Volume Fraction Optimization of Four-Parameter FGM Beams Resting on Elastic Foundation

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Abstract: In this paper, volume fraction optimization of Functionally Graded (FG) beams resting on elastic foundation for maximizing the first natural frequency is investigated. The two-constituent functionally graded beam consists of ceramic and metal. These constituents are graded through the beam thickness according to a generalized power-law distribution. One of the advantages of generalized power-law distribution is the ability of controlling the materials volume fraction of FG structures for considered applications. The primary optimization variables are the four parameters in the power-law distribution. Since the optimization processes are complicated and time consuming, a novel meta–heuristic called Imperialist Competitive Algorithm (ICA) and Artificial Neural Networks (ANNs) are implemented to improve the speed of optimization problem. The performance of ICA is evaluated in comparison with Genetic Algorithm (GA). Results show the success of combination of ANN and ICA for design of material profile of FG beam. Results also show that the combination of ANN and ICA can reduce the run time considerably.

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1   INTRODUCTION

A new class of materials known as functionally graded materials (FGMs) has attracted much attention as advanced structural materials in many structural members. FGMs are composite materials that are microscopically inhomogeneous, and the mechanical properties vary continuously in one (or more) direction(s). Beams and columns supported along their length are very common in structural configurations. Beams structures are often found to be resting on earth in various engineering applications. These include railway lines, geotechnical areas, highway pavement, building structures, offshore structures, transmission towers and transversely supported pipe lines.

This motivated many researchers to analyze the behaviour of beam structures on various elastic foundations [1-6]. Recently, Yas and Sobhani Aragh have used a four-parameter power law distribution to study the free vibration analysis of functionally graded fiber orientation cylindrical panels [7]. Static and free vibration analyses of continuously graded fiber-reinforced cylindrical panels using generalized power-law distribution are also presented by Sobhani Aragh and Yas [8].

Optimization is the task of finding one or more solutions which correspond to minimizing (or maximizing) one or more specified objectives, which satisfies all constraints (if any). Optimization is implemented for various objective functions in mechanical problems, such as buckling loads [9], weight (either as a constraint or as an objective to be minimized) [10], [11], stiffness [12], fundamental frequencies [9], deflection [10], etc. When the search space becomes large, enumeration is soon no longer feasible simply because it would take far too much time. This necessitates using a specific technique to find the optimal solution.

In the present work, Imperialist Competitive Algorithm (ICA) is implemented that has recently been introduced by Atashpaz-Gargary and Lucas (2007) for dealing with different applications, such as designing PID controller, characterizing materials properties, error rate beam forming, designing vehicle fuzzy controller, etc [13-16]. M. Abouhamze et al. [9] optimized stacking sequence of laminated cylindrical panels with respect to the first natural frequency and critical buckling load. They used genetic algorithm and neural network for optimization. The concept of neural networks has been introduced to different branches of engineering, analytical procedure of structural design, structural optimization problems and functionally graded materials [9], [17-20].

The aim of this study is to present useful results on optimization of four-parameter power-law distribution

for maximizing the first natural frequency of FG beam using imperialist competitive algorithm and neural network. As a simple modelling technique, in this work, ANN is employed to reproduce the fundamental frequency parameter in order to reduce the time of the optimization process. The fundamental frequency parameter of beam is obtained by using numerical technique termed the generalized differential quadratur (GDQ) method based on the DQ technique [21].

2   PROBLEM DESCRIPTION

Consider a FG beam resting on two-parameter elastic foundation as shown in Fig. 1 where $k(x)$, and $k_1(x)$ are Winkler foundation modulus and second parameter foundation modulus respectively.
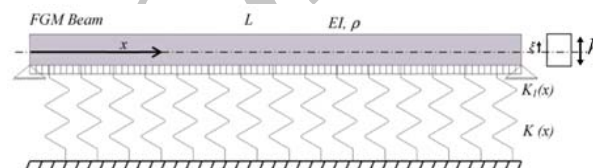

Fig. 1   FG beam supported on two-parameter elastic foundation

In this work materials distribution is as follows [7]:

$$p_{fgm} = (p_c - p_m)V_c + p_m \tag{1}$$
$$V_c = (1 - a\,(\frac{1}{2} + \eta) + b\,(\frac{1}{2} + \eta)^c)^p \quad , \quad -\frac{1}{2} \leq \eta = \frac{\varsigma}{h} \leq \frac{1}{2}$$

Where P_c, P_m represent mechanical properties of the ceramic and metal respectively, and parameters a, b and c must be chosen so that $0 \leq V_c \leq 1$. For FG beam resting on two-parameter elastic foundation in the absence of body force, the governing equation can be expressed as:

$$-(D_{fgm} - \frac{B_{fgm}^2}{A_{fgm}})\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}(k_1(x)\frac{\partial w}{\partial x}) - k(x)w - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \quad 0 < x < L \tag{2}$$

Where

$$(A_{fgm}, B_{fgm}, D_{fgm}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(1, z, z^2)dz \quad , \quad Q_{11} = \frac{E_{fgm}}{1 - v_{fgm}^2}$$

To obtain the natural frequency, Eq. (2) is formulated as an eigenvalue problem by using the following periodic function $w(x,t) = W(x)e^{-i\omega t}$, where $W(x)$ is the mode shape of the transverse motion of the beam.

A semi–analytical procedure with the aid of DQ technique was recently developed [22], [23]. According to the principles of this method, the nth-order partial derivative of a continuous function $f(x, z)$ with respect to x at a given point x_i can be approximated by a weighted linear summation of all the functional values along the discretised domain of x. If N discrete points are sampled in the domain of x, the nth-order partial derivative should be

$$\frac{\partial f^{\,n(x_i,z)}}{\partial x^{\,n}} = \sum_{k=1}^{N} c_{ik}^{n} f(x_{ik}, z) \quad , \tag{3}$$
$$i = 1, 2, ..., N \quad , \quad n = 1, 2, N-1$$

Where c_{ij}^{n}, are the weighting coefficients relating the nth derivative to the functional values at x_i and N is the number of sampling points.

3   NEURAL NETWORK MODELLING

ANN modeling is an equation-free, data-driven modeling technique that tries to emulate the learning process in the human brain by using many examples. ANN can be defined as a massive parallel-distributed information processing system that has a natural propensity for recognizing and modeling complicated input-output systems. The basic element of an NN is the artificial neuron as shown in Fig. 2 which consists of three main components namely as weights, bias, and an activation function.
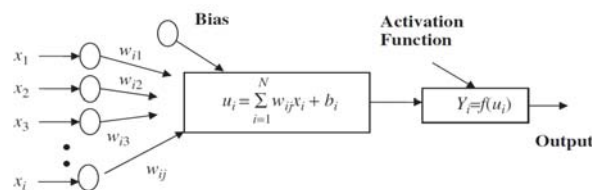

Fig. 2   Basic elements of an artificial neuron

Each neuron receives inputs $x^1; x^2; . . . ; x^n$, attached with a weight w^i which shows the connection strength for that input for each connection. Each input is then multiplied by the corresponding weight of the neuron connection. A bias bi can be defined as a type of connection weight with a constant nonzero value added to the summation of inputs and corresponding weights u, given by

$$u_i = \sum_{j=1}^{H} w_{ij} x_j + b_i \tag{4}$$

The summation u_i is transformed using a scalar-to-scalar function called an ''activation or transfer function'', $F(u_i)$ yielding a value called the unit's ''activation'', given by:

$$Y_i = f(u_i) \tag{5}$$

Activation functions serve to introduce nonlinearity into NNs which makes NNs so powerful. NNs are commonly classified by their network topology (i.e. feedback, feed forward) and learning or training algorithms (i.e. supervised, unsupervised). There is no well-defined rule or procedure to have optimal network architecture. In this study, the feed forward multi-layer perceptron (MLP) network has been applied. MLP networks are one of the most popular and successful neural network architectures which are suited to a wide range of applications such as prediction and process modeling. The neural network architecture adopted in the present work has two hidden layers which has high accuracy and has been used for various applications. Fig. 3 illustrates the topology of a simple, fully connected four-layer MLP network.
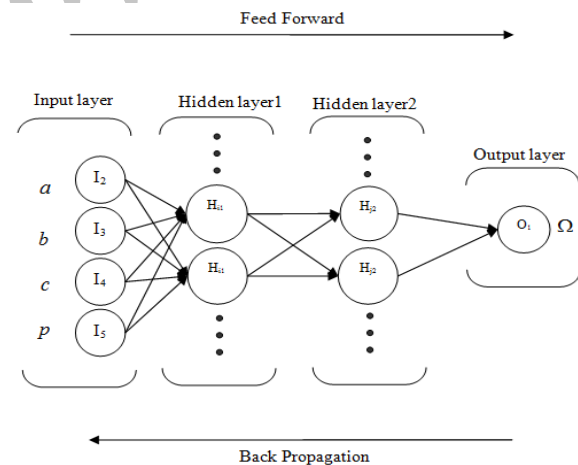

Fig. 3   Schematic diagram of ANN model

4   IMPERIALIST COMPETITIVE ALGORITHM

Imperialist Competitive Algorithm is a novel global search heuristic for optimization that uses imperialism an imperialistic competition process as a source of inspiration. Like other evolutionary algorithms, ICA starts with an initial population called countries that are divided in two types: imperialists (in optimization terminology, countries with least cost) and colonies (the remained countries). In ICA, the more powerful imperialist, have the more colonies. Based on the power of countries (the counterpart of fitness value in

Genetic Algorithm which is inversely proportional to its cost), all of them are divided among the mentioned imperialists. Starting the competition, imperialists attempt to achieve more colonies and colonies in each of them start to move toward their relevant imperialist which this movement is shown in Fig. 4. In this movement θ, x are random numbers with uniform distribution as illustrated in Eq. (6) and d is the distance between colony and the imperialist. In Eq. (6), β and γ are parameters that modify the area that colonies randomly search around the imperialist. Weak empires will lose their power and will be eliminated from the competition (in other words, they will be collapsed) and the powerful ones will be improved and remain. At last, only one imperialist will remain that in this stage, colonies have the same position and power as the imperialist.

$$x \sim U(0\ ,\ \beta \times d)\ \ ,\ \ \theta \sim U(-\gamma, \gamma) \tag{6}$$
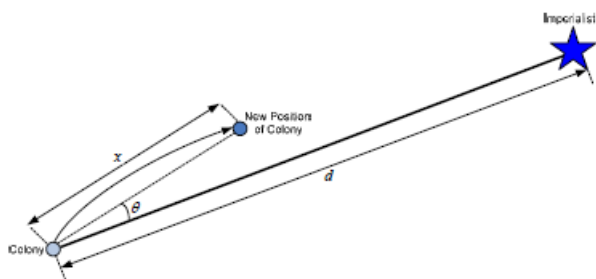

Fig. 4   Motion of colonies toward their relevant imperialist
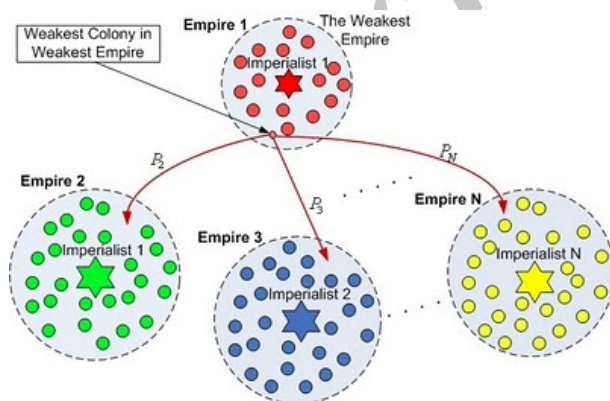

Fig. 5   Imperialistic competition

In imperialistic competition which is shown in Fig. 5, all empires try to take possession of colonies of other empires and control them. Based on the total power of empires which is observed in Eq. (7), the more powerful an empire, the more likely it will possess the

colonies. Imperialistic competition can be modelled as Fig. 5.

$$T.C._{.n} = Cost\,(imperialist_n) + \zeta\,mean$$
$$\{Cost\,(colonoes\ of\ impire_n)\} \tag{7}$$

The main steps in the ICA are summarized as follows [24]:
- Select some random points on the function and initialize the empires.
- Move the colonies toward their relevant imperialist (Assimilating).
- If there is a colony in an empire which has lower cost than that of imperialist, exchange the positions of that colony and the imperialist.
- Compute the total cost of all empires (related to the power of both imperialist and its colonies).
- Pick the weakest colony (colonies) from the weakest empire and give it (them) to the empire that has the most likelihood to possess it (imperialistic competition).
- Eliminate the powerless empires.
- If there is just one empire, stop, if not go to 2.

Also, the flowchart of the ICA is shown in Fig. 6. More details about the proposed algorithm are found in reference [24].
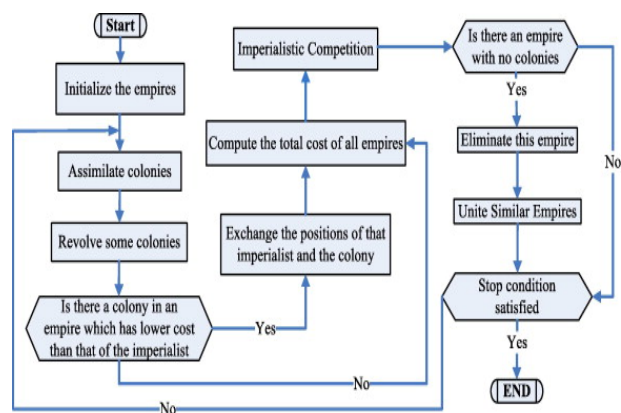

Fig. 6   Flowchart of the Imperialistic competitive algorithm

5   RESULT AND DISCUSION

5.1. Free vibration
First, verification study of the result is considered for an isotropic beam resting on Winkler elastic foundation in Table 1. As observed there is good agreement between the present results with similar ones obtained by Zhou Ding [1].

Table 1 Comparison of the frequency parameters of an isotropic beam resting on parabolic type of Winkler elastic foundation $(k(x) = k_0(1 - \beta x^2), \lambda_i^4 = \frac{\rho A L^4}{E I_0} \omega_i^2)$

k_0	β		λ_1	λ_2	λ_3
1000	**0.4**	**Zhou[1]**	5.597	7.022	9.675
		Present	5.596	7.023	9.674
2000	**0.6**	**Zhou[1]**	6.444	7.521	9.874
		Present	6.445	7.520	9.873

This section introduces some results and considerations about the free vibration problem of functionally graded beams on elastic foundations by means of GDQ method. In this study, ceramic and metal are particle mixed to form the functionally graded material. The relevant material properties for the constituent materials are listed in Table 2.

Table 2 Mechanical properties of the materials

|	Aluminum	Alumina
ρ (Kg/m^3)	2707	3800
E (Gpa)	70	380

The influence of the index p on the first natural frequency of simply supported FG beam on elastic foundations is shown in Table 3. As can be seen from this table, increasing the values of the parameter index p up to infinity reduces the contents of ceramic phase and at the same time increases the percentage of metal phase. In other words, by considering the relation (1), it is possible to obtain the homogeneous isotropic material when the power-law exponent is set equal to zero ($p=0$) or equal to infinity ($p=\infty$).

Table 3 The first non-dimensional natural frequency of FG beams resting on elastic foundation with simply supported ends ($k_1 =$, $a = 0.6$, $c = 3$)

k	$p = 0$ Ceramic beam	$p = 5$	$p = 10$	$p = \infty$ metal beam
10	20.6907	17.0530	15.5887	11.2655
100	22.1860	19.2393	18.0926	14.7280
900	32.5901	32.7938	32.8174	31.8891

The influence of the index p on the natural frequency is also shown in Figs 7-9 for two sets of boundary conditions, that is, simply supported-simply supported (S-S) and clamped- simply supported (C-S) boundary conditions. In these figures it is assumed that $k = 900$. It is expected that the values of natural frequency parameter be between the natural frequencies of metal

($p=0$) and ceramic ($p=\infty$) beams; but it is seen from Figs. 7-9 that these values sometimes exceed the limit cases. Various parameters such as the boundary condition, the materials distribution profile, mechanical properties of materials, elastic foundations modulus, etc may influence on this fact. Thus, it is possible to obtain dynamic characteristics similar or better than the isotropic ceramic limit case by choosing suitable values of the four parameter *a, b, c, p*.
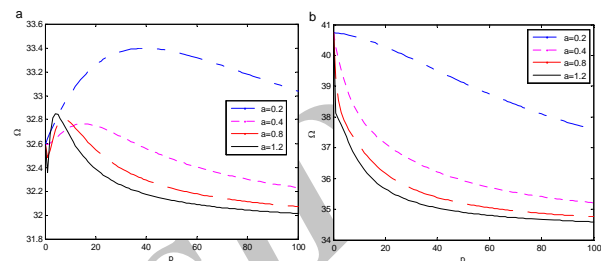
Fig. 7 Variation of the natural frequency parameter vs. the power-law exponent p for various values of the parameter *a* and different boundary conditions $(0.2 \leq a \leq 1.2$, $b = 0.2$, $c = 2)$
(**a:** S-S   **b:** C-S)
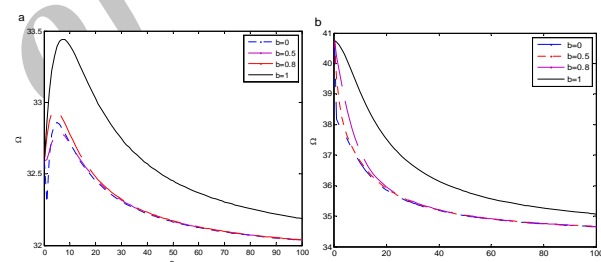
Fig. 8 Variation of the natural frequency parameter vs. the power-law exponent p for various values of the parameter *b* and different boundary conditions $(a = 1$, $0 \leq b \leq 1$, $c = 1)$
(**a:** S-S   **b:** C-S)
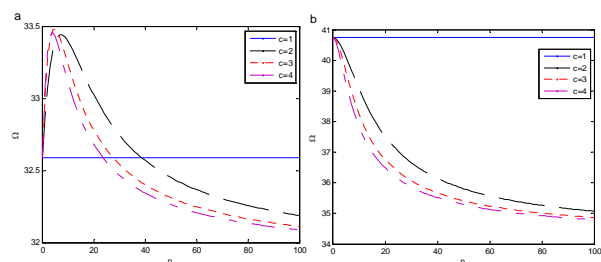
Fig. 9 Variation of the natural frequency parameter vs. the power-law exponent p for various values of the parameter *c* and different boundary conditions $(a = 1$, $b =$, $1 \leq c \leq 4)$
(a: S-S   b: C-S)

5.2. Optimization procedure

The objective of optimization in this paper is to find the best values of the parameters *a, b, c, p* in the four-parameter power-law distribution. The boundary conditions of the FG beam are considered simply

supported and the elastic foundation modulus are assumed to be $k_1 = $, $k = 900$. The parameters are considered in the following ranges:

$$-15 \leq a \leq 15 , \; -15 \leq b \leq 15, \; 0 \leq c \leq 15 , \; 0 \leq p \leq 15$$

The optimization problem is defined as:

$$\text{Minimize} \qquad f(a,b,c,p) = -\Omega \tag{8}$$

It should be noted that since ICA minimizes the objective function basically, fundamental frequency parameter has been multiplied by minus in the above equation. If GDQ method is applied, the optimization process becomes so complicated and time consuming. For example even if the increment of the parameters (*a, b, c, p*) is assumed to be 0.25, the formed discrete space contains more than 50,000,000 design choices to be searched for an optimum point. Also, if it is assumed that the process of one search takes 0.1 second in average, the optimization process takes about 1400 hours.

Thus in the present work, ANN and ICA are implemented for increasing the speed of optimization. For this purpose, 200 training examples have been given to ANN as inputs. The MLP network has been used having two hidden layers. There is no well-defined rule or procedure to have optimal network architecture. A program was developed in MATLAB which handles the trial and error process automatically. The program tries various number of neurons in hidden layers from two up to fifteen for 1000 epochs for 6 times for different back propagation training algorithms. 12 neurons for first hidden layer and 8

neurons for second hidden layer, also Levenberg–Marquardt (LM) algorithm are chosen for the network because it performs better than other cases. The ability of trained network to reproduce the fundamental frequency parameter is shown in Fig. 10 and table 4 for 81 and 5 test points respectively which were selected far from the training point randomly. As observed there is a very good agreement between the results obtained by GDQ with similar one obtained by ANN where they are so close to each other. Since the neural network has been accurately designed, it can be implemented as objective function in imperialist competitive algorithm by simulating fundamental frequency parameter.
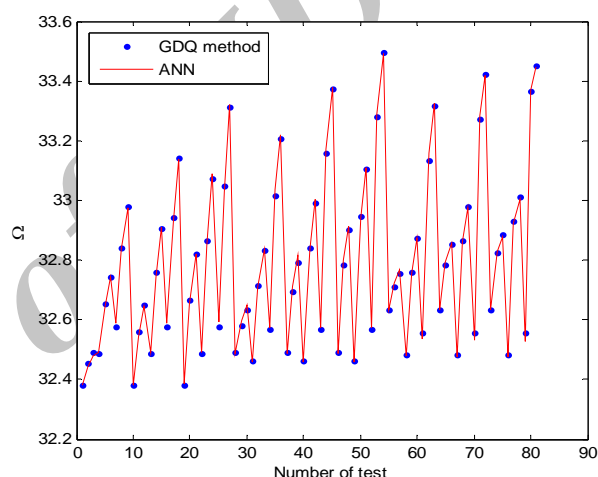

Fig. 10   Mapping nonlinear data to a higher dimensional feature space

Table 4   Comparison of various predicted values of fundamental frequency parameter versus GDQ data

Number of test	Value of parameters				Ω		Related Error %
	a	*b*	*c*	*p*	GDQ	ANN	
1	1.3	-0.1	4.5	7	32.7480	32.7471	0.003%
2	1	0.23	3.3	5	32.8452	32.8448	0.001%
3	0.35	-0.7	6.5	4	32.3090	32.2964	0.039%
4	1.3	0.7	5	9	32.6573	32.6603	0.009%
5	1.5	0.65	2.9	6	32.7486	32.7471	0.005%

Here, optimization is investigated for the FG beam. Table 5 shows the parameters of ICA used to find the optimal solution. The algorithm reached to optimal

solution of [1.30 1.28 3.21 3.03] which leads the -33.4770 for value of the objective function.

Table 5 **P**arameters of ICA approach

Parameters	Value
Number of initial countries	60
Number of initial imperialists	5
β	2
γ	0.4
ζ	0.06

The performance of ICA is compared with Genetic Algorithm. Comparison shows that the results obtained by ICA are more optimized than genetic algorithm. This fact is shown in Table 6. It should be mentioned that the process of optimization in ICA took less than 5 minutes. It means CPU time is reduced by a considerable amount. Fig. 11 shows the optimized material profile for the maximum frequency parameter. According the parameters obtained from ICA, density of FG beam will be $\rho = 3035 \, kg/m^3$. Comparing with Fig. 8 (if $p=0$, $\Omega = \Omega_c = 325901$ and $\rho = \rho_c = 3800 \, kg/m^3$), one may come to this conclusion that by choosing suitable values of *a, b, c, p*, frequency parameter can be

obtained more than the frequency parameter of the similar beam made of 100% ceramic and at the same time lighter.
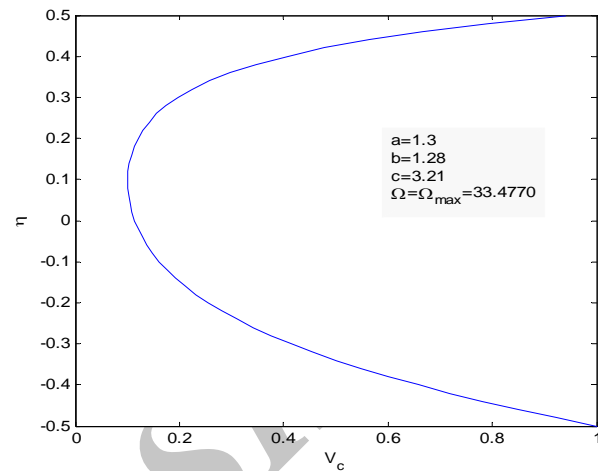
Fig. 11   Optimized material profile for maximum natural frequency

Table 6   Comparison between the Imperialist competitive Algorithm and genetic algorithm

Algorithm	Optimum parameters				$-\Omega$
	a	*b*	*c*	*p*	
ICA	1.30	1.28	3.21	3.03	33.47
GA	0.717	0.717	3.35	5.62	33.41

6   CONCLUSION

In this research, free vibrations of four-parameter FG beam on elastic foundation were studied through using GDQ method. The effect of the power-law exponent, power-law distribution choice on the natural frequencies of FG beams was investigated. Next, volume fraction optimization of FG beam resting on elastic foundations with respect to the first natural frequency was studied. Imperialist competitive algorithm and artificial neural networks were performed to obtain the best material profile through the thickness to maximize the fundamental natural frequency. Some conclusions may be made from this research work:

❖ Interesting result shows that although frequency parameter of the ceramic beam is more than the metal one, the frequency parameter of the FG beam doesn't necessarily increase with the increase in ceramic volume fraction. In other words, by choosing suitable values of *a, b, c,* and *p*,

frequency parameter may be obtained more than the frequency parameter of the similar beam made of 100% ceramic and at the same time lighter.

❖ By using four-parameter power law distribution, it is possible to control the materials volume fraction of FG structures for considered applications.

❖ ICA may be applied for engineering optimization problems especially for those related to the FGM structures.

❖ The performance of ICA is better than other nature inspired technique Genetic Algorithm. In other words, the results obtained by ICA are more optimized than GA.

❖ Combination of NN and ICA reduces the CPU time considerably with losing negligible accuracy.

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