Analyze of 3D Elasto-Static Problems by Meshless Local Petrov-Galerkin Method

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Abstract: A truly meshless local Petrov-Galerkin (MLPG) method is developed for solving 3D elasto-static problems. Using the general MLPG concept, this method is derived through the local weak forms of the equilibrium equations, by using test functions, namely, the Heaviside function. The moving least squares (MLS) are chosen to construct the shape functions, for the MLPG method. The penalty approximation is used to impose essential boundary condition. Several numerical examples are included to demonstrate that the present method is very promising for solving the elastic problems.

Keywords: Meshless Local Petrov-Galerkin- 3D Elasto-Static- Moving Least Squares

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1 INTRODUCTION

Compared with the finite element method's convenience and flexibility in use, it has been plagued for a long time, by the inherent problems such as locking, poor derivative solutions, etc. It is well known that the accuracy of the FEM relies on the quality of the mesh and the element type. First, a good-quality of the mesh cannot be always achieved, especially when adaptive refinement and adaptive re-meshing are required for 3D problems. It has also been found that only simple quadrilateral or hexahedral elements have achieved considerable success for explicit dynamic analysis. However, the use of such elements is limited by the mesh generation. In contrast, the truly meshless local Petrov-Galerkin (MLPG) approach has become very attractive as a very promising method for solving 3D problems. The MLPG concept was presented in [1]. The main advantage of this method, over the widely used finite element methods, is that it does not need any mesh, either for the interpolation of the solution variables or for the integration of the weak forms. The many research in solving PDEs demonstrates that the MLPG method, and its variants, have become some of the most promising alternative methods for computational mechanics.

The MLPG method in the present study employs a local symmetric weak form (LSWF), the shape functions are obtained from the MLS approximation and the Heaviside function is used as the test function. Although the MLS approximations have some drawbacks in dealing with the essential boundary conditions, they can be straightforwardly applied to 3D cases, by using the numerical techniques developed for 2D problems. One of the major advantages of the MLS is that, the shape functions are constructed from the local points only, with the high order continuities. Hence, this method leads to lesser cost in assembling the system equations. In the general MLPG approach, the local test domains can be arbitrary, such as spheres, cubes, and ellipsoids in 3D. However, the local sub-domains become very complicated, for the points which are located on, or near, the global boundaries, because of the intersection between the simple sub-domain and the boundary surfaces. In the present study, a method is developed to define the local sub-domains as spheres, with the use of a transformation which maps a circle on a semi-sphere for numerical integrations.

The MLPG method has been demonstrated to be quite successful in solving various partial differential equations. The MLPG concept was presented first by Atluri and Zhu [1]. They are solved elasto-static problems in two dimensional domains. Lin and Atluri [2] introduced an up winding scheme to analyze steady state convection–diffusion problems, and Liu and Gu [3] coupled the MLPG method with either the finite element or the boundary element method to enhance the efficiency of the MLPG method. Ching and Batra [4] augmented the polynomial basis functions with singular fields to determine deformations and stress fields near the crack tip for generally 2D mixed-mode problems. Gu and Liu [5] and Batra and Ching [6] used the Newmark family of methods to analyze 2D transient elasto-dynamic problems. The bending of a thin plate has been studied by Gu and Liu [7] and Long and Atluri [8]. Although the several research successes in solving boundary value problems in two dimensional domains illustrate that the MLPG method and its variants are much comparative with the Galerkin finite element method, there are only a few works that study the application of the MLPG methods in 3D problems. Han and Atluri [9] used the MLPG approach for the solution of the 3D problems in elasto-statics. They are also applied the MLPG method in 3D elastic fracture problem [10] and 3D elasto-dynamics problems [11].

2 THE MOVING LEAST SQUARES

The MLS method of interpolation is generally considered to be one of the best schemes to interpolate random data with a reasonable accuracy. Although the nodal shape functions that arise from the MLS approximation have a very complex nature, they always preserve completeness up to the order of the chosen basis, and robustly interpolate the irregularly distributed nodal information. The MLS scheme has been widely used in domain discretization methods. With the MLS, the distribution of function u in ˜s can be approximated as,

$$u(x) = \boldsymbol{P}^T(x)\boldsymbol{a}(x) \quad \forall x \in \Omega_s \tag{1}$$

where $\boldsymbol{P}^T(x) = [\, p_1(x), p_2(x), \ldots , p_m(x)]$ is a monomial basis of order m; and $\boldsymbol{a}(x)$ is a vector containing coefficients, which are functions of the global cartesian coordinates $[x_1, x_2, x_3]$, depending on the monomial basis. They are determined by minimizing a weighted discrete L_2 norm, defined, as:

$$J(x) = \sum_{i=1}^{m} w_i(x)[\boldsymbol{P}^T(x_i)\boldsymbol{a}(x) - \hat{u}_i]^2$$

$$= [\boldsymbol{P}\cdot\boldsymbol{a}(x) - \hat{u}]^T \boldsymbol{W}[\boldsymbol{P}\cdot\boldsymbol{a}(x) - \hat{u}] \tag{2}$$

Where $w_i(x)$ are the weight functions and $\hat{u}_i$ are the fictitious nodal values. The stationarity of J in Eq. (2), with respect to $\boldsymbol{a}(x)$ leads to following linear relation between $\boldsymbol{a}(x)$ and $\hat{u}$.

$$\boldsymbol{A}(x)\boldsymbol{a}(x)=\boldsymbol{B}(x)\,\hat{u} \tag{3}$$

Where matrices $\boldsymbol{A}(x)$ and $\boldsymbol{B}(x)$ are defined by

$$\boldsymbol{A}(x) = \boldsymbol{P}^T W\boldsymbol{P}\ ,\ \boldsymbol{B}(x) = \boldsymbol{P}^T W \quad \forall x \in \partial\Omega_x \tag{4}$$

Once coefficients $\boldsymbol{a}(x)$ in Eq. (3) are determined, one may obtain the approximation from the nodal values at the local scattered points, by substituting them into Eq. (1), as

$$u(x) = \Phi^T(x)\hat{u} \qquad \forall x \in \partial\Omega_x \tag{5}$$

Where $\Phi(\mathbf{x})$ is the so-called shape function of the MLS approximation, defined as,

$$\Phi(x) = \boldsymbol{P}^T(x)\boldsymbol{A}^{-1}(x)\boldsymbol{B}(x) \tag{6}$$

The weight function in Eq. (2) defines the range of influence of node I. Normally it has a compact support. Numerical practices of [1, 2] have shown that a quadratic spline weight function works well. Hence in this article, the quadratic spline weight function is used. Thus we have

$$w_i(x)=\begin{cases}1-6\left(\frac{d_i}{r_i}\right)^2+8\left(\frac{d_i}{r_i}\right)^3-3\left(\frac{d_i}{r_i}\right)^4 & 0\le d_i\le r_i\\ 0 & d_i\ge r_i\end{cases} \tag{7}$$

Where d_i is the distance between points x and nod x_i and r_i is the size of support for the weight functions. It can be seen that the quadratic spline weight function is C^1 continuous over the entire domain.

3 LOCAL SYMMETRIC WEAK-FORMS OF ELASTICITY

Consider a linear elastic body in a 3D domain Ω, with a boundary $\partial\Omega$. The solid is assumed to undergo infinitesimal deformations. The equations of balance of linear and angular momentum can be written as:

$$\sigma_{ij,j}+f_i=0;\quad \sigma_{ij}=\sigma_{ji}\ ;\quad (\ )_{,i}\equiv\frac{\partial}{\partial\xi_i} \tag{8}$$

Where σ_{ij} is the stress tensor, which corresponds to the displacement field u_i and f_i is the body force. The corresponding boundary conditions are given as follows,

$$u_i=\overline{u_i}\quad \text{on } \Gamma_u$$
$$t_i=\sigma_{ij}n_j=\overline{t_i}\quad \text{on } \Gamma_t \tag{9}$$

Where $\overline{u_i}$ and $\overline{t_i}$ are the prescribed displacements and tractions, respectively, on the displacement boundary Γ_u and on the traction boundary Γ_t, and n_i is the unit outward normal to the boundary Γ.

The strain-displacement relations are:

$$\varepsilon_{kl}=\frac{1}{2}(u_{k,l}+u_{l,k}) \tag{10}$$

The constitutive relations of isotropic linear elastic homogeneous Solid are:

$$\sigma_{ij}=E_{ijkl}\varepsilon_{kl}=E_{ijkl}u_{k,l} \tag{11}$$

Where

$$E_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

with λ and μ being the Lame's constants.

A generalized local weak form of the differential equation (7) over a local sub-domain Ω_s can be written as:

$$\int_{\Omega_s}(\sigma_{ij,j}+f_i)v_i\,d\Omega=0 \tag{12}$$

Where u_i and v_i are the trial and test functions, respectively. By applying the divergence theorem and the boundary conditions, Eq. (12) may be rewritten in a symmetric weak form as:

$$\int_{L_s}t_iv_i\,d\Gamma+\int_{\Gamma_{su}}t_iv_i\,d\Gamma+\int_{\Gamma_{st}}\overline{t}_iv_i\,d\Gamma-\int_{\Omega_s}(\sigma_{ij}v_{i,j}-f_iv_i)\,d\Omega=0 \tag{13}$$

where Γ_{su} is a part of the boundary $\partial\Omega_s$ of Ω_s, over which the essential boundary conditions are specified. In general, $\partial\Omega_s=\Gamma_s\cap L_s$, with Γ_s being a part of the local boundary located on the global boundary, and L_s being the other part of the local boundary which is inside the solution domain. $\Gamma_{su}=\Gamma_s\cap\Gamma_u$ is the intersection between the local boundary $\partial\Omega_s$ and the global displacement boundary Γ_u; $\Gamma_{st}=\Gamma_s\cap\Gamma_t$ is a part of the boundary over which the natural boundary conditions are specified. Therefore, a local symmetric weak form (LSWF) in linear elasticity can be written as:

$$\int_{\Omega_s}(\sigma_{ij}v_{i,j})\,d\Omega-\int_{L_s}t_iv_i\,d\Gamma-\int_{\Gamma_{su}}t_iv_i\,d\Gamma=\int_{\Gamma_{st}}\overline{t}_iv_i\,d\Gamma+\int_{\Omega_s}f_iv_i\,d\Omega \tag{14}$$

If a Heaviside step function is used as the test function for the nodes on the natural boundary or inside the domain, i.e., $u_i^{(I)}\notin\Gamma_{su}$ one may simplify Eq. (14) for $u_i^{(I)}$ as:

$$-\int_{L_s}t_i\,d\Gamma-\int_{\Gamma_{su}}t_i\,d\Gamma=\int_{\Gamma_{st}}\overline{t}_i\,d\Gamma+\int_{\Omega_s}f_i\,d\Omega \tag{15}$$

If the penalty approach is used to impose essential boundary condition we have:

$$-\int_{L_s}t_i\,d\Gamma-\int_{\Gamma_{su}}t_i\,d\Gamma+\alpha\int_{\Gamma_{su}}u_i\,d\Gamma=\int_{\Gamma_{st}}\overline{t}_i\,d\Gamma+\alpha\int_{\Gamma_{su}}\overline{u}_i\,d\Gamma+\int_{\Omega_s}f_i\,d\Omega \tag{16}$$

In this equation coefficient $\alpha>>1$ is used to impose essential boundary condition.

The shape of sub-domains in this study is chosen to be spherical and for numerical integrations, we use a transformation which maps a circle on a semi-sphere.

4 RESULTS OF NUMERICAL EXAMPLES

In this section the meshless local Petrov–Galerkin method is applied to compute three-dimensional elasto-static problems. Three problems in three-dimensional linear elasticity are solved to illustrate the effectiveness of the present method. The numerical results are discussed consequently.

A. *Example 1*

In this case a cube under the hydrostatic pressure is considered. The MLPG approach is applied for this elasto-static problem with boundary conditions are presented in figure 1 as

$$\sigma_x = \sigma_y = \sigma_z = -p \qquad (17)$$
$$\tau_x = \tau_y = \tau_z = 0$$

The analytical solutions for this problem are

$$u_x = -\frac{p}{3\lambda + 2\mu}(x - \frac{a}{2})$$
$$u_y = -\frac{p}{3\lambda + 2\mu}(y - \frac{b}{2}) \qquad (18)$$
$$u_z = -\frac{p}{3\lambda + 2\mu}(z)$$

The node distribution with 27 nodes are presented in figure 2 for the case of *a=b=c=2*. The displacements are presented in figure 3, figure 4 and figure 5 for the case of $p = 1, \lambda = \frac{1}{3.6}, \mu = \frac{1}{2.4}$. As shown in these figures, the MLPG results agree with the values obtained by analytical solution. The convergence of the MLPG approach is demonstrated in these figures.

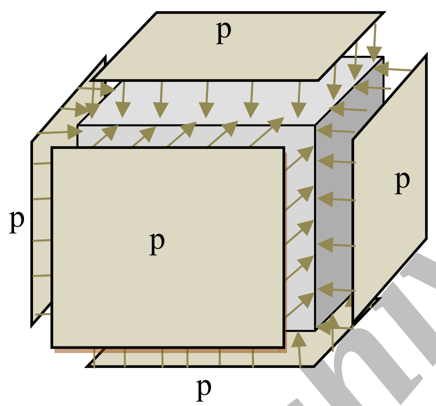

Fig. 1 Geometric and boundary conditions

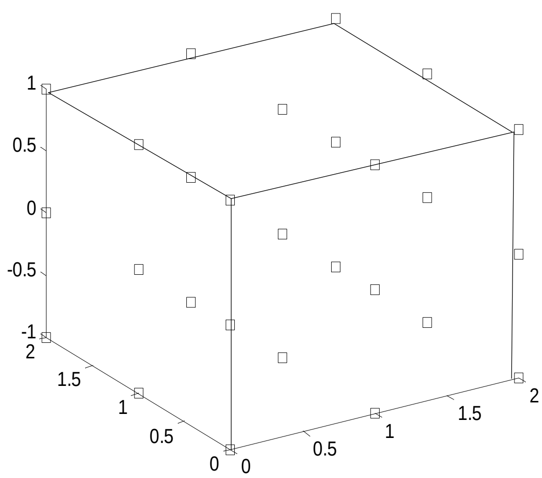

Fig. 2 The node distribution for Example 1

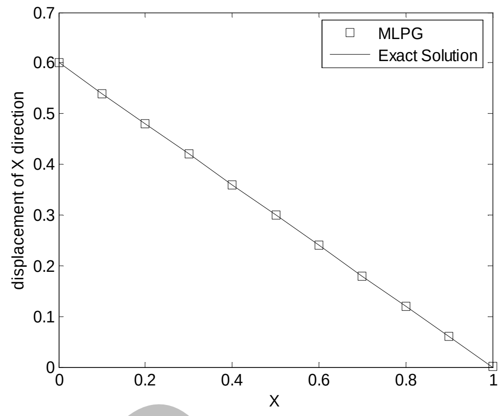

Fig. 3 Displacement of X direction at Z=-h/2 & Y=b

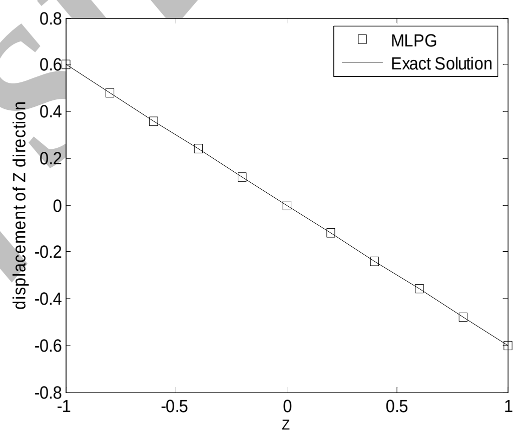

Fig. 4 Displacement of Z direction at X=L/2 & Y=b/2

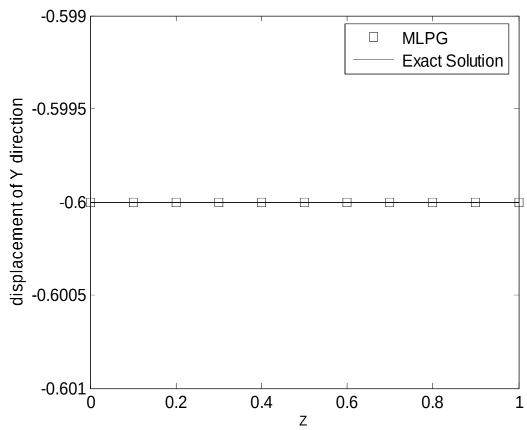

Fig. 5 Displacement of Y direction at Z=-h/2 & Y=b

B. *Example 2*

The second example is a beam under pure bending as illustrated in figure 6. The node distribution with 225 nodes are presented in figure 7 for the case of *L=24, b=2, h=2*.

The exact solutions for this problem are

$$
\begin{aligned}
u_x &= -\frac{xz}{R} \\
u_y &= -\frac{\nu z}{R}\left(y - \frac{b}{2}\right) \\
u_z &= -\frac{1}{2R}\left[x^2 + \nu\left(z^2 - \left(y - \frac{b}{2}\right)^2\right)\right] \\
\frac{1}{R} &= \frac{M}{E I_y}
\end{aligned}
\qquad (19)
$$

Where I_y is the bending stiffness of the plate, as,

$$I_y = \frac{bh^3}{12},$$

$$
E = \begin{cases} E & \text{for plane strain} \\ \frac{1+2\upsilon}{(1+\nu)^2} E & \text{for plane stress} \end{cases} \qquad (20)
$$

and

$$
\nu = \begin{cases} \nu & \text{for plane strain} \\ \frac{\nu}{1+\nu} & \text{for plane stress} \end{cases}
$$

The displacements are presented in figure 8 to figure 11 for the plane stress case with *E=1, ν=0.2* and *M=1*. As shown in these figures, the MLPG results agree with the values obtained by analytical solution.

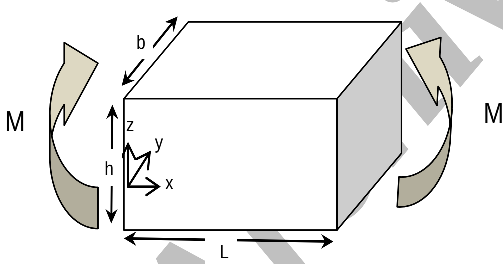

Fig. 6 Geometric and boundary conditions for Example 2

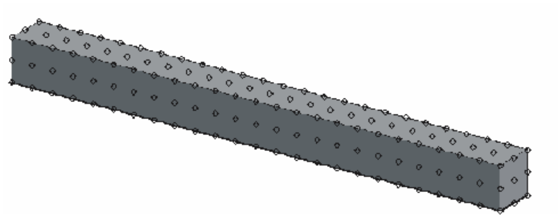

Fig. 7 The node distribution for Example 2

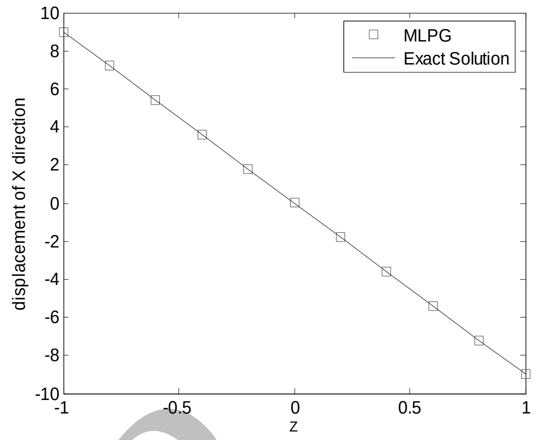

Fig. 8 Displacement of X direction at X=L/2 & Y=b/2

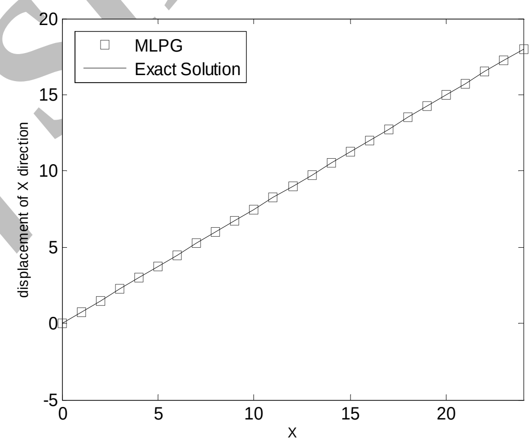

Fig. 9 Displacement of X direction at Z=-h/2 & Y=b/2

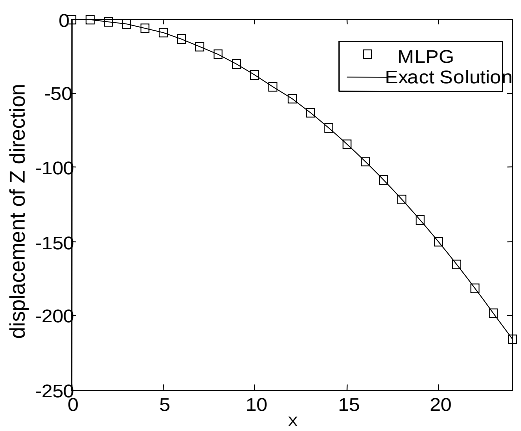

Fig. 10 Displacement in Z direction at Z=h/2 & Y=b/2

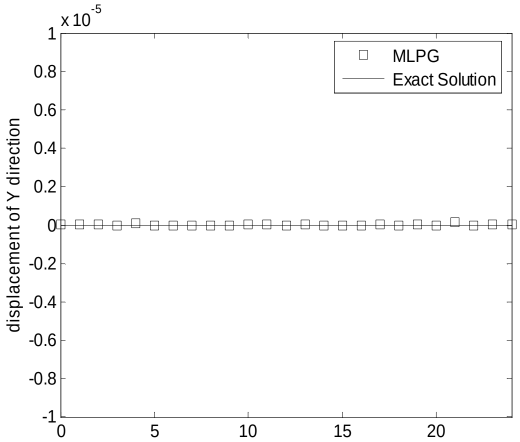

Fig. 11 Displacement of Y direction at Z=-h/2 & Y=b/2

C. *Example 3*

The cantilever beam under a transverse load problem is shown in figure 12. The exact solutions for this problem are

$$u_x = -\frac{Pz}{6bEI}[3x(2L-x)+(2+\nu)(z^2-\frac{h^2}{4})]$$

$$u_z = \frac{P}{6bEI}\,[x^2(3L-x)+3\nu(L-x)z^2+\frac{4+5\upsilon}{4}h^2x] \qquad (21)$$

Where I is the bending stiffness of the plate, as

$$I = \frac{bh^3}{12},$$

$$E = \begin{cases} E & \text{for plane strain} \\ \frac{1+2\nu}{(1+\nu)^2}E & \text{for plane stress} \end{cases} \qquad (22)$$

and

$$\nu = \begin{cases} \nu & \text{for plane strain} \\ \frac{\nu}{1+\nu} & \text{for plane stress} \end{cases}$$

The corresponding stresses are:

$$\sigma_x = -\frac{Pz}{bI}(L-x)$$
$$\sigma_z = 0 \qquad (23)$$
$$\sigma_{xz} = -\frac{P}{2bI}(z^2-\frac{h^2}{4})$$

The problem is solved for the plane stress case with *P=1, E=1, b=h=2, L=24* and *ν=0.2*. Regular uniform nodal configurations with nodal distances are used, as figure 13 shows the configuration with a nodal distance of 1.0. The number of nodes is 225.

First, a uniform tension load is applied to the free end of the cantilever beam. The problem is solved by using the mentioned MLPG method with MLS approximation. The numerical results are shown in figure 14 to figure 18, which agree with the analytical solution well.

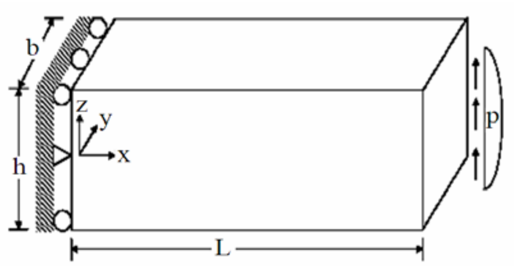

Fig. 12 Cantilever beam with an end load

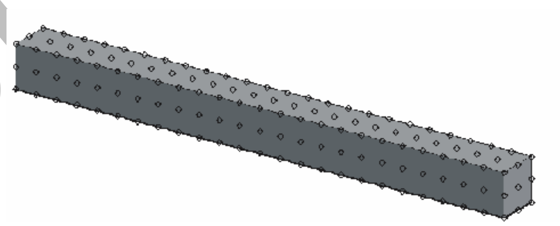

Fig. 13 The node distribution for Example 3

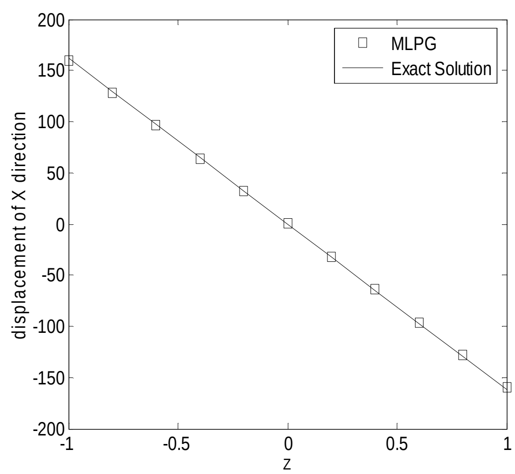

Fig. 14 Displacement of X direction at X=L/2 & Y=b/2

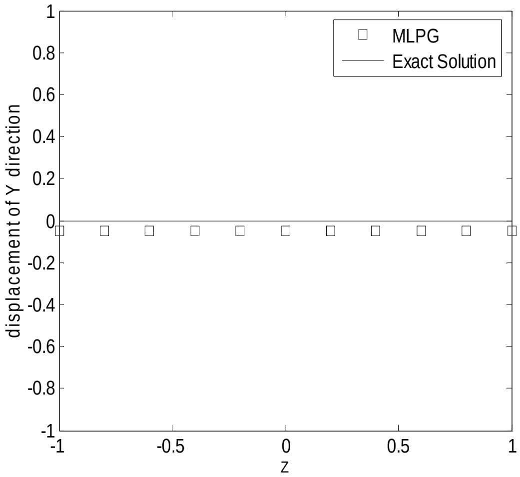

Fig. 15 Displacement of Y direction at X=L/2 & Y=b/2

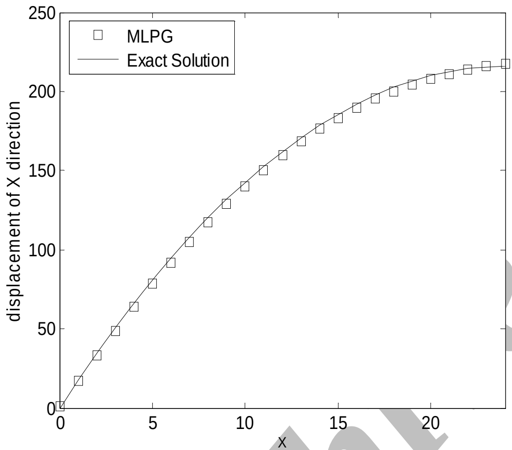

Fig. 16 Displacement of X direction at Z=-h/2 & Y=b/2

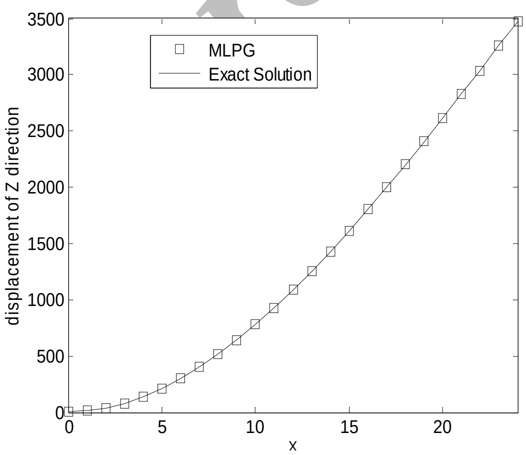

Fig. 17 Displacement of Z direction at Z=-h/2 & Y=b/2

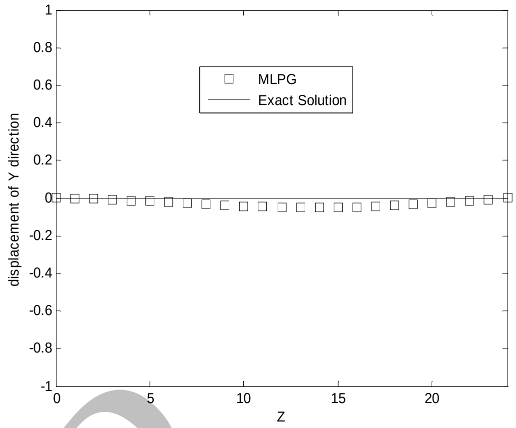

Fig.18 Displacement of Y direction at Z=-h/2 & Y=b/2

5 CONCLUSION

A meshless local Petrov-Galerkin (MLPG) method is developed for 3D static problems, based on the local symmetric weak forms. The MLS approximation is used for constructing the trial functions. The penalty approximation is used to impose essential boundary condition. A simple heaviside step function is chosen for test function. The numerical results demonstrate the high accuracy of this method while comparing with the exact solution.

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