THE STRUCTURE OF (α, β) -derivations of triangular rings^{*}

M. N. GHOSSEIRI*

Department of Mathematics, Kurdistan University P. O. Box 416, Sanandaj, I. R. of Iran Email: mnghosseiri@yahoo.com

Abstract – Let *R* and S be reduced rings with identities whose idempotents are central, and let *M* be an (*R*, *S*)-bimodule such that ann_r (*M*)=0. In this paper, we determine first the structure of automorphisms of the triangular ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, and then for all automorphisms α , β of *T* we determine the structure of (α,β) -derivations of *T*.

Keywords – Triangular ring, automorphism, (α, β) -derivation

1. INTRODUCTION

Recently many authors have considered (α,β) - derivations and generalized (α,β) -derivations of rings. We refer the interested readers to [1] and [2] and the references therein.

Motivated by [3], we describe the (α, β) -derivations of the triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, whose components satisfy certain conditions.

Let R and S be reduced rings with identities whose idempotents are central, M be an (R, S)bimodule such that ann_r (M)=0, and T be the triangular ring

$$T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} : r \in R, m \in M, s \in S \right\}.$$

I denotes the identity matrix and $E_{i,j}$ denotes the usual unitary matrix. The zero matrix and the identically zero functions are denoted by 0, and we assume that the ring automorphisms conserve identities. If *f* and *h* are automorphisms of *R* and *S* respectively, by an (f, h)-automorphism of *M* we mean an additive bijective mapping *g* on *M* such that for all $r \in R, m \in M, s \in S, g(rms) = f(r)g(m)h(s)$.

Clearly, if f and g are the identity automorphisms, then g is an (R, S)-bimodule automorphism. It is easy to see that if f and h are automorphisms of R and S, respectively, and g is an (f, h)-automorphism of M, then the mapping

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^{**}Corresponding author

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$$\Phi: T \to T, \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \mapsto \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}$$

is an automorphism of T. Φ is called *the* automorphism *induced by* f, g, and h. If A is an invertible matrix in T, then the *inner* automorphism induced by A is denoted by Inn_A . If α, β are automorphisms of T, then by an (α, β) – *derivation* of T, we mean an additive mapping d on T such that for each $x, y \in T, d(xy) = \alpha(x)d(y) + d(x)\beta(y)$. If α is the identity mapping, d is a β – *derivation* and if α, β are both identities, d is a *derivation* of T. For each $C \in T$, the mapping $I_C: T \to T$ given by $I_C(P) = CP - PC$ is easily seen to be a derivation of T. I_C is called the *inner derivation* induced by C. We determine first the structure of automorphisms of T, and then the structure of (α, β) -derivations of T. This result has attracted the attention of mathematicians in the fields of ring theory and functional analysis.

2. THE STRUCTURE OF AUTOMORPHISMS OF T

Theorem 2. 1. Let the ring T be as above and let α be an automorphism of T. Then there exist automorphisms f and h of R and S, respectively, an (f, h)-automorphism g of M, an invertible matrix $A \in T$, and an automorphism Φ of T induced by f, g, and h such that for each $P \in T$,

$$\alpha(P) = Inn_A \circ \Phi(P)$$

Proof: Let $m \in M, \alpha(E_{11}) = \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix}$, and $\alpha(mE_{12}) = \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}$, where a, m_1 , and b are some elements of R, M, and S, respectively, and $k : M \to R, g : M \to M$, and $t : M \to S$ are some functions determined by α . We have

$$\alpha(E_{22}) = I - \alpha(E_{11}) = \begin{bmatrix} 1 - a & -m_1 \\ 0 & 1 - b \end{bmatrix};$$

$$\begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} = \alpha(E_{11}) = \alpha(E_{11}^2) = \alpha(E_{11})\alpha(E_{11}) = \begin{bmatrix} a^2 & am_1 + m_1b \\ 0 & b^2 \end{bmatrix}.$$
(1)

Also,

$$\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} = \alpha(mE_{12}) = \alpha(E_{11}mE_{12}) = \alpha(E_{11})\alpha(mE_{12})$$
$$= \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}$$
$$= \begin{bmatrix} ak(m) & ag(m) + m_1t(m) \\ 0 & bt(m) \end{bmatrix};$$
$$(2)$$
$$\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} = \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22})$$

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$$= \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} \begin{bmatrix} 1 - a & -m_1 \\ 0 & 1 - b \end{bmatrix}$$
$$= \begin{bmatrix} k(m) - k(m)a & -k(m)m_1 + g(m) - g(m)b \\ 0 & t(m) - t(m)b \end{bmatrix}.$$
(3)

From (1), (2), and (3) it follows that

$$a^{2} = a$$
, $b^{2} = b$, $k(m) = ak(m)$, $t(m) = bt(m)$,
 $k(m)a = 0$, $t(m)b = 0$.

Since the idempotents of R and S are central, then these relations imply that k = t = 0. So $\alpha(mE_{12}) = g(m)E_{12}. \quad (\lceil r_1 \quad m_2 \rceil)$

Next, assume that
$$\alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right) = E_{11}$$
 for some $r_1 \in R, m_2 \in M$, and $s_1 \in S$. Then we have
 $\alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right) = \alpha \left(E_{11} \begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right)$
 $= \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} E_{11} = a E_{11},$
since,

Hence,

$$\alpha(s_1 E_{22}) = \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} - \begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right)$$
$$= \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right) - \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right) = (1-a)E_{11}.$$
(4)

Now, let $t \in M$ be arbitrary. Then

$$0 = g(t)E_{12}(1-a)E_{11} = \alpha(tE_{12}s_1E_{22}) = \alpha(ts_1E_{12}).$$

So, $ts_1 = 0$ for all $t \in M$. Thus by assumption, $s_1 = 0$, and, by (4), a = 1. Next, we prove that g is onto. Let $m \in M$ and assume that

$$\alpha \left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix} \right) = m E_{12}.$$

Then

$$\alpha \left(\begin{bmatrix} x^2 & xm' + m'z \\ 0 & z^2 \end{bmatrix} \right) = \alpha \left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix}^2 \right) = \left(\alpha \begin{bmatrix} x & m' \\ 0 & z \end{bmatrix} \right)^2 = 0.$$

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Since α is one-to-one, we have $x^2 = 0$, $z^2 = 0$, and since *R* and *S* are reduced, x = 0 and z = 0. Thus, $g(m')E_{12} = \alpha(m'E_{12}) = mE_{12}$, proving that g is onto.

Now, we show that b = 0. Let $m \in M$. Since a = 1,

$$g(m)E_{12} = \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22})$$
$$= g(m)E_{12}\begin{bmatrix} 0 & -m_1 \\ 0 & 1-b \end{bmatrix} = g(m)(1-b)E_{12}.$$

Therefore, g(m)b = 0 for all $m \in M$. Since g is onto and ann_r (M)=0, it follows that b=0. Consequently, for each $m \in M$ we have

$$\alpha(E_{11}) = \begin{bmatrix} 1 & m_1 \\ 0 & 0 \end{bmatrix}, \quad \alpha(E_{22}) = \begin{bmatrix} 0 & -m_1 \\ 0 & 1 \end{bmatrix}, \quad \alpha(mE_{12}) = g(m)E_{12}. \tag{5}$$

Let
$$x \in R$$
 and set $\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying α to $xE_{11} = xE_{11}E_{11}$ and using (5) we find
$$\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_1m_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix},$$
(6)

for some function $f: R \to R$. Similarly, applying α to $xE_{22} = E_{22}xE_{22}$, where $x \in S$, we observe that there exists a function *h* on *S* such that

$$\alpha(xE_{22}) = \begin{bmatrix} 0 & -m_1h(x) \\ 0 & h(x) \end{bmatrix}.$$
(7)

Since
$$\alpha$$
 is additive, then so are $f, g, \text{ and } h$.
Therefore, by (5), (6), and (7), for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have
 $\alpha \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix}$. (8)
Let $x \in R, y \in S$, and $m' \in M$. Since α is onto, there exists $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ such that
 $\begin{bmatrix} x & m' \\ 0 & y \end{bmatrix} = \alpha \left(\begin{bmatrix} r & m \\ o & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix}$,

proving that f and g are onto. Relation (8) and the fact that α is one-to-one imply that f, g, h are one-to-one.

Our next step is to show that f and h are homorphisms and g is an (f, g)-automorphism. Since α is additive, so are f, g, h. Let $x, y \in R$. Then

$$\alpha(xyE_{11})\alpha(xE_{11}yE_{11}) = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f(y) & f(y)m_1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f(x)f(y) & f(x)f(y)m_1 \\ 0 & 0 \end{bmatrix}.$$

So, f(xy) = f(x)f(y). Similarly, for each $x, y \in S$, h(xy) = h(x)h(y). Now, let $r \in R, m \in M$, and $s \in S$. Then

$$g(rms)E_{12} = \alpha(rmsE_{12}) = \alpha(rE_{11}mE_{12}sE_{22}) = \alpha(rE_{11})\alpha(mE_{12})\alpha(sE_{22})$$

$$= \begin{bmatrix} f(r) & f(r)m_{1} \\ 0 & 0 \end{bmatrix} g(m)E_{12} \begin{bmatrix} 0 & m_{1}h(s) \\ 0 & h(s) \end{bmatrix}$$

$$= f(r)g(m)h(s)E_{12},$$
Hence $g(rms) = f(r)g(m)h(s).$
Finally, for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have
$$\alpha \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & f(r)m_{1} - m_{1}h(s) + g(m) \\ 0 & h(s) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -m_{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} 1 & m_{1} \\ 0 & 1 \end{bmatrix}$$

$$= A\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) A^{-1} = Inn_{A} \circ \Psi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right),$$
where $A = \begin{bmatrix} 1 & -m_{1} \\ 0 & 1 \end{bmatrix}$ and $\Phi: T \to T$ is given by
$$\Phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}.$$

Using the properties of f, g, and h one can easily verify that Φ is an automorphism of T induced by the automorphisms f, g, and h. This completes the proof.

3. THE STRUCTURE OF (α, β) -derivations of the ring *t*

Let the ring T be as above and consider the automorphisms

$$\alpha = Inn_A \circ \Phi, \ \beta = Inn_B \circ \Psi$$

of *T*, where $A = \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & m_2 \\ 0 & 1 \end{bmatrix}$ are some (invertible) matrices in *T*, Φ is an automorphism of *T* induced by automorphisms *f*, *g*, *h* of *R*, *M*, *S*, respectively, and Ψ is an automorphism of *T* induced by automorphisms f', g', h', of *R*, *M*, *S*, respectively.

Theorem 3. 1. Assume that the automorphisms α, β of the ring T are as above. If d is an (α, β) derivation of T, then there exists a (Φ, Ψ) -derivation ϕ of T and a matrix C in T such that for each $P \in T$,

$$d(P) = \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C$$

In particular, if d is an (α, α) -derivation of T, then

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Proof: Recall that for every $x, y \in T$,

$$d(xy) = \alpha(x)d(y) + d(x)\beta(y).$$
(9)

Assume that $d(E_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying d to $E_{11} = E_{11}^2$ and noting that d(I) = 0, it follows that

$$d(E_{11}) = x_2 E_{12}$$
 and $d(E_{22}) = -d(E_{11}) = -x_2 E_{12}.$ (10)

Let $m \in M$ and set $d(mE_{12}) = \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix}$. Since $mE_{12} = E_{11}mE_{12} = mE_{12}E_{22}$, using (9) and (10) we infer that

$$d(mE_{12}) = y_2 E_{12} = \tau(m) E_{12}, \tag{11}$$

for some function $\tau: M \to M$. Let $x \in R$ and set $d(xE_{11}) = \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}$. Since $xE_{11} = E_{11}xE_{11} = xE_{11}E_{11}$, using (9) and previous calculations we arrive at

$$d(xE_{11}) = \begin{bmatrix} \delta(x) & -\delta(x)m_2 + f(x)x_2 \\ 0 & 0 \end{bmatrix},$$
 (12)

for some function $\delta: R \to R$. Since d is additive, then so is δ . Let $x, y \in R$. Then the identity $xyE_{11} = xE_{11}yE_{11}$ and (9) imply that

$$\delta(xy) = f(x)\delta(y) + \delta(x)f'(y).$$

That is, δ is an (f, f')-derivation of R. Similar computations show that there exists an (h, h')derivation γ of *S* such that for each $x \in S$,

$$d(xE_{22}) = \begin{bmatrix} 0 & m_1\gamma(x) - x_2h'(x) \\ 0 & \gamma(x) \end{bmatrix}.$$
(13)

Now we prove some properties of τ . Since d is additive, then so is τ . Let $r \in R$, $m \in M$. Then

$$\tau(rm)E_{12} = d(rmE_{12}) = d(rE_{11}mE_{12})$$
$$= \alpha(rE_{11})d(mE_{12}) + d(rE_{11})\beta(mE_{12})$$
$$= \begin{bmatrix} f(r) & -f(r)m_1 \\ 0 & 0 \end{bmatrix} \tau(m)E_{12}$$

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$$+\begin{bmatrix}\delta(r) & -\delta(r)m_2 + f(r)x_2\\0 & 0\end{bmatrix}g'(m)E_{12}$$
$$= (f(r)\tau(m) + \delta(r)g'(m))E_{12}.$$

Hence,

$$\tau(rm) = f(r)\tau(m) + \delta(r)g'(m).$$

Similarly, for each $m \in M$ and $s \in S$ we find

$$\tau(ms) = \tau(m)h'(s) + g(m)\gamma(s).$$
Define the function $\phi: T \to T$ by
$$\phi\left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}\right) = \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix}.$$
(14)
Let $P = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$ and $P' = \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}$ be in *T*. Using previous computations we have
$$\phi(PP') = \phi\left(\begin{bmatrix} rr' & rm' + ms' \\ 0 & ss' \end{bmatrix}\right) = \begin{bmatrix} \delta(rr') & \tau(rm' + ms') \\ 0 & \gamma(ss') \end{bmatrix}$$

$$= \begin{bmatrix} f(r)\delta(r') + \delta(r)f'(r') & f(r)\tau(m') + \delta(r)g'(m') + \tau(m)h'(s') + g(m)\gamma(s') \\ 0 & h(s)\gamma(s') + \gamma(s)h'(s') \end{bmatrix}$$
$$= \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} \delta(r') & \tau(m') \\ 0 & \gamma(s') \end{bmatrix} + \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} \begin{bmatrix} f'(r') & g'(m') \\ 0 & h'(s') \end{bmatrix}$$
$$= \Phi(P)\phi(P') + \phi(P)\Psi(P').$$

Therefore, ϕ is a (Φ, Ψ) -derivation of *T*. Finally, define $C = -x_2 E_{12}$ and let $P = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$ be in *T*. Then by (11), (12), (13), and (14) we have

$$d(P) = \begin{bmatrix} \delta(r) & -\delta(r)m_2 + f(r)x_2 + \tau(m) + m_1\gamma(s) - x_2h'(s) \\ 0 & \gamma(s) \end{bmatrix}$$
$$= \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} + \begin{bmatrix} 0 & -\delta(r)m_2 + f(r)x_2 + m_1\gamma(s) - x_2h'(s) \\ 0 & 0 \end{bmatrix}$$
$$= \phi(P) + \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} - \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} \begin{bmatrix} 1 & m_2 \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & -x_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f'(r) & g'(m) \\ 0 & h'(s) \end{bmatrix} - \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} 0 & -x_2 \\ 0 & 0 \end{bmatrix}$$

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$$= \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C.$$

In particular, if $\alpha = \beta$, then A=B and $\Phi = \Psi$. So,

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Remark. The proof of the above theorem shows that when the ring T reduces to the ordinary triangular ring, *i.e.*, R = S = M, where R is a ring with identity, then the result is in accordance with [3, Theorem, P. 263].

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