

THE STRUCTURE OF (α, β) -DERIVATIONS OF TRIANGULAR RINGS*

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Abstract – Let R and S be reduced rings with identities whose idempotents are central, and let M be an (R, S) -bimodule such that $\text{ann}_r(M)=0$. In this paper, we determine first the structure of automorphisms of the triangular ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, and then for all automorphisms α, β of T we determine the structure of (α, β) -derivations of T .

Keywords – Triangular ring, automorphism, (α, β) -derivation

1. INTRODUCTION

Recently many authors have considered (α, β) -derivations and generalized (α, β) -derivations of rings. We refer the interested readers to [1] and [2] and the references therein.

Motivated by [3], we describe the (α, β) -derivations of the triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, whose components satisfy certain conditions.

Let R and S be reduced rings with identities whose idempotents are central, M be an (R, S) -bimodule such that $\text{ann}_r(M)=0$, and T be the triangular ring

$$T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} : r \in R, m \in M, s \in S \right\}.$$

I denotes the identity matrix and $E_{i,j}$ denotes the usual unitary matrix. The zero matrix and the identically zero functions are denoted by 0, and we assume that the ring automorphisms conserve identities. If f and h are automorphisms of R and S respectively, by an (f, h) -automorphism of M we mean an additive bijective mapping g on M such that for all $r \in R, m \in M, s \in S, g(rms) = f(r)g(m)h(s)$.

Clearly, if f and g are the identity automorphisms, then g is an (R, S) -bimodule automorphism. It is easy to see that if f and h are automorphisms of R and S , respectively, and g is an (f, h) -automorphism of M , then the mapping

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$$\Phi : T \rightarrow T, \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \mapsto \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}$$

is an automorphism of T . Φ is called *the automorphism induced by f , g , and h* . If A is an invertible matrix in T , then the *inner automorphism induced by A* is denoted by Inn_A . If α, β are automorphisms of T , then by an (α, β) -*derivation* of T , we mean an additive mapping d on T such that for each $x, y \in T, d(xy) = \alpha(x)d(y) + d(x)\beta(y)$. If α is the identity mapping, d is a β -*derivation* and if α, β are both identities, d is a *derivation* of T . For each $C \in T$, the mapping $I_C : T \rightarrow T$ given by $I_C(P) = CP - PC$ is easily seen to be a derivation of T . I_C is called the *inner derivation* induced by C . We determine first the structure of automorphisms of T , and then the structure of (α, β) -derivations of T . This result has attracted the attention of mathematicians in the fields of ring theory and functional analysis.

2. THE STRUCTURE OF AUTOMORPHISMS OF T

Theorem 2. 1. Let the ring T be as above and let α be an automorphism of T . Then there exist automorphisms f and h of R and S , respectively, an (f, h) -automorphism g of M , an invertible matrix $A \in T$, and an automorphism Φ of T induced by f, g , and h such that for each $P \in T$,

$$\alpha(P) = \text{Inn}_A \circ \Phi(P).$$

Proof: Let $m \in M, \alpha(E_{11}) = \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix}$, and $\alpha(mE_{12}) = \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix}$, where a, m_1 , and b are some elements of R, M , and S , respectively, and $k : M \rightarrow R, g : M \rightarrow M$, and $t : M \rightarrow S$ are some functions determined by α . We have

$$\alpha(E_{22}) = I - \alpha(E_{11}) = \begin{bmatrix} 1-a & -m_1 \\ 0 & 1-b \end{bmatrix};$$

$$\begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} = \alpha(E_{11}) = \alpha(E_{11}^2) = \alpha(E_{11})\alpha(E_{11}) = \begin{bmatrix} a^2 & am_1 + m_1b \\ 0 & b^2 \end{bmatrix}. \quad (1)$$

Also,

$$\begin{aligned} \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} &= \alpha(mE_{12}) = \alpha(E_{11}mE_{12}) = \alpha(E_{11})\alpha(mE_{12}) \\ &= \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} \\ &= \begin{bmatrix} ak(m) & ag(m) + m_1t(m) \\ 0 & bt(m) \end{bmatrix}; \end{aligned} \quad (2)$$

$$\begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} = \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22})$$

$$\begin{aligned}
 &= \begin{bmatrix} k(m) & g(m) \\ 0 & t(m) \end{bmatrix} \begin{bmatrix} 1-a & -m_1 \\ 0 & 1-b \end{bmatrix} \\
 &= \begin{bmatrix} k(m) - k(m)a & -k(m)m_1 + g(m) - g(m)b \\ 0 & t(m) - t(m)b \end{bmatrix}. \tag{3}
 \end{aligned}$$

From (1), (2), and (3) it follows that

$$a^2 = a, \quad b^2 = b, \quad k(m) = ak(m), \quad t(m) = bt(m),$$

$$k(m)a = 0, \quad t(m)b = 0.$$

Since the idempotents of R and S are central, then these relations imply that $k = t = 0$. So $\alpha(mE_{12}) = g(m)E_{12}$.

Next, assume that $\alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right) = E_{11}$ for some $r_1 \in R, m_2 \in M$, and $s_1 \in S$. Then we have

$$\begin{aligned}
 \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right) &= \alpha \left(E_{11} \begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right) \\
 &= \begin{bmatrix} a & m_1 \\ 0 & b \end{bmatrix} E_{11} = aE_{11}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \alpha(s_1 E_{22}) &= \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} - \begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right) \\
 &= \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & s_1 \end{bmatrix} \right) - \alpha \left(\begin{bmatrix} r_1 & m_2 \\ 0 & 0 \end{bmatrix} \right) = (1-a)E_{11}. \tag{4}
 \end{aligned}$$

Now, let $t \in M$ be arbitrary. Then

$$0 = g(t)E_{12}(1-a)E_{11} = \alpha(tE_{12}s_1E_{22}) = \alpha(ts_1E_{12}).$$

So, $ts_1 = 0$ for all $t \in M$. Thus by assumption, $s_1 = 0$, and, by (4), $a = 1$.

Next, we prove that g is onto. Let $m \in M$ and assume that

$$\alpha \left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix} \right) = mE_{12}.$$

Then

$$\alpha \left(\begin{bmatrix} x^2 & xm' + m'z \\ 0 & z^2 \end{bmatrix} \right) = \alpha \left(\begin{bmatrix} x & m' \\ 0 & z \end{bmatrix}^2 \right) = \left(\alpha \begin{bmatrix} x & m' \\ 0 & z \end{bmatrix} \right)^2 = 0.$$

Since α is one-to-one, we have $x^2 = 0$, $z^2 = 0$, and since R and S are reduced, $x = 0$ and $z = 0$. Thus, $g(m')E_{12} = \alpha(m'E_{12}) = mE_{12}$, proving that g is onto.

Now, we show that $b = 0$. Let $m \in M$. Since $a = 1$,

$$\begin{aligned} g(m)E_{12} &= \alpha(mE_{12}) = \alpha(mE_{12}E_{22}) = \alpha(mE_{12})\alpha(E_{22}) \\ &= g(m)E_{12} \begin{bmatrix} 0 & -m_1 \\ 0 & 1-b \end{bmatrix} = g(m)(1-b)E_{12}. \end{aligned}$$

Therefore, $g(m)b = 0$ for all $m \in M$. Since g is onto and $\text{ann}_r(M) = 0$, it follows that $b = 0$. Consequently, for each $m \in M$ we have

$$\alpha(E_{11}) = \begin{bmatrix} 1 & m_1 \\ 0 & 0 \end{bmatrix}, \quad \alpha(E_{22}) = \begin{bmatrix} 0 & -m_1 \\ 0 & 1 \end{bmatrix}, \quad \alpha(mE_{12}) = g(m)E_{12}. \quad (5)$$

Let $x \in R$ and set $\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying α to $xE_{11} = xE_{11}E_{11}$ and using (5) we find

$$\alpha(xE_{11}) = \begin{bmatrix} x_1 & x_1 m_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix}, \quad (6)$$

for some function $f: R \rightarrow R$. Similarly, applying α to $xE_{22} = E_{22}xE_{22}$, where $x \in S$, we observe that there exists a function h on S such that

$$\alpha(xE_{22}) = \begin{bmatrix} 0 & -m_1 h(x) \\ 0 & h(x) \end{bmatrix}. \quad (7)$$

Since α is additive, then so are f , g , and h .

Therefore, by (5), (6), and (7), for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have

$$\alpha \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1 h(s) + g(m) \\ 0 & h(s) \end{bmatrix}. \quad (8)$$

Let $x \in R$, $y \in S$, and $m' \in M$. Since α is onto, there exists $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ such that

$$\begin{bmatrix} x & m' \\ 0 & y \end{bmatrix} = \alpha \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & f(r)m_1 - m_1 h(s) + g(m) \\ 0 & h(s) \end{bmatrix},$$

proving that f and g are onto. Relation (8) and the fact that α is one-to-one imply that f , g , h are one-to-one.

Our next step is to show that f and h are homomorphisms and g is an (f, g) -automorphism. Since α is additive, so are f , g , h . Let $x, y \in R$. Then

$$\alpha(xyE_{11})\alpha(xE_{11}yE_{11}) = \begin{bmatrix} f(x) & f(x)m_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f(y) & f(y)m_1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f(x)f(y) & f(x)f(y)m_1 \\ 0 & 0 \end{bmatrix}.$$

So, $f(xy) = f(x)f(y)$. Similarly, for each $x, y \in S$, $h(xy) = h(x)h(y)$. Now, let $r \in R, m \in M$, and $s \in S$. Then

$$\begin{aligned} g(rms)E_{12} &= \alpha(rmsE_{12}) = \alpha(rE_{11}mE_{12}sE_{22}) = \alpha(rE_{11})\alpha(mE_{12})\alpha(sE_{22}) \\ &= \begin{bmatrix} f(r) & f(r)m_1 \\ 0 & 0 \end{bmatrix} g(m)E_{12} \begin{bmatrix} 0 & m_1h(s) \\ 0 & h(s) \end{bmatrix} \\ &= f(r)g(m)h(s)E_{12}, \end{aligned}$$

Hence $g(rms) = f(r)g(m)h(s)$.

Finally, for each $\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \in T$ we have

$$\begin{aligned} \alpha \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) &= \begin{bmatrix} f(r) & f(r)m_1 - m_1h(s) + g(m) \\ 0 & h(s) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix} \\ &= A\Phi \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) A^{-1} = Inn_A \circ \Psi \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right), \end{aligned}$$

where $A = \begin{bmatrix} 1 & -m_1 \\ 0 & 1 \end{bmatrix}$ and $\Phi : T \rightarrow T$ is given by

$$\Phi \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix}.$$

Using the properties of f, g , and h one can easily verify that Φ is an automorphism of T induced by the automorphisms f, g , and h . This completes the proof.

3. THE STRUCTURE OF (α, β) -DERIVATIONS OF THE RING T

Let the ring T be as above and consider the automorphisms

$$\alpha = Inn_A \circ \Phi, \quad \beta = Inn_B \circ \Psi$$

of T , where $A = \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & m_2 \\ 0 & 1 \end{bmatrix}$ are some (invertible) matrices in T , Φ is an automorphism of T induced by automorphisms f, g, h of R, M, S , respectively, and Ψ is an automorphism of T induced by automorphisms f', g', h' , of R, M, S , respectively.

Theorem 3. 1. Assume that the automorphisms α, β of the ring T are as above. If d is an (α, β) -derivation of T , then there exists a (Φ, Ψ) -derivation ϕ of T and a matrix C in T such that for each $P \in T$,

$$d(P) = \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C.$$

In particular, if d is an (α, α) -derivation of T , then

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Proof: Recall that for every $x, y \in T$,

$$d(xy) = \alpha(x)d(y) + d(x)\beta(y). \quad (9)$$

Assume that $d(E_{11}) = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$. Applying d to $E_{11} = E_{11}^2$ and noting that $d(I) = 0$, it follows that

$$d(E_{11}) = x_2 E_{12} \text{ and } d(E_{22}) = -d(E_{11}) = -x_2 E_{12}. \quad (10)$$

Let $m \in M$ and set $d(mE_{12}) = \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix}$. Since $mE_{12} = E_{11}mE_{12} = mE_{12}E_{22}$, using (9) and (10) we infer that

$$d(mE_{12}) = y_2 E_{12} = \tau(m)E_{12}, \quad (11)$$

for some function $\tau : M \rightarrow M$.

Let $x \in R$ and set $d(xE_{11}) = \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}$. Since $xE_{11} = E_{11}xE_{11} = xE_{11}E_{11}$, using (9) and previous calculations we arrive at

$$d(xE_{11}) = \begin{bmatrix} \delta(x) & -\delta(x)m_2 + f(x)x_2 \\ 0 & 0 \end{bmatrix}, \quad (12)$$

for some function $\delta : R \rightarrow R$. Since d is additive, then so is δ . Let $x, y \in R$. Then the identity $xyE_{11} = xE_{11}yE_{11}$ and (9) imply that

$$\delta(xy) = f(x)\delta(y) + \delta(x)f'(y).$$

That is, δ is an (f, f') -derivation of R . Similar computations show that there exists an (h, h') -derivation γ of S such that for each $x \in S$,

$$d(xE_{22}) = \begin{bmatrix} 0 & m_1\gamma(x) - x_2h'(x) \\ 0 & \gamma(x) \end{bmatrix}. \quad (13)$$

Now we prove some properties of τ . Since d is additive, then so is τ . Let $r \in R$, $m \in M$. Then

$$\begin{aligned} \tau(rm)E_{12} &= d(rmE_{12}) = d(rE_{11}mE_{12}) \\ &= \alpha(rE_{11})d(mE_{12}) + d(rE_{11})\beta(mE_{12}) \\ &= \begin{bmatrix} f(r) & -f(r)m_1 \\ 0 & 0 \end{bmatrix} \tau(m)E_{12} \end{aligned}$$

$$\begin{aligned}
 & + \begin{bmatrix} \delta(r) & -\delta(r)m_2 + f(r)x_2 \\ 0 & 0 \end{bmatrix} g'(m)E_{12} \\
 & = (f(r)\tau(m) + \delta(r)g'(m))E_{12}.
 \end{aligned}$$

Hence,

$$\tau(rm) = f(r)\tau(m) + \delta(r)g'(m).$$

Similarly, for each $m \in M$ and $s \in S$ we find

$$\tau(ms) = \tau(m)h'(s) + g(m)\gamma(s).$$

Define the function $\phi : T \rightarrow T$ by

$$\phi \left(\begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \right) = \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix}. \tag{14}$$

Let $P = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$ and $P' = \begin{bmatrix} r' & m' \\ 0 & s' \end{bmatrix}$ be in T . Using previous computations we have

$$\begin{aligned}
 \phi(PP') & = \phi \left(\begin{bmatrix} rr' & rm' + ms' \\ 0 & ss' \end{bmatrix} \right) = \begin{bmatrix} \delta(rr') & \tau(rm' + ms') \\ 0 & \gamma(ss') \end{bmatrix} \\
 & = \begin{bmatrix} f(r)\delta(r') + \delta(r)f'(r') & f(r)\tau(m') + \delta(r)g'(m') + \tau(m)h'(s') + g(m)\gamma(s') \\ 0 & h(s)\gamma(s') + \gamma(s)h'(s') \end{bmatrix} \\
 & = \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} \delta(r') & \tau(m') \\ 0 & \gamma(s') \end{bmatrix} + \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} \begin{bmatrix} f'(r') & g'(m') \\ 0 & h'(s') \end{bmatrix} \\
 & = \Phi(P)\phi(P') + \phi(P)\Psi(P').
 \end{aligned}$$

Therefore, ϕ is a (Φ, Ψ) -derivation of T . Finally, define $C = -x_2E_{12}$ and let $P = \begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$ be in T .

Then by (11), (12), (13), and (14) we have

$$\begin{aligned}
 d(P) & = \begin{bmatrix} \delta(r) & -\delta(r)m_2 + f(r)x_2 + \tau(m) + m_1\gamma(s) - x_2h'(s) \\ 0 & \gamma(s) \end{bmatrix} \\
 & = \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} + \begin{bmatrix} 0 & -\delta(r)m_2 + f(r)x_2 + m_1\gamma(s) - x_2h'(s) \\ 0 & 0 \end{bmatrix} \\
 & = \phi(P) + \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} - \begin{bmatrix} \delta(r) & \tau(m) \\ 0 & \gamma(s) \end{bmatrix} \begin{bmatrix} 1 & m_2 \\ 0 & 1 \end{bmatrix} \\
 & \quad + \begin{bmatrix} 0 & -x_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f'(r) & g'(m) \\ 0 & h'(s) \end{bmatrix} - \begin{bmatrix} f(r) & g(m) \\ 0 & h(s) \end{bmatrix} \begin{bmatrix} 0 & -x_2 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$= \phi(P) + A\phi(P) - \phi(P)B + C\Psi(P) - \Phi(P)C.$$

In particular, if $\alpha = \beta$, then $A=B$ and $\Phi = \Psi$. So,

$$d(P) = \phi(P) + I_A \circ \phi(P) + I_C \circ \Phi(P).$$

Remark. The proof of the above theorem shows that when the ring T reduces to the ordinary triangular ring, i.e, $R = S = M$, where R is a ring with identity, then the result is in accordance with [3, Theorem, P. 263].

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