# Characterizations of Graded Distributive Modules

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**Abstract**

Let $R$ be a $\mathbb{Z}$- graded commutative ring with identity. Several characterizations of graded distributive modules will be investigate.

***Keywords:*** Graded Distributive Modules, Krull Associated Primes, Weak Bourbaki Associated Primes.

## 1 Introduction

Let $R$ be a $\mathbb{Z}$-graded commutative ring with non-zero identity and $M$ be a $\mathbb{Z}$-graded $R$-module. We shall say that $M$ is a *graded distributive* module (for brevity a g.d module) if the lattice of its graded submodules is distributive, i.e., if $(X+Y)\cap Z=(X\cap Z)+(Y\cap Z)$ for all graded submodules $X,Y,Z$ of $M$ (or equivalently, $(X\cap Y)+Z=(X+Z)\cap(Y+Z)$ for all graded submodules $X,Y,Z$ of $M$). The notion of distributive modules has been introduced and studied independently by T. M. K. Davison [2] and W. Stephenson [12]. There are many important and considerable research on the structure and characterization of distributive modules (see for example [3, 4, 5, 13, 14]), however, to the best of author knowledge there are few results concerning the graded version of this concept [6, 10].

In this paper we will give several characterizations of g.d modules. In fact, among other things, we prove:

**Theorem A** *Let $M$ be a tortion free graded $R$ - module. The following statements are equevalent.*

(i) *$M$ is a g.d $R$-module.*

(ii) *Every $^*$closed submodule of $M$ is $^*$irreducible.*

(iii) *For any $i\in\mathbb{Z}$, any $x,y\in M_i$ and any $^*$maximal ideal $\mathfrak{m}$, the graded submodules $Rx(\mathfrak{m})$ and $Ry(\mathfrak{m})$ are comparable with respect to inclusion.*



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(iv) *For each graded submodule* $N$ *of* $M$ *and each* $^*$*maximal ideal* $\mathfrak{m}$ *containing* $N:_R M$, $^*E_R(M/N(\mathfrak{m}))$ *is *indecomposable.*

**Theorem B** *For a graded torsion free* $R$*-module* $M$, *the following statements are equivalent.*

(i) $M$ *is a g.d module.*

(ii) *For each proper graded submodule* $N$ *of* $M$, $N=\bigcap_{\mathfrak{p}\in mK(M/N)} N(\mathfrak{p})$ *is an* $^*$*irreducible decomposition of* $N$.

To prove Theorems **A** and **B**, we need a series of assertions. We present the necessary notation and definitions. Let $R=\oplus_{i\in \mathbb{Z}} R_i$ and let $M=\oplus_{i\in \mathbb{Z}} M_i$. Then the elements of $M_i$ are called homogeneous of degree $i$. The set of all homogeneous elements of $R$ (resp. $M$) is denoted by $H(R)$ (resp. $H(M)$). Given multiplicatively closed subset $S\subseteq H(R)$, the ring of fractions $S^{-1}R$ turns into a graded ring by setting

$$(S^{-1}R)_i=\{r/s : r\in H(R), s\in S, i=deg(r)-deg(s)\}$$

for each $i\in \mathbb{Z}$, where $deg(r)$ represents the degree of the homogeneous element $r$. We recall that $S^{-1}M$ can be defined as $S^{-1}R\otimes_R M$, which is a graded $S^{-1}R$- module. In the case that $\mathfrak{p}$ is a graded prime ideal and $S=H(R)\setminus\mathfrak{p}$, the graded ring $S^{-1}R$ (resp. graded $S^{-1}R$-module $S^{-1}M$) is denoted by $R_{(\mathfrak{p})}$ (resp. $M_{(\mathfrak{p})}$), and is called the homogeneous localization of $R$ (resp. $M$) at $\mathfrak{p}$. A graded ideal $\mathfrak{m}$ is called $^*$*maximal* if it is maximal in the lattice of all graded ideals of $R$. The ring $R$ is called $^*$*quasi local* if it has a unique $^*$maximal ideal. Let $N$ be a graded submodule of $M$ and let $\mathfrak{p}$ be a graded prime ideal of $R$. We set $N(\mathfrak{p})=\cup_{s\in H(R)\setminus\mathfrak{p}}(N:_M s)$, which is a graded submodule of $M$ containing $N$. We note that when $M$ is a torsion free $R$-module (that is when $\{x\in M : rs=0$ for some nonzero $r\in R\}=\{0\}$), then evidently $N(\mathfrak{p})=N_{(\mathfrak{p})}\cap M$. We set

$$Z(N)=\{a\in R : N\subset (N:_M a)\}.$$

Then $R\setminus Z(N)$ is a multiplicatively closed subset of $R$. We say that $N$ is a $^*$closed submodule of $M$ if $Z(N)$ itself forms an ideal $\mathfrak{p}$ of $R$. In this case $\mathfrak{p}$ is a prime ideal of $R$ and we say that $N$ is $\mathfrak{p}-^*$closed. Indeed then $Z(N)$ is a graded prime ideal. To see this let $a=a_m+...+a_n\in Z(N)$ be the decomposition of $a$ as a sum of homogeneous elements $a_i$. Then there exists $x\in H(M)\setminus N$ such that $a_m x...+a_n x=ax\in N$. Since $N$ is graded this gives that $a_i x\in N$ for each $i=m,...,n$. It follows that $N\subset N:_M a_i$ for each $i=m,...,n$, i.e., each homogeneous components of $a$ belongs to $Z(N)$.

Finally we say that $N$ is an $^*$*irreducible* submodule of $M$ precisely when for graded submodules $N_1, N_2$ of $M$, $N = N_1 \cap N_2$ implies that either $N = N_1$ or $N = N_2$.

**Proposition 1** *Let $N$ be an $^*$irreducible submodule of $M$. Then $N$ is a $^*$closed submodule of $M$.*

*Proof.* Let $r, s \in Z(N)$. It follows that $N \subset (N :_M r)$ and $N \subset (N :_M s)$. Hence by assumption $N \subset (N :_M r) \cap (N :_M s) = (N :_M r - s)$, which means that $r - s \in Z(N)$. Since the product of an element of $R$ and an element of $Z(N)$ is always a element of $Z(N)$, the claim follows.

**Proposition 2** *Every graded submodule of $M$ is the intersection of $^*$closed submodules.*

Proof. Let $N$ be a graded submodule of $M$. Since the module $M$ itself, being $^*$irreducible, is $^*$closed, so the intersection of all $^*$closed submodules of $M$ containing $N$ is non-empty. Hence to prove the claim it is enough to show that for each $m \in H(M) \setminus N$ there exist an $^*$closed submodule $C$ of $M$ containing $N$ such that $m$ is not in $C$. Let $\sum = \{L \supseteq N : L \, is \, a \, graded \, submodule \, of \, M \, donot \, contain \, m\}$. Then $\sum$ is not empty and by Zorn's lemma it possesses a maximal element with respect to inclusion, say $C$. We show that $C$ is a $^*$closed submodule of $M$. Let $r, s \in Z(C)$. Then there exist $x, y \in H(M) \setminus C$ such that $rx$, $sy \in C$. Now by the maximality of $C$ we have $m \in C + Rx$ and $m \in C + Ry$. This gives that $rm \in rC + Rrx \subseteq C$ and $sm \in sC + Rsy \subseteq C$. Therefore $(r - s)m \in C$ and so $r - s \in Z(C)$. Consequently $C$ is an $^*$closed submodule of $M$.

**Lemma 3** *The following statements are equivalent.*

(i) *$M$ is a g.d $R$-module.*

(ii) $(Rx :_R y) + (Ry :_R x) = R$ *for all* $x, y \in H(M)$ *with* $\deg(x) = \deg(y)$.

*Furthermore if $R$ is $^*$quasi local, then each of the above is equivalent to*

(iii) *The set of all graded submodules of $M$ are linearly ordered with respect to inclusion.*

(iv) *The set of all graded cyclic submodules of $M$ is linearly ordered with respect to inclusion.*

*Proof.* (i)$\Rightarrow$(ii). Let $x, y \in H(M)$ be such that $\deg(x)=\deg(y)$. Then we have $x \in Rx \cap (Ry + R(x - y))$. By assumption it follows that $x \in Rx \cap Ry + Rx \cap R(x - y)$. Hence there exist $r, s \in R_0$ such that $x = ry + s(x - y)$. From this we deduce that $sy \in Rx$. On the other hand we have $(1 - s)x = (r - s)y$, which imply that $1 - s \in (Ry :_R x)$. Therefore $1 = s + (1 - s) \in (Rx :_R y) + (Ry :_R x)$, as desired.

(ii)$\Rightarrow$(i). Let $X, Y$ and $Z$ be graded submodules of $M$. Let $x \in X \cap (Y + Z)$ be a homogeneous element. Then there exist homogeneous elements $y \in Y$ and $z \in Z$ such that $x = y + z$ and $\deg(x)=\deg(y)=\deg(z)$. By assumption we have $(Rx :_R y) + (Ry :_R x) = R$. Therefore there exists $r \in R$ such that $r \in (Rx :_R y)$ and

$1-r\in(Ry:_R x)$. Hence we have $x=(1-r)x+ry+rz=sy+rz$, for some $s\in R$. Now $x\in(Rx\cap Ry)+(Rx\cap Rz)\subseteq(X\cap Y)+(X\cap Z)$. As the opposite inclusion always holds, the result follows.

(iii)$\Rightarrow$(iv) is clear.

(iv)$\Rightarrow$(iii) Assume (iii) dose not hold. Then there exist graded submodyls $X,Y$ of $M$ such that $X\not\subseteq Y$ and $Y\not\subseteq X$. This gives that there exist $x\in H(X)\setminus Y$ and $y\in H(Y)\setminus X$. Hence $Rx\not\subseteq Ry$ and $Ry\not\subseteq Rx$, which contradicts to (iv).

Furthermore if $R$ is $^*$quasi local it has been proved in [6. Lemma 5.22], that (i) and (iv) are equivalent.

**Lemma 4** ([6. Lemma 5. 24]) *$M$ is a g.d $R$-module if and only if $M_{(\mathfrak{p})}$ is a g.d $R_{(\mathfrak{p})}$-module for each graded prime ($^*$maximal) ideal $\mathfrak{p}$ of $R$.*

**Lemma 5** *For an $R$-module $M$ the following statements are equivalent.*

(i) *$M$ is a g.d $R$-module.*

(ii) *For each proper graded submodule $N$ of $M$ and each graded prime ($^*$maximal) ideal $\mathfrak{p}$ of $R$, $N_{(\mathfrak{p})}$ is an $^*$irreducible submodule of $M_{(\mathfrak{p})}$.*

Proof. (i)$\Rightarrow$(ii). Let $N$ be a proper graded submodule of $M$ and let $\mathfrak{p}$ be a graded prime ideal of $R$ which contains $N:_R M$. Let $N_{(\mathfrak{p})}=K_{(\mathfrak{p})}\cap L_{(\mathfrak{p})}$. (Note that each graded submodule of $M_{(\mathfrak{p})}$ can be written as a homogeneous localization of some graded submodule of $M$ at $\mathfrak{p}$.) By Lemma 4, $M_{(\mathfrak{p})}$ is a g.d module over the $^*$quasi local ring $R_{(\mathfrak{p})}$. Therefore by Lemma 3, either $L_{(\mathfrak{p})}\subseteq K_{(\mathfrak{p})}$ or $K_{(\mathfrak{p})}\subseteq L_{(\mathfrak{p})}$, i.e. either $N_{(\mathfrak{p})}=L_{(\mathfrak{p})}$ or $N_{(\mathfrak{p})}=K_{(\mathfrak{p})}$. Thus $N_{(\mathfrak{p})}$ is irreducible.

(ii)$\Rightarrow$(i). In view of Lemmas 3 and 4 it is enough to prove that for each graded prime ideal $\mathfrak{p}$ of $R$, any two graded submodules of $M_{(\mathfrak{p})}$ are comparable. So let $\mathfrak{p}$ be a graded prime ideal of $R$ and let $K_{(\mathfrak{p})}$, $L_{(\mathfrak{p})}$ be proper graded submodules of $M_{(\mathfrak{p})}$. We may assume that $(M:_R K\cap L)\subseteq\mathfrak{p}$. So by assumption $K_{(\mathfrak{p})}\cap L_{(\mathfrak{p})}=(K\cap L)_{(\mathfrak{p})}$ is an $^*$irreducible submodule of $M_{(\mathfrak{p})}$, so either $K_{(\mathfrak{p})}\subseteq K_{(\mathfrak{p})}\cap L_{(\mathfrak{p})}$ or $L_{(\mathfrak{p})}\subseteq K_{(\mathfrak{p})}\cap L_{(\mathfrak{p})}$. Consequently either $K_{(\mathfrak{p})}\subseteq L_{(\mathfrak{p})}$ or $L_{(\mathfrak{p})}\subseteq K_{(\mathfrak{p})}$ and the result follows.

**Lemma 6** *Let $N$ be a graded submodule of $M$ and $\mathfrak{p}$ be a graded prime ($^*$maximal) ideal of $R$. If $N_{(\mathfrak{p})}$ is an $^*$irreducible submodule of $M_{(\mathfrak{p})}$, then $N(\mathfrak{p})$ is an $^*$irreducible* submodule *of $M$. Furthermore if $M$ is torsion free, the converse holds.*

*Proof.* ($\Rightarrow$). Let $N(\mathfrak{p})=K\cap L$ for some graded submodules $K,L$ of $M$. By homogeneous localizing at $\mathfrak{p}$ and using the fact that $(N(\mathfrak{p}))_{(\mathfrak{p})}=N_{(\mathfrak{p})}$, we have $N_{(\mathfrak{p})}=K_{(\mathfrak{p})}\cap L_{(\mathfrak{p})}$. Hence by assumption either $N_{(\mathfrak{p})}=K_{(\mathfrak{p})}$ or $N_{(\mathfrak{p})}=L_{(\mathfrak{p})}$, which gives that either $N(\mathfrak{p})=K$ or $N(\mathfrak{p})=L$.

($\Leftarrow$). Let $K, L$ be graded submodules of $M$ such that $N_{(\mathfrak{p})} = K_{(\mathfrak{p})} \cap L_{(\mathfrak{p})}$. This gives that $N(\mathfrak{p}) = N_{(\mathfrak{p})} \cap M = (K_{(\mathfrak{p})} \cap M) \cap (L_{(\mathfrak{p})} \cap M) = K(\mathfrak{p}) \cap L(\mathfrak{p})$. Thus by assumption we have $N(\mathfrak{p}) = L(\mathfrak{p})$ or $N(\mathfrak{p}) = K(\mathfrak{p})$ and so by homogeneous localizing at $\mathfrak{p}$, $N_{(\mathfrak{p})} = K_{(\mathfrak{p})}$ or $N_{(\mathfrak{p})} = L_{(\mathfrak{p})}$.

From the above observations we deduce the following corollary.

**Corollary 7** *Let $M$ be a torsion free $R$-module. Then the following statements are equivalent.*

(a) *$M$ is a g.d $R$-module.*

(b) *For each graded submodule $N$ of $M$, each graded prime ($^*$maximal) ideal $\mathfrak{p}$ of $R$,*

*$N(\mathfrak{p})$ is an $^*$irreducible submodule of $M$.*

**Lemma 8** *Let $N$ be a finitely generated graded submodule of $M$ and let $\mathfrak{p}$ be a graded prime ideal of $R$. Assume that $N_{(\mathfrak{p})} \neq 0$. Then $(\mathfrak{p}N)(\mathfrak{p})$ is an $^*$closed submodule of $M$.*

Proof. We show that $Z((\mathfrak{p}N)(\mathfrak{p})) = \mathfrak{p}$. First let $r \in Z((\mathfrak{p}N)(\mathfrak{p}))$. Then there exists $m \in H(M) \setminus (\mathfrak{p}N)(\mathfrak{p})$ such that $rm \in (\mathfrak{p}N)(\mathfrak{p})$. It follows that $(\mathfrak{p}N :_R m) \subseteq \mathfrak{p}$ and that there exists $t \in H(R) \setminus \mathfrak{p}$ such that $rtm \in \mathfrak{p}N$. Hence $rt \in \mathfrak{p}$ and so $r \in \mathfrak{p}$. Consequently $Z((\mathfrak{p}N)(\mathfrak{p})) \subseteq \mathfrak{p}$. In order to prove the other inclusion, let $r \in \mathfrak{p}$. Since $N$ is finitely generated and $N_{(\mathfrak{p})} \neq 0$, using the graded version of Nakayama's Lemma (see[11, Lemma I.7.5]), we have $(\mathfrak{p}N)_{(\mathfrak{p})} \neq N_{(\mathfrak{p})}$. This gives that $(\mathfrak{p}N)(\mathfrak{p}) \neq N(\mathfrak{p})$. Since in any case we have $(\mathfrak{p}N)(\mathfrak{p}) \subseteq N(\mathfrak{p})$ and $N(\mathfrak{p}) \subseteq (\mathfrak{p}N)(\mathfrak{p}) :_M \mathfrak{p}$, so $(\mathfrak{p}N)(\mathfrak{p}) \subset (\mathfrak{p}N)(\mathfrak{p}) :_M \mathfrak{p}$. This gives that there exists $x \in H(M) \setminus (\mathfrak{p}N)(\mathfrak{p})$ such that $rx \in (\mathfrak{p}N)(\mathfrak{p})$, i.e., $(\mathfrak{p}N)(\mathfrak{p}) \subset ((\mathfrak{p}N)(\mathfrak{p}) :_M r)$. Hence $r \in Z((\mathfrak{p}N)(\mathfrak{p}))$ and the proof is complete.

Following [15, p. 72], we define a prime ideal $\mathfrak{p}$ of $R$ to be a *Krull associated prime* of $M$ if for every element $t \in \mathfrak{p}$, there exists $x \in M$ such that $t \in 0 :_R x \subseteq \mathfrak{p}$. We denote by $K(M)$ (resp. by $mK(M)$) the set of all Krull associated primes of $M$ (resp. the set of all maximal members of $K(M)$). Since $M$ is a graded $R$ module, then each element of $\mathfrak{p} \in K(M)$ must be graded; furthermore for each element $t \in \mathfrak{p}$, we can choose a homogeneous element $x$ such that $t \in 0 :_R x \subseteq \mathfrak{p}$. To see this let $\mathfrak{p} \in K(M)$. Let $t_m + \ldots + t_n = t \in \mathfrak{p}$ be the decomposition of $t$ as a sum of homogeneous elements $t_i$ of degree $i$. By assumption there exists $x_u + \ldots + x_v = x \in M$ such that $tx = 0$ and $0 :_R x \subseteq \mathfrak{p}$. So we have the equations $\sum_{i+j=s} t_i x_j = 0$ for $s = m+u, \ldots, n+v$. It follows that $t_m x_u = 0$, and by induction, $t_m^i x_{u+i-1} = 0$ for all $i \geq 1$. Therefore $t_m^l x = 0$ for sufficiently large value of $l$. As $\mathfrak{p}$ is prime ideal, we have $t_m \in \mathfrak{p}$. Iterating this procedure we see that each homogeneous component of $t$ belongs to $\mathfrak{p}$. In order to prove the second assertion, we have $\cap_{i=u}^{i=v} (0 :_R x_i) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, there exists $j$ with $(0:_R x_j) \subseteq \mathfrak{p}$. As $t \in (0:_R x_i)$ for all $i = u,...,v$; the proof of the claim is complete.

A prime ideal $\mathfrak{p}$ is called *weak Bourbaki associated prime* of $M$ if it is minimal prime divisor of $0:_R x$ for some $x \in M$. We will denote the set of all weak Bourbaki associated primes of $M$ by $wB(M)$. It is known that (see for example [9, Lemma 2.15]), the set $wB(M)$ is non empty. The fact that the set $K(M)$ is not empty and that each element of $wB(M)$ is graded follows from the following.

**Proposition 9** ([8, Theorem 1]) *With the above notation* $\mathrm{wB}(M) \subseteq \mathrm{K}(M)$.

Let ${}^*Spec(R)$ be the set of all graded prime ideals of $R$. It should be noted that for each proper submodule $N$ of $M$, $N = \cap_{\mathfrak{p} \in {}^*Spec(R)} N(\mathfrak{p})$. The components $N(\mathfrak{p})$ in this representation in general do not need to be ${}^*$closed. However if we focus our attention on the graded prime ideals which belongs to $\mathrm{mK}(M/N)$, then we have a representation of $N$ such that each component is ${}^*$closed. In fact:

**Theorem 10** *Let $N$ be a proper graded submodule of $M$. Then we have $N = \bigcap_{\mathfrak{p} \in mK(M/N)} N(\mathfrak{p})$, where the components $N(\mathfrak{p})$ are $\mathfrak{p}-{}^*$closed submodules with distinct and incomparable graded primes $Z(N(\mathfrak{p})) = \mathfrak{p}$.*

Proof. First we note that $N = \bigcap_{\mathfrak{p} \in mK(M/N)} N(\mathfrak{p})$. To see this, let $x \in H(M) \setminus N$. Let $\mathfrak{q}$ be a minimal prime divisor of $N:_R x$. Then there exists $\mathfrak{p} \in mK(M/N)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Hence $N:_R x \subseteq \mathfrak{p}$ and so $x$ is not an element of $N(\mathfrak{p})$. Consequently $x$ is not in $\bigcap_{\mathfrak{p} \in mK(M/N)} N(\mathfrak{p})$ and we deduce the claim.

Now to complete the proof it suffices to prove that if $\mathfrak{p} \in K(M/N)$, then $N(\mathfrak{p})$ is $\mathfrak{p}-{}^*$closed; i.e., $Z(N(\mathfrak{p})) = \mathfrak{p}$. To this end, assume that $r = r_m + ... + r_n$ is not an element of $\mathfrak{p}$. Then there exists $m \le j \le n$ such that $r_j$ is not in $\mathfrak{p}$. We show that $(N(\mathfrak{p}) :_M r) = N(\mathfrak{p})$. To this end, let $x \in H(M)$ such that $rx \in N(\mathfrak{p})$. Since $N(\mathfrak{p})$ is a graded submodule of $M$, this gives that all homogeneous components of $rx$ are in $N(\mathfrak{p})$, in particular $r_j x \in N(\mathfrak{p})$. This gives that there exists $s \in H(R) \setminus \mathfrak{p}$ such that $sr_j x \in N$ and so $x \in N(\mathfrak{p})$. So $Z(N(\mathfrak{p})) \subseteq \mathfrak{p}$. To prove the other inclusion let $r \in \mathfrak{p}$. Since $\mathfrak{p} \in K(M/N)$, there exists $x \in M \setminus N(\mathfrak{p})$ such that $r \in N(\mathfrak{p}) :_R x \subseteq \mathfrak{p}$. Therefore we have $N(\mathfrak{p}) \subset N(\mathfrak{p}) :_M r$ and so $r \in Z(N(\mathfrak{p}))$.

The Category of graded $R$ modules has as objects the graded $R$-modules. A morphism $f : M \to M'$ in this category is an $R$-module homomorphism satisfying $f(M_i) \subseteq M'_i$ for all $i \in \mathbb{Z}$. A graded $R$-module $E$ is called ${}^*$*injective* if it is an injective object in this category. One call an extension $N \subset M$ of graded $R$-modules ${}^*$*essential extension* if for any graded submodule $0 \neq U \subset M$ one has $U \cap N \neq 0$. In analogy to the definition in the non-graded case, $E$ is called a ${}^*$*injective hull* of $N$ if it is $^*$injective and $^*$essential extension of $N$. In view of [1, 3.6.2], any graded module $R$-module $X$ admits a unique $^*$injective hull up to isomorphism. We denote the $^*$injective hull of $X$ by $^*E(X)$. A graded $R$-module is said to be $^*$*indecomposible* precisely when it is non-zero and cannot be written as the direct sum of two proper graded submodules. By a similar argument as in the non-graded case one can see easily that a graded submodule $U$ of $M$ is $^*$irreducible if and only if $^*E(M/U)$ is $^*$indecomposible.

**Proof of Theorem A**

(i)$\Rightarrow$(ii). Let $N$ be an $^*$closed proper submodule of $M$ with $Z(N)=\mathfrak{p}$. This gives that $N(\mathfrak{p})=N$. Now the result follows by Corollary 7.

(ii)$\Rightarrow$(i). By virtue of Lemma 4, it is enough to show that for each $^*$maximal ideal $\mathfrak{m}$ of $R$, $M_{(\mathfrak{m})}$ is a g.d module over the $^*$quasi local ring $R_{(\mathfrak{m})}$. To do this, by Lemma 3, it suffices to prove that for any $x,y\in H(M)$, either $<x/1>\subseteq<y/1>$ or $<y/1>\subseteq<x/1>$. To this end, let $N=<x,y>$ and $N_{(\mathfrak{m})}\neq 0$. (If $N_{(\mathfrak{m})}=0$, then $x/1=y/1=0$ and there is nothing to prove.) Then by Lemma 8, $(\mathfrak{m}N)(\mathfrak{m})$ is an $^*$closed and so by our assumption an $^*$irreducible submodule of $M$. Hence by Lemma 6, $(\mathfrak{m}N)_{(\mathfrak{m})}$ is an $^*$irreducible submodule of $M_{(\mathfrak{m})}$. But $R_{(\mathfrak{m})}/(\mathfrak{m})R_{(\mathfrak{m})}$ is either a field or is of the form $k[t,t^{-1}]$, where $t$ is a homogeneous element of positive degree which is transcendental over $k$ (see [1, Lemma 1.5.7]). Since by [7, Lemma 1.1.1], any graded module over $k[t,t^{-1}]$ is graded free, this gives that either $N_{(\mathfrak{m})}/\mathfrak{m}N_{(\mathfrak{m})}$ is a finite dimensional vector space over the field $R_{(\mathfrak{m})}/\mathfrak{m}R_{(\mathfrak{m})}$, or is a rank one graded free module over $R_{(\mathfrak{m})}/\mathfrak{m}R_{(\mathfrak{m})}$. Therefore in any case we have either $N_{(\mathfrak{m})}=<x/1>+(\mathfrak{m}N)_{(\mathfrak{m})}$ or $N_{(\mathfrak{m})}=<y/1>+(\mathfrak{m}N)_{(\mathfrak{m})}$. Hence by the graded version of Nakayama's Lemma $N_{(\mathfrak{m})}=<x/1>$ or $N_{(\mathfrak{m})}=<y/1>$, which gives that either $<x/1>\subseteq<y/1>$ or $<y/1>\subseteq<x/1>$ and the result follows.

(i)$\Rightarrow$(iii). Suppose the contrary; i.e., there exist $i\in\mathbb{Z}$, $x,y\in M_i$ and an $^*$maximal ideal $\mathfrak{m}$ such that $Rx(\mathfrak{m})\not\subseteq Ry(\mathfrak{m})$ and $Ry(\mathfrak{m})\not\subseteq Rx(\mathfrak{m})$. It follows that $x$ is not an element of $Ry(\mathfrak{m})$ and $y$ is not an element of $Rx(\mathfrak{m})$. Our assumption together with Lemma 3, give that there exists $r\in R$, such that $rx\in Ry$ and $(1-r)y\in Rx$. Since $\mathfrak{m}$ is $^*$maximal ideal, at least one of the elements $r$, $1-r$ is not contained in $\mathfrak{m}$. So at least one of the homogeneous components of $r$ or one of the homogeneous components of $1-r$ is not contained in $\mathfrak{m}$. If the first possibility is true, it follows that $x\in Ry(\mathfrak{m})$, a contradiction. With the second possibility we come to the contradiction $y\in Rx(\mathfrak{m})$.

(iii)$\Rightarrow$(i). Assume that (i) does not hold. Then, by Lemma 3, there exist $i\in\mathbb{Z}$ and $x,y\in M_i$ and an $^*$maximal ideal $\mathfrak{m}$ of $R$ such that $(Ry:_R x)+(Rx:_R y)\subseteq\mathfrak{m}$. It follows that $y\in Ry(\mathfrak{m})\setminus Rx(\mathfrak{m})$ and $x\in Rx(\mathfrak{m})\setminus Ry(\mathfrak{m})$, contracting to (iii).



(i)$\Leftrightarrow$(iv). This follows by the paragraph before the proof of the theorem and Corollary 7.

The result of the previous Theorem is that for each graded submodule $N$ of a g.d module $M$, the representation of $N$ as an intersection of $^{*}$closed modules given in Theorem **A**, is a decomposition of $N$ into $^{*}$irreducible components. In theorem $B$, we show that this condition is in fact sufficient for $M$ to be g.d module.

**Proof of Theorem B**

(i)$\Rightarrow$(ii) follows from Theorems 10 and Theorem $A$.

(ii)$\Rightarrow$(i). By Lemma 3 and 4 it is enough to show that for each $^{*}$maximal ideal $\mathsf{m}$ of $R$ the graded cyclic submodules of $M_{(\mathsf{m})}$ are totally ordered. To this end, let $x, y \in H(M)$.

Set $N = \langle x, y \rangle$. Then by assumption $\mathsf{m}N = \bigcap_{\mathsf{p} \in mK(M/\mathsf{m}N)} (\mathsf{m}N)(\mathsf{p})$, is an $^{*}$irreducible decomposition of $\mathsf{m}N$. We claim that $\mathsf{m} \in mK(M/\mathsf{m}N)$. If this is not the case, then $(\mathsf{m}N)(\mathsf{p}) = N(\mathsf{p})$ for all $\mathsf{p} \in mK(M/\mathsf{m}N)$. This gives that $\mathsf{m}N = \bigcap_{\mathsf{p} \in MaxK(M/\mathsf{m}N)} N(\mathsf{p}) \supseteq N$.

But since $N$ is finitely generated graded $N Ú \mathsf{m}N$. Hence $\mathsf{m} \in mK(M/\mathsf{m}N)$ and $(\mathsf{m}N)(\mathsf{m})$ is an $^{*}$irreducible submodule of $M$. Thus by Lemma 6, $(\mathsf{m}N)_{(\mathsf{m})}$ is an $^{*}$irreducible submodule of $M_{(\mathsf{m})}$. The result now follows by the same argument as in the proof of the Theorem $A$ part (ii)$\Rightarrow$(i).

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