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Order dense injectivity of S-posets

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Abstract. In this paper, the notion of injectivity with respect to order dense embeddings in the category of S-posets, posets with a monotone action of a pomonoid S on them, is studied. We give a criterion, like the Baer condition for injectivity of modules, or Skornjakov criterion for injectivity of S-sets, for the order dense injectivity. Also, we consider such injectivity for S itself, and its order dense ideals. Further, we define and study some kinds of weak injectivity with respect to order dense embeddings, consider their relations with order dense injectivity. Also investigate if these kinds of injectivity are preserved or reflected by products, coproducts, and direct sums of S-posets.

1 Introduction and Preliminaries

The actions of a monoid on sets and on partially ordered sets have many explicit and implicit applications in almost every mathematical and related disciplines. Combining these two notions, one can get the more rich category **Pos**-S of partially ordered sets with actions of a pomonoid S on them. Some properties of S-posets have been studied by many authors, for example see [2–5, 11, 18].

The study of injectivity with respect to different classes of monomorphisms is crucial in many branches of mathematics. Many mathematicians studied this notion in different categories with respect to different classes of monomorphisms (for example, see [1, 7, 19]).

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It is well known that there is no non-trivial injective object with respect to monomorphisms in the categories **Pos** of posets and **Pos**-S of S-posets. The next natural subclass of monomorphisms is the class of embeddings, which are the regular (equalizer) monomorphisms. In [10, 14, 20, 21] injectivity of S-posets with respect to regular monomorphisms (embeddings) has been studied, in particular in [10] it is shown that there are enough regular injective S-posets. Another natural subclass of embeddings of S-posets is the class of *down closed embeddings*. In [15], injectivity of S-posets with respect to down closed regular monomorphisms has been studied.

In this paper, we study injectivity of S-posets with respect to the class of order dense regular monomorphisms, and call it *od-injectivity*. It will be seen that regular injectivity is equivalent to down closed regular injectivity plus order dense regular injectivity. Some homological characterizations of pomonoids over which all right od-ideals are od-injective are obtained. Also, we define some kinds of weak od-injectvity, consider their relations with od-injectivity, and investigate the behaviour of od-injectivity and weak od-injectivities with respect to products, coproducts, and direct sums of S-posets.

Now we give some preliminaries needed in the sequel. For more information see [1, 5, 7, 9, 12, 19].

Recall that a monoid (semigroup) S is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$). A (*right*) *S-poset* is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. An *S-poset map* (or *morphism*) is an action preserving monotone map between S-posets. Moreover, regular monomorphisms (equalizers) are exactly *order-embeddings*; that is, (mono-)morphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$.

Recall that an object A in a category $\mathcal{C}$ is called $\mathcal{M}$-*injective* if it is injective with respect to $\mathcal{M}$-morphisms; that is, for every morphism $f : B \to C$ in $\mathcal{M}$ and arbitrary morphism $g : B \to A$ there exists a morphism $h : C \to A$ such that $hf = g$. Also, A is said to be an $\mathcal{M}$-*retract* (or simply a *retract* if $\mathcal{M} = \mathcal{M}ono$) of its $\mathcal{M}$-extension $f : A \to B$ if f has a left inverse $g : B \to A$, called a *retraction*.

2 Order dense injectivity

In this section, we introduce a closure operator and dense monomorphisms with respect to which are the subject of study in this paper. In fact, we study injectivity with respect to this class and give a criterion to check such injectivity. Also, such injectivity is considered for S itself, and its order dense ideals. First note that, denoting the lattice of all sub S-posets of an S-poset B by Sub$B$, following [6] for the general definition of closure operators on a category (which is not a priori assumed to be idempotent), we get:

Definition 2.1. A family $C = (C_B)_{B\in\mathbf{Pos}-S}$, with $C_B : \mathrm{Sub}B \to \mathrm{Sub}B$, taking the subalgebra $A \leq B$ to $C_B(A)$, is called a *closure operator* on **Pos**-S if it satisfies the following laws:

(c_1) (*Extension*) $A \leq C_B(A)$,

(c_2) (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,

(c_3) (*Continuity*) $f(C_B(A)) \leq C_D(f(A))$, for all morphisms $f : B \to D$.

Now, one has two usual classes of monomorphisms related to a closure operator as follows:

Definition 2.2. Let $A \leq B$ be in **Pos**-S. We say that A is C-*closed* in B if $C_B(A) = A$, and it is C-*dense* in B if $C_B(A) = B$. Also, an S-poset map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) sub S-poset of B.

Now, we introduce the *down set closure operator* on the category of S-posets.

Definition 2.3. The *down set closure operator* $C^{\downarrow} = (C^{\downarrow}_B)_{B\in\mathbf{Pos}-S}$ on **Pos**-S is defined as

$$C^{\downarrow}_B(A) = \{b \in B : \exists a \in A, b \leq a\}$$

for any sub S-poset A of an S-poset B.

Note that, by Definition 2.2, a sub S-poset A of an S-poset B is $C^{\downarrow}$-dense, which will also be called *order dense*, in B if for each $b \in B$ there exists $a \in A$ with $b \leq a$. Also, a sub S-poset A of an S-poset B is $C^{\downarrow}$-closed, which will also be called *down closed* in B, if for each $a \in A$ and $b \in B$ with $b \leq a$ one has $b \in A$.

Definition 2.4. (1) We call an S-poset A *order dense regular injective* or briefly *od-injective* if it is injective with respect to the class of order dense embeddings.

(2) We call an S-poset A *down closed regular injective* or briefly *dc-injective* if it is injective with respect to the class of down closed embeddings.

Remark 2.5. (1) Clearly one can take order dense embeddings in the above definition of od-injectivity to be order dense inclsuions of S-posets.

(2) A (finitely generated, principal) right ideal I of the semigroup S is (finitely generated, principal) order dense in S, which we also call it a (finitely generated, principal) right od-ideal, if $C^{\downarrow}_S I = S$.

(3) If A is a regular injective S-poset then it is clearly od-injective, but the converse is not necessarily true. For example, let A be any regular injective S-poset with more than two elements. Then it has a zero bottom element $\perp$ and zero top element $\top$ (see [10]), we also assume that A does not have any other zero element. Now $A - \{\top\}$ is od-injective, for, if $B \to C$ is an order dense embedding

and $f : B \to A - \{\top\}$ is an S-poset map, then since A is regular injective, there exists an S-poset map $\overline{f} : C \to A$ which extends ιf, where $\iota : A - \{\top\} \to A$ is the inclusion map. We claim that $Im\overline{f} \subseteq A - \{\top\}$ and so $\overline{f} : C \to A - \{\top\}$ is the required S-poset map which extends f. To see this, on the contrary, let $c \in C$ be such that $\overline{f}(c) = \top$, then since B is order dense in C, there exists an element $b \in B$ such that $c \leq b$. Thus $\top = \overline{f}(c) \leq \overline{f}(b) = \iota f(b) = f(b)$, and so $f(b) = \top$, which is a contradiction. Therefore, $A - \{\top\}$ is od-injective. But, it is not regular injective since it does not have a top element.

Proposition 2.6 ([15]). *A right G-poset over a pogroup G is dc-injective if and only if it is bounded from the top by a zero element.*

Proposition 2.7. *An (S-) poset P is regular injective if and only if it is dc-injective as well as od-injective.*

Proof. Let P be a regular injective poset. Then, it is clear that P is dc-injective and od-injective. For sufficiency, let A, B, P be posets, where P is dc-injective as well as od-injective, $f : A \to B$ be an embedding and $g : A \to P$ any poset map. Consider the decomposition $A \overset{f}{\cong} f(A) \overset{od}{\hookrightarrow} \downarrow f(A) \overset{dc}{\hookrightarrow} B$ of f. Define $\overline{g} : f(A) \to P$ by $\overline{g} = gf^{-1}$. Since P is od-injective there exists a poset map $\overline{\overline{g}} : \downarrow f(A) \to P$ which extends $\overline{g}$. Now, since P is dc-injective there exists a poset map $\overline{\overline{\overline{g}}} : B \to P$ which extends $\overline{\overline{g}}$. Thus $\overline{\overline{\overline{g}}}$ is a poset map which extends g. Therefore P is regular injective. □

Recall from [10] that a non-trivial regular injective S-poset is bounded by two zero elements and recall from [15] that a non-trivial dc-injective S-poset has a zero top element. In the case of od-injectivity of S-posets, we have the following result.

Proposition 2.8. *Every non-trivial od-injective S-poset has a zero which is the bottom element.*

Proof. Let A be an od-injective S-poset. Consider the S-poset $B = \{\theta\} \cup A$ obtained by adjoining a zero bottom element θ to A. Since A is od-injective, there exists a retraction $g : B \to A$. Then, the zero element $g(\theta)$ is the bottom element of A. □

Corollary 2.9. *There exists no non-trivial od-injective pomonoid whose identity is the bottom element.*

Proof. By the above proposition, in an od-injective pomonoid S whose identity is the bottom element, the identity has to be a zero too, which means that S has to be trivial. □

Recall from [16] that, for an S-poset A and $H \subseteq A \times A$, the S-poset congruence $\theta = \theta(H)$ on A generated by H is characterized as follows:

$$a\theta(H)b \Leftrightarrow a = b \text{ or } \exists s_1, s_2, ..., s_n, t_1, t_2, ..., t_m \in S \text{ such that}$$

$$a = c_1 s_1, d_1 s_1 = c_2 s_2, d_2 s_2 = c_3 s_3, ..., d_n s_n = b; \text{ or}$$

$$b = p_1 t_1, q_1 t_1 = p_2 t_2, q_2 t_2 = p_3 t_3, ..., q_m t_m = a,$$

where $(c_i, d_i) \in H \cup H^{-1}$ and $(p_j, q_j) \in H \cup H^{-1}$ for $i = 1, 2, ..., n; j = 1, 2, ..., m$.

In the next theorem which follows a lemma, we characterize od-injectivity by absolute retractness.

First recall from [5] that the pushout of S-poset maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct $B \sqcup C = (\{1\} \times B) \cup (\{2\} \times C)$ by the congruence $\theta(H)$ generated by $H = \{((1, f(a)), (2, g(a))) : a \in A\}$ with S-poset maps $q_B = \pi u_B : B \to (B \sqcup C)/\theta$, $q_C = \pi u_C : C \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the natural epimorphism, and $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are coproduct injections.

Lemma 2.10. *In the category* **Pos**-S*, pushouts transfer order dense embeddings.*

Proof. It is known from [13] that in the category **Pos**-S, pushouts transfer regular monomorphisms. To show that pushouts transfer order dense embeddings, applying the notations preceding lemma, we should show that if f is an order dense S-poset map, then so is q_C. Let $[x]_\theta \in (B \sqcup C)/\theta$, then either $[x]_\theta = [(1, b)]_\theta$ for some $b \in B$, or $[x]_\theta = [(2, c)]_\theta$ for some $c \in C$. In the latter case, we have $[x]_\theta = [(2, c)]_\theta \leq [(2, c)]_\theta = q_C(c)$, and so the result holds. In the former case, since $b \in B$ and f is an order dense embedding, there exists $a \in A$ such that $b \leq f(a)$ and hence $[(1, b)]_\theta = q_B(b) \leq q_B(f(a)) = q_C(g(a))$. Therefore, q_C is order dense embedding. □

Theorem 2.11. *An S-poset A is od-injective if and only if it is a retract of each of its extensions in which it is order dense.*

Proof. Let A be od-injective and consider the following diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \\ A & & \end{array}$$

where $f : A \to B$ is an order dense embedding. Then there exists an S-poset map $g : B \to A$ with $gf = \mathrm{id}_A$ by od-injectivity of A. For the converse, let A be an

S-poset and every order dense embedding $A \to D$ have a left inverse S-poset map. Consider the following diagram,

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \\ A & & \end{array}$$

where $f : B \to C$ is an order dense embedding and complete it to a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{k} & Q. \end{array}$$

By Lemma 2.10, k is order dense embedding, and by the assumption, there exists an S-poset map $k' : Q \to A$ such that $k'k = \mathrm{id}_A$. Now, $\overline{g} = k'h : C \to A$ is an S-poset map which extends g. Therefore, A is od-injective. □

For a general version of the above theorem, one can see Proposition 3.12 of [8].

The following theorem gives a criterion for od-injectivity of S-posets over a pomonoid S which is a counterpart of Skornjakov criterion for injectivity of acts (see [12]).

Theorem 2.12. *Let Q be an S-poset with a zero bottom element θ. Then the following conditions are equivalent:*

(i) *Q is od-injective.*

(ii) *Q is injective with respect to the order dense embeddings of the form $B \hookrightarrow B \cup cS$ to a singly generated extension of B.*

Proof. It is clear that (i)⇒(ii).

(ii)⇒(i) Let Q be an S-poset with a zero bottom element θ and satisfy (ii). Let B be an S-poset, A be an order dense sub S-poset, and $f : A \to Q$ an S-poset map. We must show that there exists an S-poset map $\overline{f} : B \to Q$ which extends f. Consider

$$\begin{aligned} \mathcal{P} := \{(X, h) |\ & X \text{ is an order dense sub S}-\text{poset of } B, A \subseteq X \subseteq B, \\ & h : X \to Q \text{ is an S}-\text{poset map extending } f\}. \end{aligned}$$

Define a relation $\leq$ on $\mathcal{P}$ as follows:

$$(X_1, h_1) \leq (X_2, h_2) \Leftrightarrow X_1 \subseteq X_2, h_2 |_{X_1} = h_1.$$

One can easily check that $\leq$ is a partial order on $\mathcal{P}$ and any chain in $\mathcal{P}$ has an upper bound. Then, by Zorn's Lemma, there exists a maximal element (C,h) in $\mathcal{P}$. We show that $C=B$, and so $\overline{f}=h$ extends f. Let $C\neq B$. Then there exists $b\in B\setminus C$. Set $D=C\cup bS$. Now $B\hookrightarrow C\cup bS$ is an order dense embedding and by hypothesis there is an S-poset map $\overline{h}:D\to Q$ which extends h. This contradicts the maximality of (C,h) in $\mathcal{P}$, and so $C=B$. □

Remark 2.13. It is known from [12] that if S is a group, any S-act is injective if and only if it has a zero element. But, by Propositions 2.6 and 2.7 of the present paper and Remark 4.5 of [10], this fact does not hold for od-injectivity in **Pos**-S.

In the following, we consider od-injectivity of right od-ideals, and investigate when all right od-ideals are od-injective.

Lemma 2.14. *Every od-injective right od-ideal K of S is a principal ideal which is generated by an idempotent element.*

Proof. Consider the order dense embedding $K\hookrightarrow S$ and extend the identity map $\mathrm{id}:K\to K$ to $f:S\to K$, by od-injectivity. Then, we see that K is generated by $f(1)$ which is an idempotent element of K. This is because, $f(1)\in K$, so $f(1)=f(f(1))=f(1f(1))=f(1)f(1)$ and $s=f(s)=f(1s)=f(1)s$, for all $s\in K$. □

Now, applying the following definition, we characterize pomonoids S over which all od-ideals are od-injective.

Definition 2.15. A pomonoid S is called *od-regular* if every $s\in S$, for which sS is an od-ideal of S, is a regular element; that is, there exists $t\in S$ such that $sts=s$.

Theorem 2.16. *The pomonoid S is od-regular and an od-injective S-poset if and only if all principal right od-ideals of S are od-injective.*

Proof. Consider a principal right od-ideal sS of S. Since S is od-regular, s is a regular element. Thus there exists $t\in S$ such that $sts=s$. This gives that sS is a retract of S, with the retraction $\lambda_{st}:S\to sS$. Now, since S is od-injective, so is sS. For the converse, let $s\in S$ be such that sS is an od-ideal. Since sS is od-injective, there exists an S-poset morphism $f:S\to sS$ such that $f\iota=\mathrm{id}_{sS}$, where ι is the inclusion map from sS to S and id_{sS} is the identity map on sS. Consequently, one has that $s=f(s)=f(1)s$. Since $f(1)\in sS$, it follows that s is regular, and hence S is od-regular. □

Using the above theorem we clearly get the following corollary.

Corollary 2.17. *The pomonoid S is od-regular pomonoid and an od-injective S-poset all of whose finitely generated right od-ideals are principal if and only if all finitely generated right od-ideals of S are od-injective.*

Proof. Let K be a finitely generated right od-ideal of S. By hypothesis, K is principal, so K is od-injective, by Theorem 2.16. □

By Theorem 2.16 and Lemma 2.14, we have the following result.

Corollary 2.18. *The pomonoid S is od-regular and an od-injective S-poset all of whose right od-ideals are principal if and only if all right od-ideals of S are od-injective.*

3 Some kinds of weak od-injectivity

There are different types of weak regular injectivity of S-posets. In this section, we study injectivity with respect to some special kinds of order dense embeddings, and in particular, embeddings $I \hookrightarrow S$, where I is a kind of od-ideal. The following definitions of injectivity are then natural.

Definition 3.1. We call an S-poset A:

(1) (*finitely generated*, *principally*) *od-ideal od-injective* if every S-poset map $f : I \to A$ from a (finitely generated, principal) right od-ideal I of S can be extended to an S-poset map $\overline{f} : S \to A$;

(2) *finitely od-injective* (*cyclicly od-injective*) if it is injective with respect to every order dense embedding $h : F \to B$ from a finitely generated (cyclic) S-poset F.

Remark 3.2. Note that, similar to the case of od-injectivity, every non-trivial finitely (cyclicly) od-injective S-poset has a zero which is the bottom element and so there exists no non-trivial finitely (cyclicly) od-injective pomonoid whose identity is the bottom element.

Remark 3.3. Note that, for an S-poset A and $a \in A$, the map $\lambda_a : S \to A$ defined by $\lambda_a(s) = as$ and $\rho_s : A \to A$ defined by $\rho_s(a) = as$ are S-poset maps and any S-poset map $f : S \to A$ from a pomonoid S is equal to λ_a for $a = f(1)$ where 1 is the identity element of the pomonoid S. Thus, the fact that an S-poset map $f : K \to A$ from a right od-ideal of S to an S-poset can be extended to an S-poset map $\bar{f} : S \to A$ is equivalent to f being of the form λ_a for some $a \in A$. This means that:

An S-poset A is (finitely generated, principally) od-ideal od-injective if and only if for any S-poset map $f : K \to A$ where $K \subseteq S$ is a (finitely generated, principal) right od-ideal there exists an element $a \in A$ such that $f = \lambda_a$.

The following theorem characterizes the posemigroups over which all S-posets are od-ideal od-injective.

Theorem 3.4. *Each S-poset is od-ideal od-injective if and only if every right od-ideal of the posemigroup S is generated by an idempotent.*

Proof. ($\Rightarrow$) Consider the identity map id_I on a right od-ideal I of S, it is of the form λ_a for some element a in I, by hypothesis. Thus $a = \mathrm{id}_I(a) = \lambda_a(a) = aa = a^2$, and so a is an idempotent element, also $I = \lambda_a(I)$. For the converse, let $I = eS$ be a right od-ideal of S, where e is an idempotent element. Consider an S-poset map $f : I = eS \to A$. Thus $f = \lambda_a$ for $a = f(e)$. Therefore A is od-ideal od-injective. □

Recall from [14] ([15]) that an S-poset A is called *weakly regular injective* (*poideal dc-injective*) if every S-poset map $f : I \to A$ from a right (po)ideal I of S can be extended to an S-poset map $\overline{f} : S \to A$.

The following theorem shows that when all S-posets are poideal dc-injective and when od-ideal od-injectivity is equivalent to weakly regular injectivity.

Theorem 3.5. *Let S be a pomonoid whose identity element is its bottom element. Then all S-posets are poideal dc-injective, also od-ideal od-injectivity coincides with weakly regular injectivity.*

Proof. If the identity element of S is its bottom element then each ideal of S is an od-ideal and the only poideal is S itself and so all S-posets are poideal dc-injective. Therefore, by Proposition 2.7, od-ideal od-injectivity and weakly regular injectivity are the same. □

The following theorem shows when all S-posets are od-ideal od-injective and when poideal dc-injectivity is equivalent to weakly regular injectivity.

Theorem 3.6. *Let S be a pomonoid whose identity element is its top element. Then all S-posets are od-ideal od-injective and poideal dc-injectivity coincides with weakly regular injectivity.*

Proof. If the identity element of S is its top element then the only od-ideal of S is S itself and so all S-posets are od-ideal od-injective. Therefore, by Proposition 2.7, poideal dc-injectivity and weakly regular injectivity are the same. □

Lemma 3.7. *If a principal right od-ideal sS of S, $s \in S$, is principally od-ideal od-injective then s is a regular element of S.*

Proof. Let a principal right od-ideal sS of S be principally od-ideal od-injective. Then there exists an S-poset morphism $f : S \to sS$ such that $f\iota = \mathrm{id}_{sS}$. Consequently, one has that $s = f(s) = f(1)s$. Since $f(1) \in sS$, taking $f(1) = st$ for some $t \in S$,

$$s = f(s) = f(1)s = sts.$$

Therefore, s is regular. □

Theorem 3.8. *The following conditions are equivalent for a pomonoid S:*

(i) *All right S-posets are principally od-ideal od-injective.*

(ii) *All right od-ideals of S are principally od-ideal od-injective.*

(iii) *All finitely generated right od-ideals of S are principally od-ideal od-injective.*

(iv) *All principal right od-ideals of S are principally od-ideal od-injective.*

(v) *S is an od-regular pomonoid.*

In the case where S is a pomonoid whose identity element is its bottom element, the above conditions are also equivalent to:

(vi) *S is a regular pomonoid.*

(vii) *All right S-posets are principally weakly regular injective.*

Proof. (i)⇒(ii)⇒(iii)⇒(iv) are clear.

(iv)⇒(v) is obvious by Lemma 3.7.

(v)⇒(i) Each S-poset map $f : sS \to A$, from a principal right od-ideal sS to an S-poset A, is of the form λ_a for $a = f(sx)$ where $s = sxs$ by od-regularity of S. Thus, (i) holds.

(v)⇒(vi) If the identity element of S is its bottom element then each ideal of S is an od-ideal and so by the definition we get the result.

(vi)⇔(vii) follows from Corollary 5.4 of [21].

(vii)⇒(i) Since principal weakly regular injectivity implies principal od-ideal od-injectivity, (i) holds. □

Definition 3.9. A pomonoid S is called right po-PP if and only if for every $s \in S$, where sS is a right principal od-ideal of S, there exists an idempotent $e \in S$ such that $s = se$, and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$.

Theorem 3.10. *Every principally od-ideal od-injective right po-PP pomonoid whose identity element is its bottom element is regular.*

Proof. Let S be a principally od-ideal od-injective right po-PP pomonoid, and $s \in S$. Since the identity of S is its bottom element, sS is an od-ideal and since S is a right po-PP pomonoid, there exists an idempotent $e \in S$ such that $s = se$ and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$. Define a mapping $f : sS \to S$ by $f(st) = et$ for every $t \in S$. Then f is a well-defined S-poset map. Since S is principally od-ideal od-injective, f is of the form λ_y for some $y \in S$. Thus $e = f(s) = ys$. Then $s = se = sys$, so s, and hence S is regular. □

Now, we study finitely generated od-ideal od-injectivity and characterize pomonoids over which all S-posets are finitely generated od-ideal od-injective.

Theorem 3.11. *If a (finitely generated, principal) right od-ideal K of a pomonoid S is (finitely generated, principally) od-ideal od-injective then K is generated by an idempotent.*

Proof. Let K be an od-ideal od-injective right od-ideal of S. Then, by Remark 3.3, there exists an element $e \in K$ such that $\mathrm{id}_K = \lambda_e$. In other words, we have $k = ek$ for every $k \in K$. This means that $K = eS$ and e is an idempotent. □

The following theorem characterizes pomonoids over which all S-posets are finitely generated od-ideal od-injective.

Theorem 3.12. *Let S be a pomonoid whose identity element is the bottom element. Then all S-posets are finitely generated od-ideal od-injective if and only if S is a regular pomonoid all of whose finitely generated right od-ideals are principal.*

Proof. Let all S-posets be finitely generated od-ideal od-injective. Since finitely generated od-ideal od-injectivity implies principally od-ideal od-injectivity, S is regular by Theorem 3.8. Also, by Theorem 3.11 all finitely generated right od-ideals are principal.

Conversely, let S be a regular pomonoid all of whose finitely generated right od-ideals are principal. Then finitely generated od-ideal od-injectivity coincides with principally od-ideal od-injectivity, and since S is regular, all S-posets are finitely generated od-ideal od-injective by Theorem 3.8. □

Theorem 3.13. *The following conditions are equivalent:*

(i) *All right S-posets are od-ideal od-injective.*

(ii) *For every right od-ideal I, every S-poset map $f : I \to A$ is of the form λ_a for some $a \in A$.*

(iii) *S is an od-regular pomonoid all of whose right od-ideals are principal.*

If the identity element of S is the bottom element, then the above conditions are also equivalent to:

(iv) *S is a regular pomonoid all of whose right od-ideals are principal.*

Proof. (i)⇔(ii) is clear.

(ii)⇒(iii) By (ii), all right S-posets and specially all right od-ideals of S are od-ideal od-injective. So any right od-ideal of S is generated by an idempotent, by Theorem 3.11. Then similar to the proof of Theorem 2.16, S is proved to be od-regular.

(iii)⇒(ii) Assuming (iii), since all right od-ideals of S are principal, od-ideal od-injectivity coincides with principal od-ideal od-injectivity, and since S is od-regular, all right S-posets are od-ideal od-injective, by Theorem 3.8.

(ii)⇒(iv) By (ii), all right S-posets and specially all right od-ideals of S are od-ideal od-injective. So every right od-ideal of S is generated by an idempotent, by Proposition 3.11. Hence S is regular, by Theorem 3.8.

(iv)⇒(i) Assuming (iv), since all right od-ideals of S are principal, od-ideal od-injectivity coincides with principal od-ideal od-injectivity and since S is regular, all S-posets are od-ideal od-injective, by Theorem 3.8. □

The following results show when all S-posets are finitely od-injective.

Lemma 3.14. *Every finitely generated finitely od-injective S-poset is od-injective.*

Proof. Let A be a finitely generated finitely od-injective S-poset. Consider the following diagram

$$\begin{array}{ccc} A & \overset{\iota}{\hookrightarrow} & B \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \\ A & & \end{array}$$

in which B is an order dense extension of A. Using that A is finitely generated, there exists $\pi : B \to A$ such that $\pi\iota = \mathrm{id}_A$. This implies that A is an od-absolute retract. Now, by Theorem 2.11, A is od-injective. □

A posemigroup S is called *completely finitely* (*cyclicly*) *od-injective* if all S-posets are (cyclicly) finitely od-injective.

Theorem 3.15. *A posemigroup S is completely finitely od-injective if and only if all finitely generated S-posets are od-injective.*

Proof. (⇒) The proof is similar to the proof of the above lemma.

(⇐) Let A be any S-poset and $h : F \to B$ be an order dense embedding from a finitely generated S-poset F, and $f : F \to A$ be any S-poset map. Then, by hypothesis, F is od-injective and so an od-absolute retract, by Theorem 2.11. Thus there exists an S-poset map $g : B \to F$ such that $gh = \mathrm{id}_F$. Then the composite $fg : B \to A$ is an S-poset map with $(fg)h = f$. So, A is finitely od-injective. □

Recalling from [17, Theorem 2.3] that every S-poset A is uniquely decomposable into a disjoint union of down closed indecomposable sub S-posets, one has the following result.

Proposition 3.16. *An S-poset A with a zero bottom element is cyclicly od-injective if and only if for any order dense embedding $h : P \to D$ from a cyclic S-poset P into any indecomposable S-poset D, any S-poset map $P \to A$ can be extended to an S-poset map $D \to A$.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) Let A be an S-poset with a bottom zero element and $h : P \to B$ be an order dense embedding from a cyclic S-poset P, and $f : P \to A$ be any S-poset map. Consider the decomposition of $B = \coprod_{i\in I} B_i$ into its down closed indecomposable sub S-posets B_i (which exists, by [17]). Since P is cyclic, there exists $i \in I$ such that $h(P) \subseteq B_i$. Thus, by hypothesis, there exists an S-poset map $g : B_i \to A$ which extends f. Define $\overline{f} : B = \coprod_{i\in I} B_i \to A$ by

$$\overline{f}(b) = \begin{cases} g(b) & \text{if } b \in B_i \\ \theta & \text{if } b \notin B_i \end{cases}$$

where θ is the zero bottom element of A. Then $\overline{f}$ is an S-poset map which extends f. □

The following results show when all S-posets are cyclicly od-injective.

Lemma 3.17. *Every cyclic cyclicly od-injective S-poset is od-injective.*

Proof. It is similar to the proof of Theorem 3.15 by replacing finitely generated S-posets with cyclic S-posets. □

Theorem 3.18. *A posemigroup S is completely cyclicly od-injective if and only if all cyclic S-posets are od-injective.*

Proof. The proof is similar to the proof of the above lemma. □

4 Products, coproducts and direct sums of od-injective S-posets

In this section, we consider the behaviour of od-injective S-posets with respect to products, coproducts, and direct sums. First, we recall the following remark about these notions in the category of S-posets.

Remark 4.1. Let $\{A_i\}_{i\in I}$ be a family of S-posets. Then

(a) the *product* of A_i's in the category of S-posets is their cartesian product $\prod_{i\in I} A_i$ with the componentwise order and action.

(b) the *coproduct* $\coprod_{i\in I} A_i$ of A_i's in the category of S-posets is their disjoint union $\dot{\bigcup}_{i\in I} A_i$ with the order given by $x \leq y$ in coproduct if and only if $x, y \in A_i$ and $x \leq y$ in A_i, for some $i \in I$; and with the action as in A_i, for $a \in A_i,\ s \in S$.

(c) If each A_i has a unique zero element θ_i (which we denote all θ_i's by 0), then the *direct sum* $\bigoplus_{i\in I} A_i$ is the sub S-poset of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number.

Theorem 4.2. *Let $\{A_i : i \in I\}$ be a family of S-posets. Then the product $\prod_{i\in I} A_i$ is od-injective if and only if each A_i is od-injective. The same is true for all weak od-injectivities defined in the last section.*

Proof. As usual, the above types of injectivity behaves well with respect to products using the universal property of products. For the converse, let $A = \prod_{i\in I} A_i$ be od-injective, and $k \in I$. To prove that A_k is od-injective, consider the diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle f}\downarrow & & \\ A_k & & \\ {\scriptstyle p_k}\uparrow & & \\ A & & \end{array}$$

where B is an order dense sub S-poset of C, f is an S-poset map, and $p_k : A \to A_k$ is the kth projection map. Define $\overline{f} : B \to A$ by

$$\overline{f}(b)(i) = \begin{cases} f(b) & \text{if } i = k \\ \theta_i & \text{if } i \neq k \end{cases}$$

where for $i \in I$, θ_i is the zero bottom element of A_i, which exists since A has the zero bottom element by Proposition 2.8, and the ith component of that zero element is a zero element of A_i, $i \in I$. Then, since f is an S-poset map, so is $\overline{f}$. Now by od-injectivity of A, $\overline{f}$ can be extended to an S-poset map $\overline{\overline{f}} : C \to A$. Then, $p_k\overline{\overline{f}} : C \to A_k$ extends f. Therefore A_k is od-injective. □

To consider the counterpart of the above theorem for coproducts, we have:

Proposition 4.3. *Let S be a pomonoid and $\{A_i : i \in I, |I| > 1\}$ be an arbitrary family of S-posets. Then $\coprod_{i\in I} A_i$ is not (finitely, cyclicly) od-injective.*

Proof. The definition of a coproduct shows that $\coprod_{i\in I} A_i$ is not bounded from the bottom, and so by Proposition 2.8, it is not od-injective. □

Remark 4.4. (1) There exists no pomonoid S over which all S-posets are (finitely, cyclicly) od-injective. This is because for a pomonoid S, if A is an S-poset then $A \sqcup A$ is not od-injective, by the above proposition.

(2) For any pomonoid S, there exists a finitely generated (cyclic) S-poset which is not od-injective by Theorem 3.15 (Theorem 3.18) and the above proposition.

(3) Recall from [20] that a completely strongly convex pomonoid is a pomonoid over which every S-poset fulfils that all of its sub S-posets are down closed. We

show that there does not exist a completely strongly convex pomonoid. For, if S is a completely strongly convex pomonoid then the only order dense embeddings are isomorphisms and so all S-posets are od-injective which is a contradiction by (1).

Theorem 4.5. *Let $\{A_i : i \in I\}$ be a family of S-posets. If the coproduct $\coprod_{i\in I} A_i$ is (finitely generated, principally) od-ideal od-injective then each A_i is (finitely generated, principally) od-ideal od-injective.*

Proof. Let K be an (finitely generated, principal) od-ideal of S, $i \in I$, and $f : K \to A_i$ be an S-poset map. Then, considering the ith injection map $\tau_i : A_i \to \coprod_{i\in I} A_i$, $\tau_i f$ is an S-poset map and is of the form λ_a for some $a \in \coprod_{i\in I} A_i$, by (finitely generated, principally) od-ideal od-injectivity. Now, let there exist $j \in I$ such that $a \in A_j$. Then for each $s \in S, as \in A_i \cap A_j$, which is a contradiction. Thus we get $a \in A_i$, and $f = \lambda_a$. □

It is shown that the converse of the above theorem is not true in general. But in the case of principally od-ideal od-injective S-posets, the converse is also true.

Theorem 4.6. *All coproducts of principally od-ideal od-injective S-posets are principally od-ideal od-injective.*

Proof. Let $\{A_i : i \in I\}$ be a family of principally od-ideal od-injective S-posets. Notice that for each embedding $I \hookrightarrow S$ from an order dense principal right ideal I of S, and any S-poset map $f : I \to \coprod A_i$ we have $\mathrm{Im} f \subseteq A_i$ for some $i \in I$. Then f can be extended to an S-poset map $\overline{f}$, since A_i is principally od-ideal od-injective. □

In the case of (finitely generated) od-ideal od-injective S-posets we have the following.

Definition 4.7. A pomonoid S is called *left od-reversible* if every two right od-ideals of S have a nonempty intersection.

Theorem 4.8. *All coproducts of (finitely generated) od-ideal od-injective S-posets are (finitely generated) od-ideal od-injective if the pomonoid S is left reversible. The converse is also true if S is left od-reversible.*

Proof. Let $\{A_i : i \in I\}$ be a family of od-ideal od-injective S-posets. Notice that by hypothesis, for any S-poset map $f : K \to \coprod A_i$ where K is a right od-ideal of S we have $\mathrm{Im} f \subseteq A_i$ for some $i \in I$. This is because, if otherwise, $\mathrm{Im} f \subseteq A_i \cup A_j$, for some $i \neq j \in I$, then taking $J = f^{-1}(A_i)$, $L = f^{-1}(A_j)$, we have $K = J \cup L$, $J \cap L = \emptyset$. Also, J, L are right ideals of S. This decomposition of K contradicts left reversibility of S. Thus, f is of the form λ_a, for some $a \in A_i$, since A_i is od-ideal od-injective. Therefore, $\coprod A_i$ is od-ideal od-injective.

For the converse, on the contrary, suppose S is not left od-reversible. Then there exist right od-ideals I, J of S with $I \cap J = \emptyset$. Since the one element S-poset $\mathbf{1}$ is od-ideal od-injective, so is the two element discrete S-poset $\mathbf{1} \sqcup \mathbf{1} = \{a, b\}$, in which a, b are zero elements, by hypothesis. But, the S-poset map $f : I \cup J \to \mathbf{1} \sqcup \mathbf{1}$ given by

$$f(s) = \begin{cases} a & \text{if } s \in I \\ b & \text{if } s \in J \end{cases}$$

can not be extended to S, which is a contradiction. To see that f can not be extended to S, let on the contrary, $\overline{f}$ be an extension of f. Then $\overline{f}(1) = a$ or b. If $\overline{f}(1) = a$, then for $s \in J$, $\overline{f}(s) = \overline{f}(1)s = as = a \neq b = f(s)$, which is a contradiction. Similarly, in the case $\overline{f}(1) = b$, we get a contradiction.

The proof for finitely generated od-ideal od-injective S-posets is similar. □

For the direct sum of od-injective S-posets we have:

Theorem 4.9. *Let $\{A_i : i \in I\}$ be a family of S-posets with a unique zero element.*

(i) *If the direct sum $\bigoplus_{i \in I} A_i$ is od-injective then each A_i is od-injective. The same is true for all kinds of weak od-injectivity defined in the last section.*

(ii) *The converse of* (i) *is also true for the case of all kinds of weak od-injectivity defined in the last section except od-ideal od-injectivity.*

Proof. (i) Let the direct sum $\bigoplus_{i \in I} A_i$ be od-injective, $g : A \to B$ be an embedding from an order dense S-poset A, $i \in I$, and $f : A \to A_i$ be an S-poset map. Consider the injection map $\sigma_i : A_i \to \bigoplus_{i \in I} A_i$, by $a_i \mapsto (\ldots, 0, a_i, 0, \ldots)$ where a_i is the i-th component. By hypothesis there exists an S-poset map $\overline{\sigma_i f} : B \to \bigoplus_{i \in I} A_i$ which extends $\sigma_i f$. Then $p_i \overline{\sigma_i f} : B \to A_i$ extends f, where p_i is the i-th projection map.

(ii) For the converse, let each A_i be a finitely od-injective S-poset. Let F be a finitely generated S-poset which is an order dense sub S-poset of B and $f : F \to \bigoplus_{i \in I} A_i$ be an S-poset map. Assume that F is generated by $\{x_1, x_2, \ldots, x_n\}$. Then, since only finitely many components of each $f(x_i)$ are nonzero, $\mathrm{Im} f$ is contained in a direct sum of finitely many A_i, say $i_1, i_2, \ldots, i_m$. Then, since the direct sum $A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_m}$, which is in fact a product, is finitely od-injective by Theorem 4.2, there exists $f' : B \to \bigoplus_{j=i_1}^{i_m} A_j$ which extends $f : F \to \bigoplus_{j=i_1}^{i_m} A_j$. Finally, $\sigma f' : B \to \bigoplus_{i \in I} A_i$ extends f, where $\sigma : \bigoplus_{j=i_1}^{i_m} A_j \to \bigoplus_{i \in I} A_i$ is the injection map, $(a_j)_{j=i_1}^{i_m} \mapsto (\ldots, a_{i_1}, \ldots, a_{i_2}, \ldots, a_{i_m}, \ldots)$, where each $a_j, j = i_1, \ldots, i_m$ is the j-th component and other components are 0. □

In the case of od-ideal od-injective S-posets we have the following theorem which comes after a definition.

Definition 4.10. A pomonoid S is said to be *od-Noetherian* if and only if every right od-ideal of S is finitely generated.

It is easy to check that a pomonoid S is od-Noetherian if and only if it satisfies the *ascending chain condition* for right od-ideals, that is, for every ascending chain

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

of right od-ideals of S there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \ldots$.

Theorem 4.11. *Each direct sum of od-ideal od-injective S-posets is od-ideal od-injective if and only if S is od-Noetherian.*

Proof. If S is od-Noetherian, a similar argument to that of the above theorem gives the result. Conversely, let $\{0\} = I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ be an ascending chain of right od-ideals of S. Consider the Rees factor S-posets $S/\theta(I_n \times I_n)$, and let E_n be the regular injective S-poset in which $S/\theta(I_n \times I_n)$ can be embedded. Then $E = \bigoplus_{n\in\mathbb{N}} E_n$ is od-ideal od-injective by hypothesis. Take $I = \bigcup_{n\in\mathbb{N}} I_n$ which is clearly an od-ideal, and consider the natural epimorphisms $f_n : S \to S/\theta(I_n \times I_n)$. Define an S-poset map $f : I \to E$ by $f(s) = (f_n(s))_{n\in\mathbb{N}}$. Notice that for each $s \in S$ only finitely many components of $f(s)$ are nonzero, because $s \in I_k$ for some $k \in \mathbb{N}$, and so $f_n(s) = I_n = 0_{S/\theta(I_n\times I_n)}$, for all $n \geq k$. Now, since E is od-ideal od-injective, $f = \lambda_a$ for some $a \in E$. Let $a = (a_n)_{n\in\mathbb{N}}$ where for some $k \in \mathbb{N}$, $a_n = 0$ for all $n \geq k$. Then for each $s \in I$, since $f(s) = as$, we get $f_k(s) = 0$. So, $I \subseteq I_k$, which gives $I = I_k$. □

The condition that weak injectivity implies injectivity is known as the *Baer Criterion* for injectivity. Here we study some Baer conditions and give some conditions under which a special kind of od-injectivity implies od-injectivity itself.

First, we give some examples to show that some kinds of od-injectivity do not imply some other kinds.

Example 4.12. (1) Principal od-ideal od-injectivity does not imply finitely generated od-ideal od-injectivity. To see this, let $S = \{0, 1, e, f\}$ be the commutative idempotent pomonoid in which 1 is the bottom identity element, 0 is the top zero element, e, f are incomparable, and $ef = fe = 0$. Then, the right od-ideal $K = \{0, e, f\}$ of S can be seen to be principally od-ideal od-injective, but K can not be finitely generated od-ideal od-injective since it is not generated by an idempotent.

(2) Finitely generated od-ideal od-injectivity does not imply od-ideal od-injectivity. To see this, take the pomonoid $S = (\mathbb{N}, \max)$ with the order

$$1 \geq 2 \geq 3 \geq 4 \geq \cdots .$$

Then, S, which is not generated by an idempotent, can not be od-ideal od-injective, by Theorem 3.11. But at the same time each finitely generated right od-ideal of S is of the form $\uparrow n = \{x \in \mathbb{N} : x \geq n\} = \{1, 2, \cdots, n\}$, and is generated by an idempotent. This implies that S is finitely generated od-ideal od-injective.

(3) Od-ideal od-injectivity does not imply od-injectivity. To see this, consider the pomonoid $S = (\mathbb{N}, \max)$ with the ordinary order of natural numbers. Then S is od-ideal od-injective since each od-ideal of S is generated by an idempotent, but it is not od-injective, since it does not contain a zero element.

(4) Principally od-ideal od-injectivity does not imply cyclicly od-injectivity. To see this, let $T = \{e, f\}$ be a right zero semigroup, $R = T^1$ with the order $e, f \leq 1$. Then by Theorem 3.8, R is principally od-ideal od-injective. But, by Remark 3.2, R is not cyclicly od-injective because it does not have any zero element.

(5) Od-ideal od-injectivity does not imply finitely od-injectivity. To see this, consider the pomonoid $S = (\mathbb{N}, \max)$ with the ordinary order of natural numbers. Then S is od-ideal od-injective by Theorem 3.13, since S is a regular pomonoid all of whose right od-ideals are principal, but it is not finitely od-injective since it does not contain a zero element.

Theorem 4.13. *Every finitely generated od-ideal od-injective S-poset is od-ideal od-injective if and only if S is od-Noetherian.*

Proof. Let A be a finitely generated od-ideal od-injective S-poset, and $f : I \to A$ be an S-poset map from a right od-ideal I of S. Since S is od-Noetherian, I is finitely generated. Now, since A is finitely generated od-ideal od-injective there exists an S-poset map $g : S \to A$ which extends f. Then g is of the form λ_a for $a = g(1)$. Thus f is also of the form λ_a and hence A is od-ideal od-injective. For the converse, let $\{A_i : i \in I\}$ be a family of od-ideal od-injective S-posets. Since each od-ideal od-injective S-poset is finitely generated od-ideal od-injective, each A_i is finitely generated od-ideal od-injective. Then by Theorem 4.9, $\bigoplus_{i \in I} A_i$ is finitely generated od-ideal od-injective, and so it is od-ideal od-injective, by hypothesis. Now, by Theorem 4.11, S is od-Noetherian. □

Theorem 4.14. *The following conditions are equivalent:*

(i) *Every od-ideal od-injective S-poset is regular injective and S is od-Noetherian.*

(ii) *Every finitely generated od-ideal od-injective S-poset is regular injective.*

Proof. (i) ⇒ (ii) Since S is od-Noetherian, every finitely generated od-ideal od-injective S-poset is od-ideal od-injective, and then regular injective by hypothesis.

(ii) ⇒ (i) Let $\{A_i\}_{i \in I}$ be a family of od-ideal od-injective S-posets. Since every od-ideal od-injective S-poset is finitely generated od-ideal od-injective, $\bigoplus_{i \in I} A_i$ is finitely generated od-ideal od-injective, by Theorem 4.9. Then $\bigoplus_{i \in I} A_i$ is regular injective, by hypothesis. Hence S is od-Noetherian by Theorem 4.11. Every od-ideal od-injective S-poset is finitely generated od-ideal od-injective and hence regular injective by hypothesis. □

Theorem 4.15. *All principally od-ideal od-injective S-posets over a pomonoid S are finitely generated od-ideal od-injective if and only if all finitely generated right od-ideals of S are principal.*

Proof. To prove necessity, consider a finitely generated right od-ideal $I = \bigcup_{j=1}^{n} s_j S$. By assumption and since principally weakly regular injectivity implies principally od-ideal od-injectivity, its principally weakly regular injective extension $I^{(2)}$, which is constructed in [21], is finitely generated od-ideal od-injective. Hence there exists an S-poset morphism $g : S \to I^{(2)}$ such that the diagram

$$\begin{array}{ccc} I & \xhookrightarrow{\iota} & S \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ I^{(2)} & & \end{array}$$

is commutative, where ι and f are inclusion mappings. Then, for every $s_j \in I$,

$$s_j = f(s_j) = g\iota(s_j) = g(s_j) = g(1)s_j$$

and hence

$$I = \bigcup_{j=1}^{n} s_j S = \bigcup_{j=1}^{n} g(1)s_j S \subseteq g(1)S.$$

Now $g(1) \in I_n$ for some $n \in \mathbb{N}_0$ where I_n is as defined in Theorem 4.10 of [21]. If $n = 0$ then $g(1) \in I$. Otherwise, by n times applying Lemma 5.7 of [21], we get an element $d \in I$ such that $I \subseteq dS$. So in both cases $I = sS$ for some $s \in I$. Sufficiency, is obvious. □

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رسته‌ها و ساختارهای کلی جبری با کاربردها
جلد ۳، شماره ۱، تیر ۱۳۹۴

انژکتیوی S-مجموعه‌های مرتب نسبت به نشاننده‌های چگال ترتیبی

لیلا شهباز

مطالعه انژکتیوی نسبت به کلاس‌های مختلف تکریختی‌ها در بسیاری از شاخه‌های ریاضیات با اهمیت است. ریاضیدانان بسیاری این مفهوم را در رسته‌های مختلف و نسبت به تکریختی‌های متفاوت مطالعه کرده‌اند. در این مقاله مفهوم انژکتیوی نسبت به نشاننده‌های چگال ترتیبی S- مجموعه‌های مرتب، مجموعه‌های مرتب با کنش یکنوای تکواره‌ی مرتب S روی آنها مورد مطالعه قرار می‌گیرد. محکی، مشابه محک بئر برای انژکتیوی مدول‌ها یا محک اسکورنیاکوف برای انژکتیوی S-مجموعه‌ها، برای انژکتیوی چگال ترتیبی ارائه می‌دهیم. همچنین، این نوع انژکتیوی را برای خود و ایده‌آل‌های چگال ترتیبی آن بررسی می‌کنیم. به علاوه، نوعی انژکتیوی ضعیف نسبت به نشاننده‌های چگال ترتیبی معرفی و رابطه‌ی آن را با انژکتیوی چگال ترتیبی مورد بررسی قرار می‌دهیم. همچنین، بررسی می‌کنیم که آیا این انواع انژکتیوی عمل‌های ضرب، همضرب و مجموع مستقیم S-مجموعه‌های مرتب را حفظ یا منعکس می‌کنند.

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