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FUZZY INNER PRODUCT AND FUZZY NORM OF HYPERSPACES

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Abstract. We introduce and study fuzzy (co-)inner product and fuzzy (co-)norm of hyperspaces. In this regard by considering the notion of hyperspaces, as a generalization of vector spaces, first we will introduce the notion of fuzzy (co-)inner product in hyperspaces and will apply it to formulate the notions of fuzzy (co-)norm and fuzzy (co-)orthogonality in hyperspaces. In particular, we will prove that to every fuzzy hyperspace there is an associated unique fuzzy inner product in a natural way.

1. Introduction

Hyperstructure theory was born in 1934, when Marty defined hypergroups, began to analyse their properties and applied them to groups, and rational algebraic functions [14]. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied mathematics (for example see [1]-[10], [23], [24]). In 1990, M.S. Tallini introduced the notion of hypervector spaces(or hyperspace) [23] and studied the basic properties and applied them to geometry (for more see [24]-[26]).

Following the introduction of fuzzy set by L. A. Zadeh in 1965 ([31]), he developed it's theory which can be found in mathematics and other applied subjects. Recently, fuzzy set theory has been developed in the context of hyperalgebraic structure theory (for example see [1-7], [10-12], [15-16]).

In [14], [17] and [19-20] notions of fuzzy fields and fuzzy vector spaces were introduced and studied. The author in [1-7] introduced the notions of fuzzy hyper subspaces, fuzzy balanced and fuzzy convex sets, fuzzy dimension and fuzzy basis of fuzzy hyperspaces as a generalization of fuzzy subspaces. Also in [7] the author introduced the notion of norm in fuzzy hyperspaces, which we will develope it in this paper. Here we introduce fuzzy inner product, fuzzy co-inner product, fuzzy norm and fuzzy co-norm of fuzzy hyperspaces. In this regards, first we introduce the notion of fuzzy (co-) inner product, then we apply it to formulate fuzzy (co-)norm and fuzzy normality in hyperspaces and finally, we investigate the basic properties of these notions.

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2. **Preliminaries**

In this section we gather all definitions and properties which we require in (resp. fuzzy) hyperspaces and fuzzy subsets, and set the notions.

Let H be a nonempty set. By $P_*(H)$ we mean the family of all nonempty subsets of H.

A map $. : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or *join operation*.

The join operation is extended to nonempty subsets of H in a natural way, where $A.B$ or AB is given by

$$AB = \bigcup \{ab \mid a \in A \text{ and } b \in B\}.$$

A *hypergroup* is a structure $(H, .)$ that satisfies two axioms,

(Reproduction) $aH = H = Ha$, for all $a \in H$;

(Associativity) $a(bc) = (ab)c.$, for all $a, b, c \in H$.

Let H be a hypergroup and K a nonempty subset of H. Then K is a *subhypergroup* of H if it is a hypergroup with respect to the hyperoperation "." restricted to K.

Hence it is clear that a nonempty subset K of H is a subhypergroup if and only if $aK = Ka = K$, for all $a \in K$, under the hyperoperation on H.

Definition 2.1. [13] Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K as a quadruple $(V, +, \circ, K)$, where $\circ$ is a mapping

$$\circ : K \times V \longrightarrow P_*(V)$$

such that the following conditions are satisfied:

(1) $\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) \subseteq a \circ x + a \circ y$ (right distributivity)

(2) $\forall a, b \in K, \forall x \in V, (a + b) \circ x \subseteq a \circ x + b \circ y$ (left distributivity)

(3) $\forall a, b \in K, \forall x \in V, a \circ (b \circ x) = (ab) \circ x,$

(4) $\forall a \in K, \forall x \in V, a \circ (-x) = -a \circ x,$

(5) $\forall x \in V, x \in 1 \circ x.$

For simplicity of notation sometimes we write ax instead of $a \circ x$.

Remark 2.2. (i) In the right of (1) and (2) the sum is in the sense of Frobenius, that is, we consider the set of all the sums of elements of $a \circ x$ and $a \circ y$. Moreover, the left member of (3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$.

(ii) We say that $(V, +, \circ, K)$ is *strongly left distributive* iff

$$\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) = a \circ x + a \circ y,$$

and *anti distributive*, iff

$$\forall a \in K, \forall x, y \in V, \quad a \circ (x + y) \supseteq a \circ x + a \circ y,$$

In a similar way we define the strongly right distributive law.

(iii) Let $\Omega = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V, +)$. In [26] it is shown that if V is either strongly right or left distributive, then Ω is a subgroup of $(V, +)$.

Proposition 2.3. *[26] Let V be a strongly left distributive hypervector space over a field K. Then the following hold:*
(1) *Ω is a subgroup of of $(V, +)$.*
(2) $\forall a \in K, a \circ 0 = \Omega = a \circ \Omega$.
(3) *$\forall x \in V, 0 \circ x \supseteq \Omega$ is a subhypergroup of $(V, +)$.*
(4) *$\forall x \in V, 0 \circ x$ is a subhypergroup of $(V, +)$.*

Definition 2.4. [1] Let V be a hyperspace over a field K and ν be a fuzzy subfield of K. A fuzzy subset μ_V of V is said to be a *fuzzy hyperspace* of V over the fuzzy field ν_K if, for all $x, y \in V$, and all $a \in K$, the following conditions are satisfied:
(i) $\mu_V(x + y) \geq \mu_V(x) \wedge \mu_V(y)$;
(ii) $\mu_V(-x) \geq \mu_V(x)$;
(iii) $\bigwedge_{y \in a \circ x} \mu_V(y) \geq \mu_V(x) \wedge \nu_K(a)$;
(iv) $\nu_K(1) \geq \mu_V(0)$;
where the operation $\bigwedge$ denotes the infimum of the values in the unite interval $[0, 1]$.

We say that μ_V is a fuzzy hyperspace over the fuzzy field ν_K. Hereafter except for ambiguous cases we shall drop the subscripts on μ and ν.

Remark 2.5. (i) In Definition 3.1 if we consider $\nu = \chi_K$, the characteristic function of K, then μ is a fuzzy hyperspace.

(ii) In the sequel, unless otherwise specified, we always assume that V is a hyperspace over the field K.

Dually the notion of an *anti fuzzy hyperspace* can be defined as follows:

Definition 2.6. A fuzzy subset μ of V is said to be an *anti-fuzzy hyperspace* of V over ν, if for all $x, y \in V$ and $a \in K$, the following conditions are satisfied:
(i) $\mu(x + y) \leq max\{\mu(x), \mu(y)\}$;
(ii) $sup_{z \in a \circ x} \mu(z) \leq max\{\nu(a), \mu(x)\}$.

Proposition 2.7. *If μ_V is a fuzzy hyperspace over the fuzzy field ν_K, then*
(i) $\nu(0) \geq \mu(0)$;
(ii) $\mu(0) \geq \mu(x), \quad \forall x \in V$;
(iii) $\nu(0) \geq \mu(x), \quad \forall x \in V$.

Proof. The proof is an immediate consequence of Definition 2.1. □

Proposition 2.8. *[1] Let V be a strongly left distributive hyperspace over a field K and ν_K be a fuzzy field. Let $\mu \in FS(V)$. Then μ_V is a fuzzy hyperspace over ν_K iff*
(i) $\bigwedge_{z \in \alpha \circ x + \beta \circ y} \mu(z) \geq ((\nu(\alpha) \wedge \mu(x)) \bigwedge (\nu(\beta) \wedge \mu(y))), \quad \forall x, y \in V, \forall \alpha, \beta \in K$;
(ii) $\nu(1) \geq \mu(x), \quad \forall x \in V$.

Definition 2.9. For a fuzzy subset μ of A, the level subset μ_t is defined by

$$\mu_t = \{x \in A : \mu(x) \geq t\}, \ t \in [0, 1].$$

We denote the set of all fuzzy subsets of X by FS(X).

3. Fuzzy (co-)inner Product of Hyperspaces

Throughout this note we assume that V is a hyperspace over the field K.

Definition 3.1. A fuzzy subset ν of a field K is called a *fuzzy subfield* if $\forall x, y \in K$, the following conditions are satisfied:
(1) $\nu(x+y) \geq min\{\nu(x), \nu(y)\}$;
(2) $\nu(-x) \geq \nu(x)$;
(3) $\nu(xy) \geq min\{\nu(x), \nu(y)\}$;
(4) $\nu(x^{-1}) \geq \nu(x)$;
(5) $\nu(0) = 1 = \nu(1)$.

Proposition 3.2. *[22] A fuzzy subset ν of a field K is a fuzzy subfield of K if and only if ν_t is a subfield of K for every $t \in [0, \nu(0)]$.*

Definition 3.3. Let V be a hyperspace over a field K and ν be a fuzzy subfield of K. A fuzzy subset μ of V is said to be a *fuzzy hyperspace* of V with respect to ν, if for all $x, y \in V$ and $a \in K$, the following conditions are satisfied:
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$;
(ii) $inf_{z \in ax} \mu(z) \geq min\{\nu(a), \mu(x)\}$;
(iii) $\mu(0) = 1$.

In the above definition if we set $\nu = \chi_F$, the characteristic function of F, then we say that μ is a fuzzy hyperspace of V.

Proposition 3.4. *[22] Let ν be a fuzzy subfield of K. Then a fuzzy subset μ of V is a fuzzy hyperspace of V if and only if μ_t is a subhyperspace of V over ν_t, for every $t \in [0, 1]$.*

Proposition 3.5. *[22] A fuzzy subset ν of a field K is a fuzzy subfield of K if and only if ν_t is a subfield of K for every $t \in [0, \nu(0)]$.*

Lemma 3.6. *Let μ be a fuzzy hyperspace of V. Then the following statements are satisfies:*
(i) $\mu(0) \geq \mu(x), \forall x \in V$;
(ii) $\mu(x-y) = \mu(0) \Longrightarrow \mu(x) = \mu(y)$;
(iii) *if μ has inf-property, then $inf_{z \in aox} \mu(z) = \mu(x), \forall a \in K \backslash \{0\}$;(we say that μ has inf-property if for every non-empty subset S of V $inf_{x \in S} \mu(x) = \mu(a)$, for some $a \in S$);*
(iv) *if $\mu(x) > \mu(y)$, then $\mu(x+y) = \mu(v)$;*
$(v) \mu(x) = \mu(0), \forall x \in \Omega$.

Proof. (i) Since $-x \in (-1)ox$, then from Definition 3.3 we have $\mu(-x) \geq \mu(x)$. On the other hand, by Definition 2.1 (4) we have $x \in (-1)o(-x)$, then $\mu(x) \geq \mu(-x)$. Thus $\mu(x) = \mu(-x)$, and hence $\mu(0) = \mu(x-x) \geq \mu(x)$, by Definition 3.3.

(ii) $\mu(x-y) \geq min(\mu(x), \mu(-x)) = (\mu(x), \mu(x)) = \mu(x)$. But Definition 3.1 and inf-property of μ imply that $\mu(z_0) = inf_{z \in aox} \mu(z) \geq \mu(x)$, where $\mu(z_0) = inf_{z \in aox} \mu(z)$. On the other hand, we have $x \in 1ox = a^{-1}o(aox)$, thus $x \in a^{-1}oz$, for some $z \in aox$. Then $\mu(x) \geq \mu(z) \geq \mu(z_0)$, and so $\mu(z_0) = \mu(x)$.

(iv) this is easy.

$(v) x \in \Omega = 0o\underline{0}$ implies that $\mu(x) \geq \mu(0)$ by Definition 3.3 (ii), and hence by Lemma 3.6 (i), the equality holds. □

Corollary 3.7. *[1] Let V be strongly left distributive. A fuzzy subset μ of V is a fuzzy hyperspace if and only if, $\forall x, y \in V, \forall a, b \in K$,*

$$inf_{z \in aox+boy} \mu(z) \geq min(\mu(x), \mu(y)).$$

Theorem 3.8. *Let $\mu \in FS(V)$. Then μ is a fuzzy hyperspace of V, if and only if μ_t is a subhyperspace of V, for every $t \in Im\mu$.*

Proof. Let μ be a fuzzy hyperspace. Let $t \in Im\mu$ and $x, y \in \mu_t$. Then

$$\mu(x - y) \geq min(\mu(x), \mu(y)) \geq t,$$

that is, $x - y \in V$. Also for $a \in K$ and $x \in V$ we have $inf_{z \in aox} \mu(z) \geq \mu(x)$, that is $aox \subseteq \mu_t$. Thus μ_t is a subhyperspace of V.

Conversely, let μ_t be a subhyperspace, for every $t \in Im(\mu)$. Let $x_1, x_2 \in V, a \in K$ and $t = min(\mu(x_1), \mu(x_2))$. Then $x, y \in \mu_t$, and hence $x_1 - x_2 \in \mu_t$ and $aox_1 \subseteq \mu_t$. Thus $\mu(x_1 - x_2) \geq min(\mu(x_1), \mu(x_2))$ and $inf_{z \in aox_1} \mu(z) \geq \mu(x_1)$. Therefore μ is a fuzzy hyperspace. □

Example 3.9. (1) In $(\mathbb{R}^2, +)$ we define the product times a scalar in $\mathbb{R}$ by setting:

$$\forall P \in \mathbb{R}^2, \forall a \in \mathbb{R} : \begin{cases} a \circ P = \text{line } OP, \ P \neq \underline{0} \\ a \circ \underline{0} = \{\underline{0}\} \end{cases}$$

where $\underline{0} = (0, 0)$. It is easy to see that $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space. Now define the fuzzy subset μ on $\mathbb{R}^2$ as follows: $\mu(0, 0) = 1, \mu((0, y)) = 1/2, \forall y \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}, \mu((x, o)) = 1/3$ and $\mu(x, y) = 0$, otherwise. Then μ is a fuzzy hyperspace of $\mathbb{R}^2$, since the only level subsets of μ are $\mu_1 = \{(0, 0)\}, \mu_{1/2} = \{(x, 0) | x \in \mathbb{R}\}, \mu_{1/2} = \{(0, y) | y \in \mathbb{R}\} \mu_{1/3} = \mathbb{R}^2$, which are clearly subhyperspaces of $\mathbb{R}^2$. Thus by Theorem 3.9 μ is a fuzzy hyperspace.

(2) Let $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ be a strictly increasing sequence of subhyperspaces of V and $\{t_i\}_{i=1}^n$ be a strictly decreasing sequence in $[0, 1]$. Define μ on V as follows:

$$\mu(x) = t_i \ \text{ if } \ x \in V_i \backslash V_{i-1}, \ \ where \ \ t_{i-1} < t_i, i = 1, 2, ... \ \text{ and } \ \mu(x) = 0, if x \in V \backslash \cup_{i=1}^{\infty} V_i.$$

It is easy to verify that $\mu_{t_{i+1}} \subseteq \mu_{t_i}$ and the only level subhyperspaces of V are V, and $\mu_{t_i} = V_i, i = 1, 2, ...$ Then by Theorem 3.9 μ is a fuzzy hyperspace of V.

(3) Let $V = V_0 \supset V_1 \supset ... \supset V_i \supset ...$ be a strictly increasing sequence of subhyperspaces of V and $\{t_i\}_1^n$ be an increasing sequence in $[0, 1]$. Define fuzzy subset μ on V by $\mu(x) = t_{i-1}$, if $x \in V_{i-1} \backslash V_i$ where

$$t_{i-1} < t_i, i = 1, 2, 3, ... \ \text{ and } \ \mu(x) = 1 \text{if } \ x \in \cap_i^{\infty} V_i.$$

Again by Theorem 3.9 μ is a fuzzy hyperspace of V.

Definition 3.10. Let ν be a fuzzy subfield of K. A fuzzy subset μ of $V \times V$ is called a *fuzzy inner product hyperspace* if $\forall x, y, z \in V$ and $\forall a \in K$, the following conditions hold:

(i) $\mu(x+y,z) \geq min\{\mu(x,z),\mu(y,z)\}$;
(ii) $inf_{t\in a\circ x}\mu(t,y) \geq min\{\nu(a),\mu(x,y)\}$;
(iii) $\mu(x,y)=\mu(\overline{y},\overline{x})$;
where $\overline{x}$ is the conjugate of x.

Definition 3.11. Let ν be a fuzzy subfield of K and μ be a fuzzy inner product hyperspace in V. Then the fuzzy subset $\parallel . \parallel$ of V is called a *fuzzy norm function* in V if $\forall x,y\in V$ and $a\in K$, the following conditions are satisfied:

(i) $\parallel x \parallel \leq \parallel 0 \parallel$;
(ii) $inf_{z\in aox}\parallel z \parallel \geq min\{\mu(x,z),\nu(\overline{a}),\parallel x \parallel\}$;
(ii) $\parallel x+y \parallel \geq min\{\mu(x,y),\mu(y,x),\parallel x \parallel,\parallel y \parallel\}$.

Remark 3.12. The condition (iii) in Definition 3.9 of a fuzzy inner product hyperspace is meaningless, since it is stated in terms of conjugate of an element and conjugation, in general is undefined in an arbitrary field. Similarly, the condition (ii) in Definition 3.10 is meaningful only if the conjugate is defined in the field K. Therefore in view of the foregoing discussion we modify the above definitions as follow.

Definition 3.13. (modified version of Definition 3.10). Let ν be a fuzzy subfield of K. A fuzzy subset θ of $V\times V$ is called a fuzzy inner product hyperspace with respect to ν if $\forall x,y,z\in V$ and $\forall a\in K$, the following conditions hold:

(i) $\theta(x+y,z) \geq min\{\theta(x,z),\theta(y,z)\}$;
(ii) $inf_{t\in aox}\theta(t,y) \geq min\{\nu(a),\theta(x,y)\}$;
(iii) $\theta(x,y)=\theta(y,x)$.

The number $\theta(x,y)$ is called the inner product of elements x,y.

Definition 3.14. (modified version of Definition 3.11) Let ν be a fuzzy subfield of K and μ be a fuzzy inner product hyperspace in V. Then the fuzzy subset $\parallel . \parallel$ of V is called a fuzzy norm function in V if $\forall x,y\in V$ and $a\in K$, the following conditions are satisfied:

(i) $\parallel x \parallel \leq \parallel 0 \parallel$;
(ii) $inf_{z\in aox}\parallel z \parallel \geq min\{\mu(x,z),\nu(a),\parallel x \parallel\}$;
(iii) $\parallel x+y \parallel \geq min\{\mu(x,y),\mu(y,x),\parallel x \parallel,\parallel y \parallel\}$.

Henceforth the terms "fuzzy inner product hyperspace", and "fuzzy norm" will be referred to in the sense of Definitions 3.13 and 3.14, respectively.

Proposition 3.15. *Let ν be a fuzzy subfield of K and θ be a fuzzy inner product hyperspace in V. Then*
$\forall x,y,\in V$ and $\forall a\in K$ the following statements are satisfied:

(i) $\theta(0,y) \geq \theta(x,y)$;
(ii) $\theta(x,0) \geq \theta(x,y)$;
(iii) $\theta(0,0) \geq \theta(x,y)$,
(iv) $inf_{z\in a\circ x}\theta(x,z) \geq min\{\nu(a),\theta(x,y)\}; and$
$(v)\theta(x,y+z) \geq min\{\theta(x,y),\theta(y,z)\}$.

Proof. (*i*) $\theta(0,y) = \theta(x-x,y) \geq min\{\theta(x,y), \theta(-x,y)$, by Definition 3.13 (i). Also by Definition 2.1 (5), we have $-x \in 1o(-x) = (-1)ox$. Thus $\theta(-x,y) \geq min\{\nu(-1), \theta(x,y)\} = \theta(x,y)$, by Definitions 3.1 and 3.13.

(*ii*) $\theta(x,0) = \theta(,x) \geq \theta(x,y)$, by (*i*).
(*iii*) $\theta(0,0) \geq \theta(0,y) \geq \theta(x,y)$,, by(*i*).
(*iv*) $inf_{z \in aox}\theta(x,z) = inf_{z \in aox}\theta(z,x) \geq min\{\nu(a), \theta(x,z)\}$,, by Definition 3.14.
(*v*) $\theta(x+y,z) = \theta(y+z,x) \geq min\{\theta(y,x), \theta(z,x)\} = \{\theta(x,y), \theta(x,z)\}$. □

Definition 3.16. Let θ be a fuzzy inner product hyperspace in V and let $x, y \in V$. Then the element x is said to be *orthogonal* to y if $\theta(x,y) = 0$.

Proposition 3.17. *Let μ be a fuzzy hyperspace of V with respect to a fuzzy subfield ν of K. Let θ be a fuzzy subset in $V \times V$ defined by $\theta(x,y) = min\{\mu(x), \mu(y)\}$. Then*
$\forall x, y \in V$ and $\forall a \in K$ the following statements are satisfied:
(*i*) $\theta(x,y) = \theta(y,x)$;
(*ii*) $\theta(x+y,z) \geq min\{\theta(x,z), \theta(y,z)\}$;
(*iii*) $inf_{t \in aox}\theta(t,y) \geq min\{\nu(a), \theta(x,y)\}$;
(*iv*) $\theta(0,0) \geq max\{\theta(0,y), \theta(x,0)\} \geq \theta(x,y)$.

Proof. (*i*) is an immediate consequence of definition θ.

(*ii*)

$$\begin{aligned}
\theta(x+y,z) &= min\{\mu(x+y), \mu(z)\}, \\
&\geq min\{min\{\mu(x), \mu(y)\}, \mu(z)\}, \\
&= min\{\mu(x), \mu(y), \mu(z)\}, \\
&= min\{min\{\mu(x), \mu(z)\}, min\{\mu(y), \mu(z)\}\}, \\
&= min\{\theta(x,z), \theta(y,z)\}.
\end{aligned}$$

(*iii*)

$$\begin{aligned}
inf_{t \in aox}\theta(t,y) &= inf_{t \in aox}\{\mu(t), \mu(y)\}, \\
&= min\{inf_{t \in aox}\mu(t), \mu(y)\}, \\
&\geq min\{min\{\nu(a), \mu(x)\}, \mu(y)\}\}, \\
&= min\{\nu(a), \theta(x,y)\}.
\end{aligned}$$

For (*iv*)

$$\begin{aligned}
\theta(x,0) &= min\{\mu(x), \mu(0)\} = \mu(x), \text{ since } \mu(0) \geq \mu(x), \\
&\geq min\{\mu(x), \mu(y)\} = \theta(x,y) \qquad (1)
\end{aligned}$$

Similarly, we conclude that

$$\theta(0,x) \geq \theta(x,y) \qquad (2)$$

Now, if we take $x = 0$ in (1) and $y = 0$ in (2), we obtain

$$\theta(0,0) \geq max\{\theta(0,y), \theta(x,0)\},$$

which complete the proof. □

Remark 3.18. From parts $(i)-(iii)$ of the above proposition it follows that θ is a fuzzy inner product hyperspace in V with respect to the fuzzy subfield ν. Thus with every fuzzy linear subhyperspace of V a unique fuzzy inner product hyperspace in V is associated in a natural way.

Proposition 3.19. *Let μ be a fuzzy hyperspace of V with respect to a fuzzy subfield ν of K. Let θ be a fuzzy subset in $V \times V$ defined by $\theta(x,y) = min\{\mu(x), \mu(y)\}$. Then*
$\forall x, y, \in V$ and $\forall a \in K$ the following statements are satisfied:

(i) $\theta(x,y) = \theta(y,x)$;
(ii) $\theta(x+y,z) \geq min\{\theta(x,z), \theta(y,z)\}$;
(iii) $inf_{t \in aox}\theta(t,y) \geq min\{\nu(a), \theta(x,y)\}$;
(iv) $\theta(0,0) \geq max\{\theta(0,y), \theta(x,0)\} \geq \theta(x,y)$.

Proof. (i) is an immediate consequence of definition θ. (ii),

$$\begin{aligned} \theta(x+y,z) &= min\{\mu(x+y), \mu(z)\}, \\ &\geq min\{min\{\mu(x), \mu(y)\}, \mu(z)\}, \\ &= min\{\mu(x), \mu(y), \mu(z)\}, \\ &= min\{min\{\mu(x), \mu(z)\}, min\{\mu(y), \mu(z)\}\}, \\ &= min\{\theta(x,z), \theta(y,z)\}. \end{aligned}$$

For (iii)

$$\begin{aligned} inf_{t \in aox}\theta(t,y) &= inf_{t \in aox}\{\mu(t), \mu(y)\}, \\ &= min\{inf_{t \in aox}\{min\{\nu(a), \mu(y)\}\}, \\ &\geq min\{min\{\nu(a), \mu(x)\}, \mu(y)\}\}, \\ &= min\{\nu(a), \theta(x,y)\}. \end{aligned}$$

For (iv)

$$\begin{aligned} \theta(x,0) &= min\{\mu(x), \mu(0)\} = \mu(x), \text{ since } \mu(0) \geq \mu(x), \\ &\geq min\{\mu(x), \mu(y)\} = \theta(x,y) \end{aligned} \quad (3)$$

Similarly, we conclude that

$$\theta(0,x) \geq \theta(x,y) \quad (4)$$

Now, if we taking $x = 0$ in (1) and $y = 0$ in (2), we obtain

$$\theta(0,0) \geq max\{\theta(0,y), \theta(x,0)\},$$

which complete the proof. □

Remark 3.20. From parts $(i)-(iii)$ of the above proposition it follows that θ is a fuzzy inner product hyperspace in V with respect to the fuzzy subfield ν. Thus with every fuzzy linear subhyperspace of V there is associated a unique fuzzy inner product hyperspace in V in a natural way.

Example 3.21. Consider Example 3.9(1), and define ν as the $\chi_{\mathbb{R}}$. Define $\theta(x, y) = min\{\mu(x), \mu(y)\}$. Then by Proposition 3.18, θ is a fuzzy inner product on the hyperspace $\mathbb{R}^2$ in Example 3.9.

4. Fuzzy co-inner product, Fuzzy co-norm and Fuzzy co-orthogonality

In this section we formulate the dual notions of fuzzy inner product, fuzzy co-norm and fuzzy co-orthogonality in hyperspaces.

Definition 4.1. Let ν be a fuzzy subfield of F. A fuzzy subset $\overline{\theta}$ of $V \times V$ is said to be a *fuzzy co-inner product hyperspace* in V with respect to ν if $\forall x, y, \in V, \forall a \in F$ the following conditions hold:

(i) $\overline{\theta}(x + y, z) \leq \max \overline{\theta}(x, z), \overline{\theta}(y, z)$,
(ii) $sup_{z \in a \circ x} \overline{\theta}(z, y) \leq min\{\nu(a), \overline{\theta}(x, y)\}$, and
(iii) $\overline{\theta}(x, y) = \overline{\theta}(y, x)$.

Henceforth, the value $\overline{\theta}(x, y)$ is referred to as a *co-inner product* of the elements x, y.

Definition 4.2. Let $\overline{\theta}$ be a fuzzy co-inner product hyperspace in V with respect to a fuzzy subfield ν of F. A *fuzzy co-norm function*, denoted by $\overline{\| . \|}$, is a fuzzy subset of V satisfying $\forall x, y \in V$ and $\forall a \in F$ the following conditions:

(i) $\overline{\| x \|} \geq \overline{\| 0 \|}$;
(ii) $sup_{y \in a \circ x} \overline{\| z \|} \leq min\{\nu(a), \overline{\| x \|}\}$;
(iii) $\overline{\| x + y \|} \leq max\{\overline{\| x \|}, \overline{\| y \|}\}$.

Proposition 4.3. *Let $\overline{\theta}$ be a fuzzy co-inner product hyperspace in V with respect to a fuzzy subfield ν of F and let $\mu(x) = \overline{\theta}(x, x) \quad \forall x \in V$. Then the following statements are equivalent:*

(i) $\mu(x) \geq \mu(0)$;
(ii) $\sup_{y \in a \circ x} \mu(x) \leq min\{\nu(a), \mu(x)\}$ *and*
(iii) $\mu(x + y) \leq max\{\mu(x), \mu(y), \overline{\theta}(x, y)\}$.

Proof. (i) Since $\sup_{y \in a \circ x} \mu(x) \leq min\{\nu(a), \mu(x)\}$, takeing $a = 0$, we obtain $\overline{\theta}(0, y) \leq min\{(\nu(0), \overline{\theta}(x, y)\} = \overline{\theta}(x, y)$, since $\nu(0) = 1$. Similarly, interchanging the role of x and y we obtain $\overline{\theta}(x, 0) \leq \overline{\theta}(x, y)$. Also,

$$\begin{aligned} \mu(0) &= \overline{\theta}(0, 0) \\ &\leq \overline{\theta}(0 \circ x, 0) \\ &\leq min\{\nu(0), \overline{\theta}(x, 0)) \\ &= \overline{\theta}(x, 0) \\ &\leq \overline{\theta}(x, x) \\ &= \mu(x). \end{aligned}$$

For (ii),

$$\begin{aligned} sup_{z \in a \circ x} \mu(z) &= sup_{z \in a \circ x} \overline{\theta}(z, z) \\ &\leq min\{\nu(a), \overline{\theta}(x, z)\} \\ &= min\{\nu(a), min\{\nu(a), \overline{\theta}(x, x)\}\} \\ &= min\{\nu(a), \overline{\theta}(x, x)\} \\ &= min\{\nu(a), \mu(x)\}. \end{aligned}$$

For (iii),

$$\begin{aligned}\mu(x+y) &= \overline{\theta}(x+y, x+y) \\ &\leq max\{\overline{\theta}(x, x+y), \overline{\theta}(y, x)\}, max\{\overline{\theta}(x, y), \overline{\theta}(y, x)\}, max\overline{\theta}(x, y), \overline{\theta}(y, y)\} \\ &= \{\overline{\theta}(x, x), \overline{\theta}(y, y), \overline{\theta}(x, y)\} \\ &= max\{\mu(x), \mu(y), \overline{\theta}(x, y)\}.\end{aligned}$$

□

Remark 4.4. From Proposition 4.3 it is clear that μ is a fuzzy co-norm and is referred to as the "fuzzy co-norm" induced by the fuzzy co-inner product space $\overline{\theta}$.

Definition 4.5. Let $\overline{\theta}$ be a fuzzy co-inner product in V with respect to a fuzzy subfield ν of F. Then $x, y \in V$ are said to be *co-orthogonal*, if $\overline{\theta}(x, y) = 0$.

Obviously, co-orthogonality is a symmetric relation.

For every nonempty subset A of V, we define the fuzzy co-orthogonal complement of A, denoted $A^{\perp}$, by

$$A^{\perp} = \{x \in V | \overline{\theta}(a, x) = 0 \forall a \in A\}.$$

Proposition 4.6. *Let V be a hyperspace and $A \subseteq B$, then*

(i) $B^{\perp} \subseteq A^{\perp}$;
(ii) $A^{\perp}$ is a subhyperspace of V.

Proof. (i) It is routine. To prove (ii), suppose that $x, y \in A^{\perp}, c \in K$,
$\overline{\theta}(x, a) = 0$ and

$$0 \leq \overline{\theta}(x+y, a) \leq max\{\overline{\theta}(x, a), \overline{\theta}(y, a)\} = 0.$$

Hence $x + y \in A^{\perp}$. Again

$$0 \leq inf_{z \in c \circ x}\overline{\theta}(z, , a) \geq min\{\nu(c), \overline{\theta}(x, a)\} = 0$$

and so $c \circ x \subseteq A^{\perp}$

Thus $A^{\perp}$ is a sub-hyperspace of V. □

In the following we show that in any fuzzy hyperspace the fuzzy co-inner product exists.

Proposition 4.7. *Let μ be a fuzzy hyperspace of V with respect to the constant fuzzy subfield $\nu(a) = 1 \forall a \in K$. Define the fuzzy subset $\eta : V \times V \longrightarrow [0, 1]$ by $\eta(x, y) = min\{1 - \mu(x), 1 - \mu(y)\}$. Then for every $x, y, z \in V$ and $a \in K$ the following hold:*

(i) $\eta(x + y, z) \leq max\{\eta(x, z), \eta(y, z)\}$;
(ii) $sup_{z \in a \circ x}\eta(z, y) \leq \eta(x, y)$;
(iii) $\eta(x, 0) = \eta(0, 0) = 0$;
(iv) $\eta(x, 0) = 1 - \mu(x), \quad \forall x \in V$.

Proof. (i),

$$\begin{aligned}\eta(x+y, z) &= min\{1 - \mu(x+y), 1 - \mu(z)\} \\ &\leq min\{max\{1 - \mu(x), 1 - \mu(y)\}, 1 - \mu(z)\} \\ &= maxmin\{1 - \mu(x), 1 - \mu(z)\}, min\{1 - \mu(y), 1 - \mu(z)\} \\ &= max\{\eta(x, z), \eta(y, z)\}\end{aligned}$$

(*ii*)

$$\begin{aligned} sup_{z\in a\circ x}\eta(z,y) &= sup_{z\in a\circ x}\eta(z,y) \\ &= sup_{z\in a\circ x}min\{1-\mu(z),1-\mu(y)\} \\ &= min\{1-sup_{z\in a\circ x}\mu(z),1-\mu(y)\} \\ &\leq min\{1-\mu(x),1-\mu(y)\} \\ &= \eta(x,z) \end{aligned}$$

(*iii*) Obvious. (*iv*) $\eta(x,0) = min\{1-\mu(x), 1-\mu(0)\} = min\{1-\mu(x), 0\} = 1-\mu(x)$. □

The next results is an immediate consequences of Definition 4.6, Proposition 4.7 and Proposition 4.8.

Proposition 4.8. *Let μ be a fuzzy hyperspace of V with respect to the constant fuzzy subfield $\nu(x) = 1 \forall\ x \in V$. Let η be the fuzzy co-inner product hyperspace of V induced by μ. Then the following are satisfied:*

(*i*) $\{0\}^{\perp} = V$.

(*ii*) *Let $U = \{x \in V | \mu(x) = \mu(0) = 1\}$, in case $\mu \neq \chi_V$, where χ_V denotes the characteristic function of V.*

Proposition 4.9. *Let θ and η be two fuzzy hyperspaces of V with respect to fuzzy subfield ν, then $\theta \cup \eta$ is also a fuzzy co-inner product hyperspace in V with respect to the fuzzy subfield ν.*

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