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EFFICIENT ANALYSIS OF STRUCTURES HOLDING TRI-DIAGONAL AND BLOCK TRI-DIAGONAL STIFFNESS MATRICES, GENERALIZING THE METHOD TO OTHER STRUCTURES USING HOUSEHOLDER AND BLOCK HOUSEHOLDER TRANSFORMATION

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ABSTRACT

A large group of structures hold tri-diagonal stiffness matrices. The eigenpairs and inverse of these matrices are found simpler than the ones of common matrices. In addition, using the householder transformation, symmetric matrices can be converted to the similar tri-diagonal matrices. Therefore, since stiffness matrices are symmetric, they can be changed to the similar tri-diagonal ones. In other words, all symmetric matrices can be converted to the tri-diagonal ones and the simpler solution of tri-diagonal matrices can be used for all stiffness matrices. Such a comparison is also true for block tri-diagonal matrices and block symmetric matrices. Although block matrices are a specific kind of common matrices, we want to study them independently because working with blocks can be more time-saving and efficient in many cases. In this paper, efficient solutions are presented for tri-diagonal and block tri-diagonal matrices. Besides, using the features of symmetric and block symmetric matrices they are converted to the tri-diagonal and block tri-diagonal ones.

Keywords: Block tri-diagonal matrices; eigenpairs; householder transformation; similar matrices; stiffness matrices; efficient solution.

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1. INTRODUCTION

Although computational capacity of computers has ameliorated during recent years, the swift solution of structures is still of great importance in the literature. The analysis of huge structures is time-consuming and occupies lots of memory. The efficient solution of regular and near-regular structures has been discussed in the reference [1-8]. The stiffness matrix of regular structures follows repetitive patterns. In these structures a sector is repeated to form the whole structure. In the near-regular structures a combination of regular and non-regular parts forms the main structure. To solve the near-regular structures, usually combined graph products and sub structuring methods are adapted. Although the matrix patterns of regular and near-regular structures are considered very special from the viewpoint of matrix theory and linear algebra, however, these patterns are very common from the structural point of view. Moreover, using the prefabricated structural components has caused more regularity in the structures. Other interesting patterns, seen in many non-regular structures, include tri-diagonal and block tri-diagonal ones. These patterns can be seen in the shear buildings and structures with several parts where the parts hold same degrees of freedom. Moreover, in arbitrary structures because stiffness matrices are symmetric, they can be converted into the tri-diagonal and block tri-diagonal matrices using the methods in the linear algebra. Therefore, the swift solution of tri-diagonal and block tri-diagonal structures can be utilized in all structures since stiffness matrices are symmetric.

In this paper some efficient methods for computing the eigenpairs and inverse of tri-diagonal and block tri-diagonal stiffness matrices are presented. Then, the methods of converting the symmetric matrices into the tri-diagonal and block tri-diagonal matrices are studied. This way the advantages of using the methods together are seen in the solution of all structures.

2. TRI-DIAGONAL MATRICES

Although the stiffness matrix of structures is symmetric, here the general form of tri-diagonal matrices is considered. Suppose matrix A holds the following pattern

$$A=\begin{bmatrix} a_1 & c_1 & & & \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & c_{n-2} & \\ & & b_{n-2} & a_{n-1} & c_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} \tag{1}$$

With no change in the eigenvalues we will have

$$S=D^{-1}AD \tag{2}$$

Where matrix S is a symmetric tri-diagonal one. Now, the components of matrix D are found

$$S=\begin{bmatrix} d_1^{-1} & & & & \\ & d_2^{-1} & & & \\ & & \ddots & & \\ & & & d_{n-1}^{-1} & \\ & & & & d_n^{-1} \end{bmatrix}\begin{bmatrix} a_1 & c_1 & & & \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & c_{n-2} & \\ & & b_{n-2} & a_{n-1} & c_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}\begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix} \qquad (3)$$

$$d_1 = 1 \to d_2 = \sqrt{\frac{c_2}{b_1}}d_1 \to d_3 = \sqrt{\frac{c_3}{b_2}}d_2 \ldots \to d_n = \sqrt{\frac{c_n}{b_{n-1}}}d_{n-1}$$

This way the symmetric matrix S is found

$$S=\begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \ddots & & \\ & \ddots & \ddots & \beta_{n-2} & \\ & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{bmatrix} \qquad (4)$$

Using Cholesky decomposition we will have

$$S = L\Delta L^T \qquad (5)$$

Where

$$L=\begin{bmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & l_{n-2} & 1 & \\ & & & l_{n-1} & 1 \end{bmatrix} \qquad (6)$$

$$\Delta=\begin{bmatrix} \delta_1 & & & & \\ & \delta_2 & & & \\ & & \ddots & & \\ & & & \delta_{n-1} & \\ & & & & \delta_n \end{bmatrix} \qquad (7)$$

Where

$$\delta_1 = \alpha_1\,, \qquad l_1 = \frac{\beta_1}{\delta_1}, \qquad \delta_i = \alpha_i - \frac{\beta_{i-1}^2}{\delta_{i-1}}\,, i = 2, \ldots, n\,, \qquad l_i = \frac{\beta_i}{\delta_i}\,, i = 2, \ldots, n$$

The decomposition can be also written as

$$S = L^{*}\Delta^{-1}(L^{*})^T \qquad (8)$$

Where

$$L^* = \begin{bmatrix} \delta_1 & & & & \\ \beta_1 & \delta_2 & & & \\ & \ddots & \ddots & & \\ & & \beta_{n-2} & \delta_{n-1} & \\ & & & \beta_{n-1} & \delta_n \end{bmatrix} \tag{9}$$

As seen, δ_i s is the only components which should be calculated and saved. Besides, the relationships corresponding to δ_is are recursive. Considering the features of recursive relationships, each component is quickly obtained using the previous component. Consequently, all components of the three matrices L^*, Δ^{-1} and $(L^*)^T$ are quickly obtained.

To solve the equation $Sx = b$, one can use the following equation with no need to invert the matrix S.

$$L^* y = b, \quad (L^*)^T x = \Delta y \tag{11}$$

As shown in reference [1], using the eigenpairs of matrix A, the equation $Ax = b$ is solved as follows

$$\mathrm{Ax} = \mathrm{b} \Rightarrow \{\varphi\}^t{}_j \mathrm{A} \{\varphi\}_j y_j = \lambda_j y_j = \{\varphi\}^t{}_j \mathrm{b}$$
$$y_j = \frac{b_j}{\lambda_j} \Rightarrow \{x\}_n = \sum_{i=1}^{n} \{\varphi\}_i y_i = \sum_{i=1}^{n} \{\varphi\}_i \frac{b_i}{\lambda_i} = \sum_{i=1}^{n} \frac{\{\varphi\}_i \{\varphi\}^t{}_i}{\lambda_i} \mathrm{b} \tag{12}$$

Consequently, for solving the relationships in Eq. (11), the approach used in Eq. (12) is utilized. Applying the approach is possible because the eigenvalues (and thus the eigenvectors) of the matrix L^* (also the matrix $(L^*)^T$) are available. These eigenvalues are the δ_i s located on the diagonal. This way the equation $Sx = b$ is simply solved for a symmetric tri-diagonal matrix. Application of the method is in the analysis of shear buildings and mass-spring systems which hold tri-diagonal stiffness matrices.

In following, the eigenvalues of tri-diagonal matrices are obtained. Previously, the eigenvalues of matrix L^* were calculated that are different from the eigenvalues of matrix S. To obtain the eigenvalues of matrix, the following Equation should be solved

$$\det(S - \lambda I) = 0 \tag{13}$$

Showing the relationship $S - \lambda I$ with $S(\lambda)$, we will have

$$\mathrm{S}(\lambda) = \mathrm{F}(\lambda)\mathrm{F}(\lambda)^{\mathrm{T}} \tag{14}$$

Where

$$F(\lambda)=\begin{bmatrix} \sqrt{\delta(\lambda)_1} & & & & \\ \frac{\beta_1}{\sqrt{\delta(\lambda)_1}} & \sqrt{\delta(\lambda)_2} & & & \\ & \ddots & \ddots & & \\ & & \frac{\beta_{n-2}}{\sqrt{\delta(\lambda)_{n-2}}} & \sqrt{\delta(\lambda)_{n-1}} & \\ & & & \frac{\beta_{n-1}}{\sqrt{\delta(\lambda)_{n-1}}} & \sqrt{\delta(\lambda)_n} \end{bmatrix} \tag{15}$$

And

$$\delta(\lambda)_1=\alpha_1-\lambda\,, \qquad \delta(\lambda)_i=\alpha_i-\lambda-\frac{\beta_{i-1}^2}{\delta(\lambda)_{i-1}}\,, i=2,\ldots,n \tag{16}$$

Using Eq. (14) and considering Eq. (13) and Eq. (15), we will have

$$\det(S-\lambda I)=\delta(\lambda)_1\delta(\lambda)_2\ldots\ldots\delta(\lambda)_n=0 \tag{17}$$

By multiplying the δ_i s sequentially, the characteristic equation is simply found. An appropriate matrix form of the characteristic equation is shown in Eq. (18).

$$\det(S-\lambda I)=\left[\begin{pmatrix}\alpha_n-\lambda & -\beta_{n-1}{}^2\\ 1 & 0\end{pmatrix}\ldots\begin{pmatrix}\alpha_2-\lambda & -\beta_1{}^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_1-\lambda & 0\\ 1 & 0\end{pmatrix}\right]_{11}=0 \tag{18}$$

Where the index 11 shows the component 11 of the matrix is the characteristic equation. Therefore, in each multiplication of two matrices one should only multiply the second matrix by the first row of the first matrix. This way the characteristic equation is found just using vector multiplication. The application of the method is in obtaining the natural frequencies of free vibration of shear buildings and mass-spring systems.

In the following section it is shown how to convert symmetric matrices into the tri-diagonal matrices. Using the conversion the simply obtained features of tri-diagonal matrices are also applicable to symmetric matrices.

3. ONVERTING SYMMETRIC MATRICES INTO TRI-DIAGONAL MATRICES USING HOUSEHOLDER TRANSFORMATION

Symmetric matrices can be converted to the symmetric tri-diagonal matrices using householder transformation. The conversion is performed in the limited steps. The initial matrix and the obtained tri-diagonal matrix hold similar eigenvalues.

Suppose ν is a column vector in which $\|\upsilon\|_2=1$. The orthogonal matrix bellow is the householder transformation corresponding to the vector ν.

$$H = I_n - 2\upsilon\upsilon^t \tag{19}$$

Algorithm:
Step 1. Put $k = 1$ and $B = A$
Step 2. Calculate

$$s = \sqrt{\sum_{i=k+1}^{n} b_{ik}^2}$$

If $s = 0$, put $k = k + 1$ and calculate s again.
Step 3. Put

$$SG = \begin{cases} -1 \; if \quad b_{k+1,k} < 0 \\ +1 \; if \quad b_{k+1,k} \geq 0 \end{cases}$$

Step 4. Consider

$$z = \frac{1}{2}(1 + \frac{SG.b_{k+1,k}}{s})$$

Step 5. Consider $\nu_i = 0$ for $i = 1, 2, \dots, k$ and $\upsilon_{k+1} = \sqrt{z}$ and $\upsilon_i = \frac{\mathrm{SG.b_{ki}}}{2s\upsilon_{\mathrm{k+1}}}$ for $i = k + 2, \dots, n$

Step 6. Put $\upsilon = (\upsilon_1, \upsilon_2, \dots, \upsilon_n)^t$ and $H = I_n - 2\upsilon\upsilon^t$
Step 7. Calculate $A = HBH$
Step 8. If $k = n + 2$, matrix A is the aimed tri-diagonal matrix, otherwise put $k = k + 1$ and $B = A$ and go to the step 2.

This way using $n - 2$ steps, a symmetric matrix is converted to a similar tri-diagonal matrix. Thus, the characteristics of tri-diagonal matrices in the fast solution can be utilized for all symmetric matrices.

Stiffness matrix of structures is composed of smaller blocks since first the stiffness matrix of members is constructed. Therefore, it is a good idea to get advantages of the block form. In many cases, working with blocks is more economical and time-saving. The blocks can be saved separately and therefore they occupy less memory. In the following sections the features of block tri-diagonal matrices are obtained and then block matrices are converted into the block tri-diagonal matrices using a block householder transformation. Thus, to increase the speed of the solution of block matrices one can utilize the characteristics of block tri-diagonal matrices for the block matrices.

4. BLOCK TRI-DIAGONAL MATRICES

Consider the block tri-diagonal matrix M

$$M=\begin{bmatrix} A_1 & C_1 & & & \\ B_1 & A_2 & \ddots & & \\ & \ddots & \ddots & C_{n-2} & \\ & & B_{n-2} & A_{n-1} & C_{n-1} \\ & & & B_{n-1} & A_n \end{bmatrix} \tag{20}$$

Where the blocks are matrices of dimension m. The determinant of matrix M is calculated as

$$\det M=(-1)^{mn}\det[T_{11}]\det[B_1 \ldots B_{n-1}] \tag{21}$$

Where matrix T_{11}, of the dimension m, is the block on the top left hand of matrix T shown bellow

$$T=\begin{bmatrix} -A_n & -C_{n-1} \\ I_m & 0 \end{bmatrix}\begin{bmatrix} -B_{n-1}^{-1}A_{n-1} & -B_{n-1}^{-1}C_{n-2} \\ I_m & 0 \end{bmatrix}\ldots\begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1} \\ I_m & 0 \end{bmatrix} \tag{22}$$

To calculate the eigenvalues, we will have

$$\det M(\lambda)=(-1)^{mn}\det[T_{11}(\lambda)]\det[B_1 \ldots B_{n-1}]=0 \tag{23}$$

And then

$$\det[T_{11}(\lambda)]=0 \tag{24}$$

Where matrix $T_{11}(\lambda)$, of the dimension m, is the block on the top left hand of matrix $T(\lambda)$ shown bellow

$$T(\lambda)=\begin{bmatrix} -A_n-\lambda I_m & -C_{n-1} \\ I_m & 0 \end{bmatrix}\begin{bmatrix} -B_{n-1}^{-1}(A_{n-1}-\lambda I_m) & -B_{n-1}^{-1}C_{n-2} \\ I_m & 0 \end{bmatrix}\ldots\begin{bmatrix} -B_1^{-1}(A_1-\lambda I_m) & -B_1^{-1} \\ I_m & 0 \end{bmatrix} \tag{25}$$

Therefore, in each multiplication of two matrices one should only multiply the second matrix by the first block row of the first matrix. Consequently, the characteristic equation is found just using vector multiplication. The application of the method is in finding the natural frequencies of the structures which are composed of part by part substructures. Regular structures are a specific kind of these structures.

In the next section using block householder transformation a method for converting symmetric block matrices into the block tri-diagonal matrices is presented. The conversion makes the opportunity of using the fast solution of block tri-diagonal matrices for the symmetric block matrices.

5. CONVERTING SYMMETRIC BLOCK MATRICES INTO THE BLOCK TRI-DIAGONAL MATRICES USING BLOCK HOUSEHOLDER TRANSFORMATION

Block householder transformation is an appropriate method for obtaining the eigenvalues of large symmetric matrices. The method is specifically useful for solving large stiffness matrices since they can be saved by their blocks, occupying less memory. The approach for changing a block matrix to a similar block tri-diagonal matrix is presented bellow
Product p of householder transformation is written as

$$H_1 H_2 \dots H_p = I - WY^T \tag{26}$$

Where $W$and Y are matrices including p columns. Consider block symmetric matrix C as follows

$$C = \begin{bmatrix} C_{1,1} & {C_{2,1}}^T & \dots & {C_{m,1}}^T \\ C_{2,1} & C_{2,2} & \dots & {C_{m,2}}^T \\ \vdots & \vdots & \ddots & C_{n-2} \\ C_{m,1} & C_{m,2} & & C_{m,m} \end{bmatrix} \tag{27}$$

Consider matrix M as the first block column of matrix C except for the diagonal block

$$M = \begin{bmatrix} C_{2,1} \\ C_{3,1} \\ \vdots \\ C_{m,1} \end{bmatrix} \tag{28}$$

C
Onsider H_1, H_2,... and H_p as householder transformations of QR decomposition of matrix M in which H_is hold the following pattern

$$H_i = I - 2u_i {u_i}^T \tag{29}$$

Householder transformations are put in the matrix WY (composed of w_i columns for W and y_i for Y)

$$\begin{gathered} w_i = 2u_i \\ y_i = H_p H_{p-1} \dots H_{i+1} u_i \end{gathered} \tag{30}$$

It can be shown

$$H_1 H_2 \dots H_p = I - WY^T \tag{31}$$

Consider transformation matrix Q which does not change eigenvalues

$$Q = \begin{bmatrix} I & 0 \\ 0 & I - WY^T \end{bmatrix} \quad (32)$$

Using the following transformation, we will have

$$Q^T C Q = \begin{bmatrix} \tilde{C}_{1,1} & R^T & 0 & \dots & 0 \\ R & \tilde{C}_{2,2} & \tilde{C}_{3,2}{}^T & \dots & \tilde{C}_{m,2}{}^T \\ 0 & \tilde{C}_{3,2} & \tilde{C}_{3,3} & \dots & \tilde{C}_{m,3}{}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{C}_{m,2} & \tilde{C}_{m,3} & \dots & \tilde{C}_{m,m} \end{bmatrix} \quad (33)$$

Where matrix R is the upper triangular matrix obtained from QR decomposition of matrix M and $\tilde{C}_{i,j}$s show the blocks of new matrix C obtained from Eq. (33). If $C^{(2)}$ is the matrix gained through eliminating the first block row and column of matrix C, we will have

$$\tilde{C}^{(2)} = (I - WY^T)^T C^{(2)} (I - WY^T) \quad (34)$$

Or

$$\tilde{C}^{(2)} = C^{(2)} - YW^T C^{(2)} - C^{(2)} WY^T + YW^T C^{(2)} WY^T \quad (35)$$

Replacing S = $C^{(2)}$W, V = W^T $C^{(2)}$W and T = S – 0.5Y, we will have

$$\tilde{C}^{(2)} = C^{(2)} - YT^T - TY^T \quad (36)$$

Therefore, the computational cost of $\tilde{C}^{(2)}$is reduced to the computational cost of some multiplications of W, V and $C^{(2)}$. These multiplications are efficiently done using vector computers. The matrix is converted into a block tri-diagonal one by applying m-1 steps. This operation is not an iterative operation but it is an exact one with limited steps. Now, the obtained block tri-diagonal matrices can be solved using the proposed method in the previous section.

6. EXAMPLE

Consider the 28-story tall building shown in Fig. 1. The front view and scheme of the building are shown in Fig. 2 and Fig. 3, respectively. According to Fig. 1, the first 22 stories are similar and the top 6 stories are the same.

Figure 1. The 28-story tall building

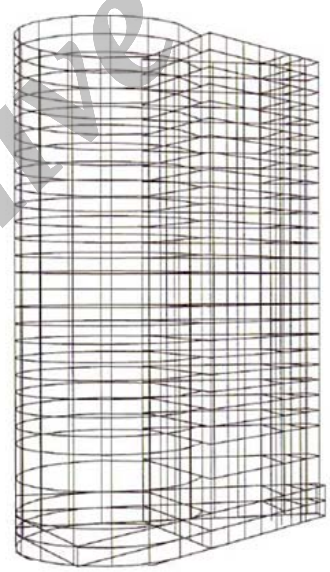

Figure 2. The scheme of the building

Considering these two parts, the stiffness matrix will have the form below

$$K=\begin{bmatrix}
A_1 & B^T{}_1 & & & & & & & & & & & & & \\
B_1 & A_1 & B^T{}_1 & & & & & & & & & & & & \\
 & B_1 & A_1 & B^T{}_1 & & & & & & & & & & & \\
 & & B_1 & A_1 & \ddots & & & & & & & & & & \\
 & & & B_1 & \ddots & B^T{}_1 & & & & & & & & & \\
 & & & & \ddots & A_1 & B^T{}_1 & & & & & & & & \\
 & & & & & B_1 & A_1 & B^T{}_1 & & & & & & & \\
 & & & & & & B_1 & A_1 & B^T{}_1 & & & & & & \\
 & & & & & & & B_1 & A_1 & B^T{}_1 & & & & & \\
 & & & & & & & & B_1 & A_2 & B^T{}_2 & & & & \\
 & & & & & & & & & B_2 & A_2 & \ddots & & & \\
 & & & & & & & & & & B_2 & \ddots & B^T{}_2 & & \\
 & & & & & & & & & & & \ddots & A_2 & B^T{}_2 & \\
 & & & & & & & & & & & & B_2 & A_2 & B^T{}_2 \\
 & & & & & & & & & & & & & B_2 & A_3
\end{bmatrix}$$

Consider Eq. (25)

$$T(\lambda)=\begin{bmatrix}-A_n-\lambda I_m & -C_{n-1}\\ I_m & 0\end{bmatrix}\begin{bmatrix}-B_{n-1}^{-1}(A_{n-1}-\lambda I_m) & -B_{n-1}^{-1}C_{n-2}\\ I_m & 0\end{bmatrix}\dots\begin{bmatrix}-B_1^{-1}(A_1-\lambda I_m) & -B_1^{-1}\\ I_m & 0\end{bmatrix}$$

In this example n = 28 and we have

$$\begin{cases} A_i=A_1, \quad B_i=B_1, \quad C_i=B^T{}_1, & 1\le i\le 22\\ A_i=A_2, \quad B_i=B_2, \quad C_i=B^T{}_2, & 23\le i\le 27\\ A_i=A_3 & i=28\end{cases}$$

Then, the characteristic equation will be obtained from the following relationships

$$T(\lambda)=\begin{bmatrix}-A_3-\lambda I_m & -B_2^T\\ I_m & 0\end{bmatrix}\begin{bmatrix}-B_2^{-1}(A_2-\lambda I_m) & -B_2^{-1}B_2^T\\ I_m & 0\end{bmatrix}^6\begin{bmatrix}-B_1^{-1}(A_1-\lambda I_m) & -B_1^{-1}B_1^T\\ I_m & 0\end{bmatrix}^{20}\begin{bmatrix}-B_1^{-1}(A_1-\lambda I_m) & -B_1^{-1}\\ I_m & 0\end{bmatrix}$$

And

$$\det[T_{11}(\lambda)]=0$$

After obtaining eigenpairs, the inverse of stiffness matrix is calculated by Eq. (12).

7. CONCLUSIONS

In this paper methods for finding the eigenpairs and inverse of tri-diagonal and block tri-diagonal matrices are presented. In these methods the characteristic equation of the matrices are efficiently calculated. Many structures hold these tri-diagonal and block tri-diagonal patterns and therefore their calculations are of great importance. Fortunately, the common structures, which do not hold the above mentioned patterns, can be easily converted into the tri-diagonal and block tri-diagonal matrices. These conversions are feasible just because the stiffness matrices are symmetric. The transformations are done using householder methods. Applying the block householder is more efficient in large structures because working with blocks occupies less memory.

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