A GEOMETRIC APPROACH IN CHARACTERIZING IPH FUNCTIONS BY THE $p$–FUNCTIONS IN A CLASS OF NORMED VECTOR SPACE

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Abstract. The similarity between convex analysis and monotonic analysis was investigated in [2], in which the $p$–function played an essential role. In this paper we study some more applications of this function to characterize IPH functions from a geometric point of view.

1. Introduction

Many functions show monotonicity in their behavior, and some special classes of monotonic functions have been investigated in this respect [5]. Recent works show close relations between monotonic analysis and convex analysis, which has led to a new subject in analysis, namely; abstract convexity, that has found so many applications in optimization and approximation theory. The core of this relation stems from this well-known theorem: every proper and lower semi-continuous convex function can be expressed as a pointwise supremum of a family of affine functions majorized by it. This motivates the question that for a given function $f$, can we find a collection $L$ of simple structured functions such that $f$ can be stated as a pointwise supremum of the elements of a subcollection $U$ of $L$ that are majorized by $f$? If so, we

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call $f$ abstract convex with respect to $L$, or briefly; $L$-convex. For some classes of monotonic functions such collections are found, specially for IPH functions\cite{1, 4, 5}, which will be introduced and examined in the sequel. To this end, we note that a partial order on a topological vector space $X$ can always be defined by a closed convex cone $K$ in $X$ in which $K \cap (-K) = \{0\}$. We define the partial order $\leq$ on $X$ by
\begin{equation}
  x \leq y \iff y - x \in K, \ (x, y \in X).
\end{equation}

Let $\text{int}K \neq \emptyset$, and $u \in \text{int}K$. Then we introduce the $p$-function by
\begin{equation}
  p(x) := \inf\{\lambda \in \mathbb{R} : x \leq \lambda u\}, \ (x \in X).
\end{equation}

Consider the function
\begin{equation}
  \|x\| := \max(p(x), p(-x)) \ (x \in X).
\end{equation}

It can be easily checked that $\|\| \,$ induces a norm on $X$. From now on, we assume that the the original topology on $X$ and the topology induced on $X$ by this norm coincide. This situation usually occurs, for example; when $X = \mathbb{R}^n, u = (1, 1, ..., 1) \in \mathbb{R}^n, K = \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$. Take $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_i > 0, i = 1, \ldots, n\}$, and $\text{int}D, \bar{D}$ and $\text{bd}D$ as interior, closure and boundary of a set $D$.

**Definition 1.1.** A function $f : X \to \mathbb{R}$ is increasing (decreasing) when $x \leq y$ implies $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$) for every pair $x, y \in X$.

**Definition 1.2.** A set $D \subset X$ is called downward if $d \in D, x \leq d \Rightarrow x \in D$.

**Definition 1.3.** A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called positively homogeneous when $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and $\lambda > 0$. A function $f$ is said to be IPH if it is both increasing and positively homogeneous.

For simplicity, we collect some of the known properties of the $p$-function as a proposition. Some more results together with proofs can be found in\cite{3} and references therein.

**Proposition 1.4.** Let $X$ be the normed vector space as stated above, and consider the $p$-function in (2). The following assertions hold:
i) $p$ is continuous;
ii) $p$ is increasing;
iii) $p$ is finite;
iv) $p$ is sublinear, that is; $p(x + y) \leq p(x) + p(y)$ for all $x, y$ in $X$;

v) $p$ is plus homogeneous, that is; $f(x + \lambda u) = f(x) + \lambda$ for all $x$ in $X$ and $\lambda \in \mathbb{R}$;

vi) $p$ is positively homogeneous (definition 1.3);

vii) $x \leq p(x)u$ for all $x \in X$;

viii) For each $x \in X$ the set $\Lambda_x = \{\lambda \in \mathbb{R} : \lambda u - x \in K\}$ is a closed segment of the form $[\lambda_x, +\infty)$ with $\lambda_x > -\infty$;

ix) The infimum in (2) is attained, namely; $p(x) = \min\{\lambda \in \mathbb{R} : \lambda u - x \in K\}$;

By (iv) and (v), $p$ is a convex function on $X$.

**Definition 1.5.** Let

$$K^y_x = \text{int}(y + K) = y + \text{int}K$$

for all $y \in X$,

we say that a point $x \in X$ is an upper boundary point of a set $G \subset X$ if $x \in \bar{G}$ while $K^y_x \subset X \setminus G$. The set of all upper boundary points of $G$ is called upper boundary of $G$ and it is denoted by $\partial^+ G$.

In our special space $X$, $\partial^+ G$ is always a subset of $\text{bd}G$, the boundary of $G$.

### 2. Application of the $p$-function in characterizing IPH functions

The $p$-function is used in the proof of all subsequent results in [2].

**Proposition 2.1.** [2, Proposition 4.1] Let $G$ be a nontrivial closed downward subset of $X$. For every $x \in X$ and $y \in \text{int}K$, the line passing through $x$ and $x + y$ defined by

$$L(x, y) = \{x - \alpha y : \alpha \in \mathbb{R}\}.$$  

intersects the upper boundary of $G$ (Definition 1.6) in the unique point $\sigma_G(x, y)$, which is introduced by

$$\sigma_G(x, y) = x - \mu y, \quad \mu = \min\{\alpha : x - \alpha y \in G\}.$$  

Now, let $G = -K$ and $\lambda = -\mu$ in (6). Then we have

**Proposition 2.2.** Let $x, y \in \text{int}K$. Then there exists a unique $\lambda > 0$ such that $p(\lambda y - x) = 0$.  

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Theorem 2.3. Let \( x, y \in \text{int}K \), and \( \lambda \) be the unique positive value as in Proposition 2.2. Then \( f : \text{int}K \to \mathbb{R}_+ \) is an IPH function if and only if \( f(x) \geq \lambda f(y) \) for all \( x, y \) in \( \text{int}K \).

Example 2.4. Let \( K = \mathbb{R}_+^n \) and \( u = (1, 1, ..., 1) \). For every \( x, y \) in \( \text{int}K = \mathbb{R}_+^n \), from (6) we deduce that \( \lambda = -\mu = (\min_{1 \leq i \leq n} x_i) \). Hence, \( f \) is IPH if and only if \( f(x) \geq (\min_{1 \leq i \leq n} \frac{x_i}{y_i}) f(y) \) for all \( x, y \) in \( \mathbb{R}_+^n \). This is just the theorem 2.1 in [1].

Consider the definition of \( \lambda = -\mu = -\min\{\alpha : -x - \alpha y \in (-K)\} > 0 \), as stated in proposition 2.2 and (6). A simple computation shows that \( 0 < \lambda = \max\{\alpha : \alpha y \leq x\} \), motivating the definition of \( l(x, y) = \max\{\alpha \in \mathbb{R}_+ : \alpha y \leq x\} \) for all \( x, y \in K \). Note that if \( x, y \in \text{int}K \), then \( l(x, y) = \lambda > 0 \). This is just the concept in [1] to characterize IPH functions on cones, and in [4] to characterize IPH functions on TVSs’ in the framework of abstract convexity.

REFERENCES