^2 E r^2}{L^2} = 0.2837 \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \quad (9)$$

where σ_{cr} is the critical stress; r is the radius of gyration.

3. Conclusions

In this analytical study, the stability of masonry column is checked. The expression for the critical buckling load with varying moment of inertia has been derived. The analysis is developed by assuming that the member does not resist tensile force and exhibits a linear gross stress-strain relationship in compression. Use of linear stress-strain relation is generally conservative for commonly used slenderness ratios in masonry structures. The main conclusions of the study are summarized below.

a) Use of slender structural members in the modern structures needs stability analysis, as economy demands optimization and reliability assessment.

b) The analytical solution obtained, makes it possible to analyze the behavior of masonry members with zero tensile strength, subjected to vertical loading with eccentricity equal to or more than one sixth of total depth of the column section. The solution provides the critical load, deflection, stresses and makes it possible to locate the cracked zone.

The critical load for a column with prismatic section and concentric load

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Whereas the expression obtained by this study given the critical load for a masonry column with eccentricity equal to $\frac{d}{6}$ and cracked section.

$$P_{cr} = 0.2837 \frac{\pi^2 EI}{L^2}$$

It is clear from the above expression that due to eccentricity and cracking, the P_{cr} value of column reduces to 28.37%.

c) In Indian standard code of practice for structural use of un-reinforced masonry IS: 1905-1987, the combined effect of slenderness and eccentricity is taken into consideration in design calculation by the factor known as stress reduction factor. The code doesn't give any direct method to check stability or determination of critical load for the column. With the help of present study, we can determine the critical load for the masonry for eccentricity more than or equal to one sixth of the depth of cross section of column.

References

1. Bureau of Indian Standard, Indian Standard Code of Practice for structural use of Un-reinforced Masonry (IS:1905-1987), 1987.
2. Bureau of Indian Standard, Hand book on Masonry design and construction (SP:20 (S and T), 1991.
3. Timoshenko SP, Gere JM. *Theory of Elastic Stability*, McGraw Hill Publication, 1961.
4. Benipal GS, A Study On the Non-Linear Elastic Behavior of Reinforced Concrete Structural Elements Under Normal Loading, Ph. D. Thesis, IIT Delhi, 1993.
5. Gurel MA, M Kiş, F Cili. Lateral stiffness of un-reinforced masonry circular columns under cracked conditions, *Turkish Journal of Engineering and Environmental Sciences* **30**(2006)57-68.
6. Filippo R, Salvatore G, Gaetano Z. Cracked non-linear masonry stability under vertical and lateral loads, *Journal of Structural Engineering, ASCE*, No. 12, **119**(1993)167-74.