"Research Note"

ON THE MOMENTS OF LKD[*]

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Abstract – The mean and variance of the Lagrangian Katz family of distribution (LKD), the basic Lagrangian Katz distribution of Type I (LKD-I) and Type II (LKD-II) are provided in an easy method. The recurrence relation between the non-central, central and factorial moments, and the recurrence relation between moments and cumulants of LKD, LKD-I and LKD-II are given. The first four cumulants of LKD-I and LKD-II are provided.

Keywords – Lagrangian Katz Family of Distribution (LKD), recurrence relations, central, non-central and factorial moments, cumulants

1. INTRODUCTION

Katz [1, 2] defined and studied a new family of probability distributions known as the Katz distributions, of which the binomial, Poisson and negative binomial distributions are special cases. Consul and Shenton [3-5] and Consul [6, 7] have exploited the Lagrange expansion of analytic function $f(t)$, under the transformation $t = ug(t)$, to define and study numerous families of Lagrangian probability distributions under different titles. Consul [8] applied the Lagrangian expansion to the probability generating function of Katz distribution to obtain the family of Lagrangian Katz distribution (LKD) whose probability mass function (pmf) is given by:

$$P(x; a, b, \beta) = P(X = x) = \frac{\frac{a}{\beta}}{\frac{a}{\beta} + \frac{xb}{\beta} + x} \binom{\frac{a}{\beta} + \frac{xb}{\beta} + x}{x} \beta^x (1 - \beta)^{\frac{a}{\beta} + \frac{xb}{\beta}} \tag{1}$$

For $x = 1,2,...$ and zero otherwise, where $a > 0, b > -\beta, \beta < 1$. Consul and Famoye [9] studied a number of interesting properties of the versatile three parameters LKD. They provided the probability generating function of the LKD and have shown that the LKD satisfy the convolution property. They considered some methods for estimating the parameters of LKD, and also provided the mean, variance, and recurrence relation among the non-central and central moments of LKD.

Following Consul and Famoye [9], let $a = c\beta$ and $b = h\beta$ in LKD be defined by (1). Thus, the probability model in (1) becomes

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$$f_X(x;\beta) = \frac{c}{c+hx+x}\binom{c+hx+x}{x}\beta^x(1-\beta)^{c+hx} \tag{2}$$

A basic Lagrangian Katz distribution of Type I (LKD-I) is defined by p.m.f.

$$P(X=x) = \frac{1}{x}\binom{\frac{xb}{\beta}+x-2}{x-1}\beta^{x-1}(1-\beta)^{\frac{xb}{\beta}}, \qquad x = 1,2,... \tag{3}$$

and zero otherwise. The basic LKD-I is the limit of zero-truncated LKD as $a \to -\beta$.

A basic Lagrangian Katz distribution of type II (LKD-II) is defined by the p.m.f.

$$P(X=x) = \frac{1}{x}\binom{\frac{xb}{\beta}+x}{x-1}\beta^{x-1}(1-\beta)^{1+\frac{xb}{\beta}}, \qquad x = 1,2,... \tag{4}$$

and zero otherwise. The basic LKD-II is the limit of zero-truncated LKD as $a \to \beta$.

Following Janardan [10], if X is a discrete random variable having a p.m.f. $f(x;\beta)$, which satisfies the differential equation:

$$\frac{df(x;\beta)}{d\beta} = \{x - C(\beta)\}B(\beta)f(x;\beta), \tag{5}$$

then the mean, variance, recurrence relation between non-central, central and factorial moments, and the recurrence relation between moments and cumulants are as follows, respectively:

$$\mu = E(X) = C(\beta) \tag{6}$$

$$\sigma^2 = \mu_2 = \frac{1}{B(\beta)}\frac{d\mu}{d\beta} \tag{7}$$

$$\mu'_{r+1} = \frac{1}{B(\beta)}\cdot\frac{d\mu'_r}{d\beta} + \mu'_r\mu \tag{8}$$

$$\mu_{r+1} = \frac{1}{B(\beta)}\cdot\left[\frac{d\mu_r}{d\beta} - r\mu_{r-1}\frac{d\mu}{d\beta}\right] \tag{9}$$

$$\mu^{(r+1)} = \frac{1}{B(\beta)}\cdot\frac{d\mu^{(r)}}{d\beta} + (\mu-r)\mu^{(r)} \tag{10}$$

$$k_{r+1} = \frac{1}{B(\beta)}\sum_{j=1}^{r}\binom{r-1}{j-1}\mu'_{r-j}\cdot\frac{dk_j}{d\beta} - \sum_{j=2}^{r}\binom{r-1}{j-2}\mu'_{r+1-j}k_j \tag{11}$$

We recall that $k_r = \frac{d}{dt}LnM_X(t)|_{t=0}$ is the rth cumulant of X, where $M_X(t)$ is the moment generating function of X. For example, if $X \sim B(n,p)$, then $k_1 = \mu = np$, $k_2 = \mu_2 = np(1-p)$, $k_3 = \mu_3 = np(1-p)(1-2p), \dots$.

The objective of this paper is to find the mean and variance of LKD, LKD-I and LKD-II in a simpler derivation. Also, we will consider the recurrence relation between non-central, central and factorial moments of LKD, LKD-I and LKD-II. Finally, recurrence relation between moments and cumulants of LKD, LKD-I and LKD-II are provided and the first four cumulants of LKD, LKD-I and LKD-II are computed.

2. MOMENTS AND RECURRENCE RELATIONS

While the direct evaluation of the mean, variance, higher order moments and cumulants of the LKD is very complicated, the approach suggested by this paper provides an easy method. Thus, differentiating (2) with respect to β yields:

$$\frac{df(x;\beta)}{d\beta} = \left[x - \frac{c\beta}{1-\beta-h\beta} \right] \frac{1-\beta-h\beta}{\beta(1-\beta)} f(x;\beta) \tag{12}$$

Comparing (12) and (5) we identify the functions as:

$$C(\beta) = \frac{c\beta}{1-\beta-h\beta} \quad , \quad B(\beta) = \frac{1-\beta-h\beta}{\beta(1-\beta)}$$

From which the mean, variance, recurrence relations between non-central moments, central moments and factorial moments and the recurrence relation between moments and cumulants of LKD are as follows, respectively:

$$\mu = \frac{c\beta}{1-\beta-h\beta} \tag{13}$$

$$\mu_2 = \sigma^2 = \frac{c\beta(1-\beta)}{(1-\beta-h\beta)^3} \tag{14}$$

$$\mu'_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu'_r}{d\beta} + \mu'_r \cdot \mu'_1, \qquad r = 1,2,... \tag{15}$$

$$\mu_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu_r}{d\beta} + r\mu_2\mu_{r-1}, \qquad r = 1,2,... \tag{16}$$

$$\mu^{(r+1)} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu^{(r)}}{d\beta} + (\frac{c\beta}{1-\beta-h\beta} - r)\mu^{(r)}, \qquad r = 1,2,... \tag{17}$$

$$k_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \sum_{j=1}^{r} \binom{r-1}{j-1}\mu'_{r-j} \frac{dk_j}{d\beta} - \sum_{j=2}^{r} \binom{r-1}{j-2}\mu'_{r+1-j}k_j \; , \tag{18}$$

where $c = \frac{a}{\beta}$ and $h = \frac{b}{\beta}$. Relations (13)-(18) are the same as Consul and Famoye's [9] results.

Remark 2. 1. By using (18), the first four cumulants of LKD can be obtained as follows:

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$$k_1 = \frac{c\beta}{1-\beta-h\beta} = \mu \tag{19}$$

$$k_2 = \frac{c\beta(1-\beta)}{(1-\beta-h\beta)^3} = \sigma^2 \tag{20}$$

$$k_3 = \frac{c\beta(1-\beta^2)}{(1-\beta-h\beta)^4} + \frac{3ch\beta^2}{(1-\beta-h\beta)^5} \tag{21}$$

$$k_4 = a(1-\beta)\left[\frac{1+4\beta+\beta^2}{(1-\beta-h\beta)^5} + \frac{10b(1+\beta)}{(1-b-\beta)^6} + \frac{15b^2}{(1-\beta-h\beta)^7}\right], \tag{22}$$

where $c = \dfrac{a}{\beta}$ and $h = \dfrac{b}{\beta}$. Relations (19) – (22) are the same as Consul and Famoye's [9] results.

Differentiating (3) with respect to β yields:

$$\frac{df(x;\beta)}{d\beta} = \left[x - \frac{1-\beta}{1-\beta-h\beta}\right]\frac{1-\beta-h\beta}{\beta(1-\beta)}f(x;\beta) \tag{23}$$

Comparing (23) and (5) we identify the functions as:

$$C(\beta) = \frac{1-\beta}{1-\beta-h\beta} \quad , \quad B(\beta) = \frac{1-\beta-h\beta}{\beta(1-\beta)}$$

From which the mean, variance, recurrence relations between non-central, central, and factorial moments and the recurrence relation between moments and cumulants of the LKD-I are as follows, respectively:

$$\mu = \frac{1-\beta}{1-\beta-h\beta} \tag{24}$$

$$\mu_2 = \sigma^2 = \frac{h\beta(1-\beta)}{(1-\beta-h\beta)^3} \tag{25}$$

$$\mu'_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu'_r}{d\beta} + \mu'_r \cdot \mu'_1, \qquad r = 1,2,... \tag{26}$$

$$\mu_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu_r}{d\beta} + r\mu_2\mu_{r-1}, \qquad r = 1,2,... \tag{27}$$

$$\mu^{(r+1)} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu^{(r)}}{d\beta} + \left(\frac{1-\beta}{1-\beta-h\beta} - r\right)\mu^{(r)}, \qquad r = 1,2,... \tag{28}$$

$$k_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \sum_{j=1}^{r}\binom{r-1}{j-1}\mu'_{r-j}\frac{dk_j}{d\beta} - \sum_{j=2}^{r}\binom{r-1}{j-2}\mu'_{r+1-j}k_j, \tag{29}$$

where $h = \dfrac{b}{\beta}$. Relations (24) - (29) are the same as Consul and Famoye's [9] results.

Remark 2. 2. By using (29), the first four cumulants of LKD-I can be obtained as follows:

$$k_1 = \frac{1-\beta}{1-\beta-h\beta} = \mu \tag{30}$$

$$k_2 = \frac{h\beta(1-\beta)}{(1-\beta-h\beta)^3} = \sigma^2 \tag{31}$$

$$k_3 = \frac{h\beta\left(1-\beta^2\right)}{(1-\beta-h\beta)^4} + \frac{3h^2\beta^2(1-\beta)}{(1-\beta-h\beta)^5} \tag{32}$$

$$k_4 = h\beta(1-\beta)\left[\frac{1+4\beta+\beta^2}{(1-\beta-h\beta)^5} + \frac{10h\beta(1+\beta)}{(1-\beta-h\beta)^6} + \frac{15h^2\beta^2}{(1-\beta-h\beta)^7}\right], \tag{33}$$

where $h = \dfrac{b}{\beta}$.

Differentiating (4) with respect to β yields:

$$\frac{df(x;\beta)}{d\beta} = \left[x - \frac{1}{1-\beta-h\beta}\right]\frac{1-\beta-h\beta}{\beta(1-\beta)} f(x;\beta) \tag{34}$$

Comparing (34) and (5) we identify the functions as:

$$C(\beta) = \frac{1}{1-\beta-h\beta} \quad , \quad B(\beta) = \frac{1-\beta-h\beta}{\beta(1-\beta)}$$

From which the mean, variance, recurrence relations between non-central, central, and factorial moments and the recurrence relation between moments and cumulants of the LKD-II are as follows, respectively:

$$\mu = \frac{1}{1-\beta-h\beta} \tag{35}$$

$$\mu_2 = \sigma^2 = \frac{(\beta+b)(1-\beta)}{(1-\beta-h\beta)^3} \tag{36}$$

$$\mu'_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu'_r}{d\beta} + \mu'_r \cdot \mu'_1, \qquad r = 1,2,... \tag{37}$$

$$\mu_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu_r}{d\beta} + r\mu_{r-1}\mu_2, \qquad r = 1,2,... \tag{38}$$

$$\mu^{(r+1)} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \cdot \frac{d\mu^{(r)}}{d\beta} + (\frac{1}{1-\beta-h\beta}-r)\mu^{(r)}, \qquad r = 1,2,... \tag{39}$$

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$$k_{r+1} = \frac{\beta(1-\beta)}{1-\beta-h\beta} \sum_{j=1}^{r}\binom{r-1}{j-1}\mu'_{r-j}\frac{dk_j}{d\beta} - \sum_{j=2}^{r}\binom{r-1}{j-2}\mu'_{r+1-j}k_j,$$ (40)

where $h = \dfrac{b}{\beta}$. Relations (35) and (36) are the same as Consul and Famoye's [9] results.

Remark 2. 3. By using (40), the first four cumulants of LKD-II can be obtained as follows:

$$k_1 = \frac{1}{1-\beta-h\beta} = \mu$$ (41)

$$k_2 = \frac{\beta(1-\beta)(1+h)}{(1-\beta-h\beta)^3} = \sigma^2$$ (42)

$$k_3 = \frac{\beta(1-\beta^2)(1+h)}{(1-\beta-h\beta)^4} + \frac{3h\beta^2(1-\beta)(1+h)}{(1-\beta-h\beta)^5}$$ (43)

$$k_4 = \beta(1-\beta)(1+h)\left[\frac{1+4\beta+\beta^2}{(1-\beta-h\beta)^5} + \frac{10h\beta(1+\beta)}{(1-\beta-h\beta)^6} + \frac{15h^2\beta^2}{(1-\beta-h\beta)^7}\right]$$ (44)

Where $h = \dfrac{b}{\beta}$.

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