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TRANSFORMATION SEMIGROUPS AND EXACT SEQUENCES*

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Abstract – This text carries out some ideas about exact and P− exact sequences of transformation semigroups. Some theorems like the short five lemma (Lemma 1.3 and Lemma 2.3) are valid here as in exact sequences of R − modules.

Keywords – a − minimal set, exact sequence, P− exact sequence, proximal relation, transformation semigroup

PRELIMINARIES

By a transformation semigroup (X,S,π) (or simply (X,S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup S (phase semigroup) with identity e and a continuous map $\pi : X\times S \to X$ $(\pi(x,s)=xs\,(\forall x\in X,\forall s\in S))$ such that:

- $\forall x\in X \quad xe=x$,
- $\forall x\in X \quad \forall s,t\in S \quad x(st)=(xs)t$.

In the transformation semigroup (X,S) we have the following definitions:

1. For each $s\in S$, define the continuous map $\pi^s : X\to X$ by $x\pi^s = xs \quad (\forall x\in X)$, then $\mathrm{E}(X,S)$ (or simply $\mathrm{E}(X)$) is the closure of $\{\pi^s \mid s\in S\}$ in X^X with pointwise convergence, moreover it is called the enveloping semigroup (or Ellis semigroup) of (X,S). We used to write s instead of π^s $(s\in S)$. $\mathrm{E}(X,S)$ has a semigroup structure [1], a nonempty subset K of $\mathrm{E}(X,S)$ is called a right ideal if $K\,\mathrm{E}(X,S)\subseteq K$.
2. A nonempty subset Z of X is called invariant if $ZS\subseteq Z$. A closed invariant subset of X is called minimal if it does not have any proper subset, which is a closed invariant subset of X. $a\in X$ is called almost periodic if $\overline{aS}$ is a minimal subset of X.
3. Let $a\in X$, A be a nonempty subset of X and K be a closed right ideal of $\mathrm{E}(X,S)$ [2].

- We say K is an $a-$minimal set if:
  - $aK = a\,\mathrm{E}(X,S)$,
  - K does not have any proper subset like L, such that L be a closed right ideal of $\mathrm{E}(X,S)$ and $aL = a\,\mathrm{E}(X,S)$.

The set of all $a-$minimal sets is denoted by $\mathrm{M}_{(X,S)}(a)$.

- We say K is an $A-$minimal set if:
  - $\forall b\in A \quad bK = b\,\mathrm{E}(X,S)$,

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- K does not have any proper subset like L, such that L be a closed right ideal of $\mathrm{E}(X,S)$ and $bL = b\,\mathrm{E}(X,S)$ for all $b \in A$.

The set of all $A-$minimal sets is denoted by $\overline{\mathrm{M}}_{(X,S)}(A)$.

- We say K is an $A-$minimal set if:
- $AK = A\,\mathrm{E}(X,S)$,
- K does not have any proper subset like L, such that L be a closed right ideal of $\mathrm{E}(X,S)$ and $AL = A\,\mathrm{E}(X,S)$.

The set of all $A-$minimal sets is denoted by $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$.

$\overline{\mathrm{M}}_{(X,S)}(A)$ and $\mathrm{M}_{(X,S)}(a)$ are nonempty. We set:

$$\overline{\mathscr{M}}(X,S) = \{B \subseteq X \mid B \neq \varnothing \wedge (\forall K \in \overline{\mathrm{M}}_{(X,S)}(A) \quad \mathrm{J}(\mathrm{F}(A,K)) \neq \varnothing)\}$$

(where for all $L \subseteq \mathrm{E}(X,S)$, $\mathrm{J}(\mathrm{F}(A,L)) = \{u \in L \mid u^2 = u \wedge (\forall b \in A \quad bu = b)\}$).

4. Let A be a nonempty subset of X; we introduce the following sets [3]:

$\mathrm{P}(X,S) = \{(x,y) \in X \times X \mid \exists p \in \mathrm{E}(X,S) \quad xp = yp\}$ (or simply $\mathrm{P}(X)$),

$\mathrm{P}_A(X,S) = \{(x,y) \in X \times X \mid \exists a \in A \quad \exists I \in \mathrm{M}_{(X,S)}(a) \quad \forall p \in I \quad xp = yp\}$ (or simply $\mathrm{P}_A(X)$),

$\underline{\underline{\mathrm{P}}}_A(X,S) = \{(x,y) \in X \times X \mid \exists I \in \underline{\underline{\mathrm{M}}}_{(X,S)}(A) \quad \forall p \in I \quad xp = yp\}$ (or simply $\underline{\underline{\mathrm{P}}}_A(X)$),

$\overline{\mathrm{P}}_A(X,S) = \{(x,y) \in X \times X \mid \exists I \in \overline{\mathrm{M}}_{(X,S)}(A) \quad \forall p \in I \quad xp = yp\}$ (or simply $\overline{\mathrm{P}}_A(X)$).

5. Let (Y,S) be a transformation semigroup, a continuous map $\phi:(X,S) \to (Y,S)$ is called a homomorphism if $\phi(xs) = \phi(x)s \quad (x \in X, s \in S)$, moreover let $\mathrm{R}(\phi) = \{(x,y) \in X \times X \mid \phi(x) = \phi(y)\}$. Bijective (resp. injective, surjective) homomorphism $\phi:(X,S) \to (Y,S)$ is called an isomorphism (resp. monomorphism, epimorphism).

By Δ_Y we mean $\{(y,y) \mid y \in Y\}$.

1. EXACT SEQUENCES OF TRANSFORMATION SEMIGROUPS

In this section you will find the definition of exact sequences (of the transformation semigroups) and some of their properties.

Definition 1. 1.

- $(X_1,S) \overset{\phi_1}{\to} (X_2,S) \overset{\phi_2}{\to} \cdots \overset{\phi_{n-1}}{\to} (X_n,S) \overset{\phi_n}{\to} (X_{n+1},S) \quad (n \geq 2)$ is called an exact sequence (of transformation semigroups) if for each $i \in \{1,\dots,n-1\}$, $\phi_i \times \phi_i(X_i \times X_i) \cup \Delta_{X_{i+1}} = \mathrm{R}(\phi_{i+1})$.
- An infinite sequence of transformation semigroup homomorphisms $\cdots \overset{\phi_{n-1}}{\to} (X_n,S) \overset{\phi_n}{\to} (X_{n+1},S) \overset{\phi_{n+1}}{\to} (X_{n+2},S) \overset{\phi_{n+2}}{\to} \cdots$ is exact if for each $i \in \mathbf{Z}$, $\phi_i \times \phi_i(X_i \times X_i) \cup \Delta_{X_{i+1}} = \mathrm{R}(\phi_{i+1})$.

Note 1. 2. Let (X,S), (X_1,S), (X_2,S) and (X_3,S) be transformation semigroups. Let C be a closed invariant subset of X, and $t \in C$, be such that $tS = \{t\}$, then:

- $(\{t\},S) \overset{\iota}{\to} (C,S) \overset{\iota}{\to} (X,S) \overset{\pi}{\to} (\frac{X}{(C \times C) \cup \Delta_X},S) \to (\{t\},S)$ is exact (where ι and π are respectively inclusion and projection maps).
- $(\{t\},S) \overset{\phi_0}{\to} (X_1,S) \overset{\phi_1}{\to} (X_2,S)$ is exact if and only if $\phi_1:(X_1,S) \to (X_2,S)$ is a monomorphism (ϕ_0 exists if and only if there exists $u \in X_1$ such that $uS = \{u\}$).
- $(X_2,S) \overset{\phi_2}{\to} (X_3,S) \to (\{t\},S)$ is exact if and only if $\phi_2:(X_2,S) \to (X_3,S)$ is an epimorphism.

- If $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}(X_3,S)\to(\{t\},S)$ is exact, then two transformation semigroups (X_3,S) and $(\frac{X_2}{(\phi_1(X_1)\times\phi_1(X_1))\cup\Delta_{X_2}},S)$ are isomorph.
- If $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}(X_3,S)$ is an exact sequence, then:
- For each $x\in X_1$, $\phi_2\phi_1(x)$ is almost periodic.
- $\phi_2\phi_1(X_1)$ is a singleton minimal subset of (X_3,S).
- If $t\in X_1\cap X_2$, then $(\{t\},S)\overset{t}{\to}(X_1,S)\overset{\iota_1}{\to}(X_1\times\{t\}\cup\{t\}\times X_2,S)\overset{\pi_2}{\to}(X_2,S)\to(\{t\},S)$ where $\iota_1(x)=(x,t)$ $(\forall x\in X_1)$ and $\pi_2(x,y)=y$ $(\forall(x,y)\in X_1\times\{t\}\cup\{t\}\times X$ is exact (attention: ι_1 is one to one and $\pi_2:X_1\times\{t\}\cup\{t\}\times X_2\to X_2$ is onto).

Lemma 1. 3. (The Short Five Lemma). Let

$$\begin{array}{ccccccccc}(\{t\},S) & \to & (X_1,S) & \overset{\mu_1}{\to} & (X_2,S) & \overset{\mu_2}{\to} & (X_3,S) & \to & (\{t\},S)\\ & & \downarrow\phi_1 & & \downarrow\phi_2 & & \downarrow\phi_3 & & \\ (\{t\},S) & \to & (Y_1,S) & \overset{\lambda_1}{\to} & (Y_2,S) & \overset{\lambda_2}{\to} & (Y_3,S) & \to & (\{t\},S)\end{array}$$

be a commutative diagram of transformation semigroup homomorphisms such that each row is an exact sequence, then (see [4]):

- If $\phi_1:(X_1,S)\to(Y_1,S)$ and $\phi_3:(X_3,S)\to(Y_3,S)$ are monomorphisms, then $\phi_2:(X_2,S)\to(Y_2,S)$ is a monomorphism.
- If $\phi_1:(X_1,S)\to(Y_1,S)$ and $\phi_3:(X_3,S)\to(Y_3,S)$ are epimorphisms, then $\phi_2:(X_2,S)\to(Y_2,S)$ is an epimorphism.
- If $\phi_1:(X_1,S)\to(Y_1,S)$ and $\phi_3:(X_3,S)\to(Y_3,S)$ are isomorphisms, then $\phi_2:(X_2,S)\to(Y_2,S)$ is an isomorphism.

Proof:

- Let $\phi_1:(X_1,S)\to(Y_1,S)$ and $\phi_3:(X_3,S)\to(Y_3,S)$ be monomorphisms, and $a,b\in X_2$ be such that $\phi_2(a)=\phi_2(b)$, we have:

$$\begin{aligned}\phi_2(a)=\phi_2(b) &\Rightarrow \lambda_2\phi_2(a)=\lambda_2\phi_2(b)\\ &\Rightarrow \phi_3\mu_2(a)=\phi_3\mu_2(b)\\ &\Rightarrow \mu_2(a)=\mu_2(b)\\ &\Rightarrow (a,b)\in \mathrm{R}(\mu_2)\\ &\Rightarrow (a,b)\in\mu_1\times\mu_1(X_1\times X_1)\cup\Delta_{X_2}\\ &\Rightarrow a=b\vee(\exists c,d\in X_1\quad (a,b)=(\mu_1(c),\mu_1(d)))\end{aligned}$$

Moreover, if $c,d\in X_1$ are such that $(a,b)=(\mu_1(c),\mu_1(d))$, then since ϕ_1 and λ_1 (see Note 1.2) are one to one, we have:

$$\begin{aligned}(a,b)=(\mu_1(c),\mu_1(d)) &\Rightarrow \phi_2\mu_1(c)=\phi_2\mu_1(d)\\ &\Rightarrow \lambda_1\phi_1(c)=\lambda_1\phi_1(d)\\ &\Rightarrow c=d\\ &\Rightarrow \mu_1(c)=\mu_1(d)\\ &\Rightarrow a=b\end{aligned}$$

thus $\phi_2:(X_2,S)\to(Y_2,S)$ is a monomorphism.

- Let $\phi_1:(X_1,S)\to(Y_1,S)$ and $\phi_3:(X_3,S)\to(Y_3,S)$ be epimorphisms, and $a\in Y_2$, since $\phi_3:(X_3,S)\to(Y_3,S)$ and $\mu_2:(X_2,S)\to(X_3,S)$ (see Note 1.2) are epimorphisms, there exists

$b \in X_2$ such that $\phi_3 \mu_2(b) = \lambda_2(a)$. Using the fact that $\phi_1 : (X_1, S) \to (Y_1, S)$ is an epimorphism, we have:

$$\begin{aligned}
\lambda_2(a) = \phi_3 \mu_2(b) \quad & \Rightarrow \quad \lambda_2(a) = \lambda_2 \phi_2(b) \\
& \Rightarrow \quad (a, \phi_2(b)) \in \mathrm{R}(\lambda_2) \\
& \Rightarrow \quad (a, \phi_2(b)) \in \lambda_1 \times \lambda_1 (Y_1 \times Y_1) \cup \Delta_{Y_2} \\
& \Rightarrow \quad a = \phi_2(b) \vee (\exists c, d \in Y_1 \quad (a, \phi_2(b)) = (\lambda_1(c), \lambda_1(d))) \\
& \Rightarrow \quad a \in \phi_2(X_2) \vee (\exists c \in Y_1 \quad \exists k \in X_1 \quad (a = \lambda_1(c) \wedge c = \phi_1(k))) \\
& \Rightarrow \quad a \in \phi_2(X_2) \vee (\exists k \in X_1 \quad a = \lambda_1 \phi_1(k)) \\
& \Rightarrow \quad a \in \phi_2(X_2) \vee (\exists k \in X_1 \quad a = \phi_2 \mu_1(k)) \\
& \Rightarrow \quad a \in \phi_2(X_2)
\end{aligned}$$

thus $\phi_2 : (X_2, S) \to (Y_2, S)$ is an epimorphism.

Definition 1. 4. Exact sequences $\cdots \overset{\mu_{n-2}}{\to} (X_{n-1}, S) \overset{\mu_{n-1}}{\to} (X_n, S) \overset{\mu_n}{\to} (X_{n+1}, S) \overset{\mu_{n+1}}{\to} \cdots$ and $\cdots \overset{\lambda_{n-2}}{\to} (Y_{n-1}, S) \overset{\lambda_{n-1}}{\to} (Y_n, S) \overset{\lambda_n}{\to} (Y_{n+1}, S) \overset{\lambda_{n+1}}{\to} \cdots$ are isomorph if there are isomorphisms $\phi_i : (X_i, S) \to (Y_i, S)$ $(i \in \mathbf{Z})$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \overset{\mu_{n-2}}{\to} & (X_{n-1}, S) & \overset{\mu_{n-1}}{\to} & (X_n, S) & \overset{\mu_n}{\to} & (X_{n+1}, S) & \overset{\mu_{n+1}}{\to} & \cdots \\
& & \downarrow \phi_{n-1} & & \downarrow \phi_n & & \downarrow \phi_{n+1} & & \\
\cdots & \overset{\lambda_{n-2}}{\to} & (Y_{n-1}, S) & \overset{\lambda_{n-1}}{\to} & (Y_n, S) & \overset{\lambda_n}{\to} & (Y_{n+1}, S) & \overset{\lambda_{n+1}}{\to} & \cdots
\end{array}.$$

Theorem 1. 5. Let $(\{t\}, S) \overset{\lambda_0}{\to} (X_1, S) \overset{\lambda_1}{\to} (X, S) \overset{\lambda_2}{\to} (X_2, S) \to (\{t\}, S)$ be an exact sequence such that $t = \lambda_i(t)$ $(i = 0,1,2)$ and for each $x \in X$, $xS = \{x\}$ if and only if $x = t$. The following statements are equivalent (see [4]):

a. There exist homomorphisms $\mu_1 : (X, S) \to (X_1, S)$ and $\mu_2 : (X_2, S) \to (X, S)$ such that $\mu_1 \lambda_1 = \mathrm{id}_{X_1}$ and $\lambda_2 \mu_2 = \mathrm{id}_{X_2}$.

b. There exists a homomorphism $\mu_2 : (X_2, S) \to (X, S)$ such that $\lambda_2 \mu_2 = \mathrm{id}_{X_2}$.

c. Two exact sequences $(\{t\}, S) \overset{\lambda_0}{\to} (X_1, S) \overset{\lambda_1}{\to} (X, S) \overset{\lambda_2}{\to} (X_2, S) \to (\{t\}, S)$ and $(\{t\}, S) \overset{t}{\to} (X_1, S) \overset{\iota_1}{\to} (X_1 \times \{t\} \cup \{t\} \times X_2, S) \overset{\pi_2}{\to} (X_2, S) \to (\{t\}, S)$ (which is introduced in Note 1.2), are isomorph, in particular two transformation semigroups (X, S) and $(X_1 \times \{t\} \cup \{t\} \times X_2, S)$ are isomorph.

Proof:

- "(b) $\Rightarrow$ (c)": Define $\phi : (X_1 \times \{t\} \cup \{t\} \times X_2, S) \to (X, S)$ by:

$$\phi(x, y) = \begin{cases} \lambda_1(x) & (x, y) \in X_1 \times \{t\} \\ \mu_2(y) & (x, y) \in \{t\} \times X_2 \end{cases} \quad \forall (x, y) \in X_1 \times \{t\} \cup \{t\} \times X_2$$

The following diagram commutes (since $\lambda_2 \phi(X_1 \times \{t\}) = \lambda_2 \lambda_1(X_1) = \{\lambda_2 \lambda_1(t)\} = \{t\} = \pi_2(X_1 \times \{t\})$):

$$\begin{array}{ccccc}
(X_1, S) & \overset{\iota_1}{\to} & (X_1 \times \{t\} \cup \{t\} \times X_2, S) & \overset{\pi_2}{\to} & (X_2, S) \\
\mathrm{id}_{X_1} \downarrow & & \downarrow \phi & & \downarrow \mathrm{id}_{X_2} \\
(X_1, S) & \overset{\lambda_1}{\to} & (X, S) & \overset{\lambda_2}{\to} & (X_2, S)
\end{array}$$

thus by Lemma 1.3 (Short Five Lemma), $\phi:(X_1\times\{t\}\cup\{t\}\times X_2,S)\to(X,S)$ is an isomorphism and the mentioned two exact sequences are isomorph.

- "(c)$\Rightarrow$(a)": Suppose two exact sequences $(\{t\},S)\overset{\lambda_0}{\to}(X_1,S)\overset{\lambda_1}{\to}(X,S)\overset{\lambda_2}{\to}(X_2,S)\to(\{t\},S)$ and $(\{t\},S)\overset{\iota}{\to}(X_1,S)\overset{\iota_1}{\to}(X_1\times\{t\}\cup\{t\}\times X_2,S)\overset{\pi_2}{\to}(X_2,S)\to(\{t\},S)$ are isomorph, there are isomorphisms $\phi_1:(X_1,S)\to(X_1,S)$, $\phi:(X,S)\to(X_1\times\{t\}\cup\{t\}\times X_2,S)$ and $\phi_2:(X_2,S)\to(X_2,S)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}(\{t\},S) & \overset{\lambda_0}{\to} & (X_1,S) & \overset{\lambda_1}{\to} & (X,S) & \overset{\lambda_2}{\to} & (X_2,S) & \to & (\{t\},S)\\ & & \phi_1\downarrow & & \downarrow\phi & & \downarrow\phi_2 & & \\ (\{t\},S) & \overset{\iota}{\to} & (X_1,S) & \overset{\iota_1}{\to} & (X_1\times\{t\}\cup\{t\}\times X_2,S) & \overset{\pi_2}{\to} & (X_2,S) & \to & (\{t\},S)\end{array}.$$

Define $\mu_2:(X_2,S)\to(X,S)$ by $\mu_2(x_2)=\phi^{-1}(t,\phi_2(x_2))$ $(\forall x_2\in X_2)$. For each $x_2\in X_2$, we have (use $\phi_2\lambda_2=\pi_2\phi$):

$$\begin{aligned}\lambda_2\mu_2(x_2)&=\lambda_2\phi^{-1}(t,\phi_2(x_2))=\phi_2^{-1}\phi_2\lambda_2\phi^{-1}(t,\phi_2(x_2))\\&=\phi_2^{-1}\pi_2\phi\phi^{-1}(t,\phi_2(x_2))=\phi_2^{-1}\pi_2(t,\phi_2(x_2))\\&=\phi_2^{-1}\phi_2(x_2)=x_2\end{aligned}$$

so $\lambda_2\mu_2=\mathrm{id}_{X_2}$. On the other hand, define $\pi_1:(X_1\times\{t\}\cup\{t\}\times X_2,S)\to(X_1,S)$ by $\pi_1(x,y)=x$ $(\forall(x,y)\in X_1\times\{t\}\cup\{t\}\times X_2)$, and $\mu_1:(X,S)\to(X_1,S)$ by $\mu_1(x)=\phi_1^{-1}\pi_1\phi(x)$ $(\forall x\in X)$. For each $x_1\in X_1$, we have (use $\phi\lambda_1=\iota_1\phi_1$) $\mu_1\lambda_1(x_1)=\phi_1^{-1}\pi_1\phi\lambda_1(x_1)=\phi_1^{-1}\pi_1\iota_1\phi_1(x_1)=\phi_1^{-1}\phi_1(x_1)=x_1$ so $\mu_1\lambda_1=\mathrm{id}_{X_1}$.

Lemma 1. 6. (The Five Lemma) Let

$$\begin{array}{ccccccccc}(X_1,S) & \overset{\mu_1}{\to} & (X_2,S) & \overset{\mu_2}{\to} & (X_3,S) & \overset{\mu_3}{\to} & (X_4,S) & \overset{\mu_4}{\to} & (X_5,S)\\ \phi_1\downarrow & & \phi_2\downarrow & & \downarrow\phi_3 & & \downarrow\phi_4 & & \downarrow\phi_5\\ (Y_1,S) & \overset{\lambda_1}{\to} & (Y_2,S) & \overset{\lambda_2}{\to} & (Y_3,S) & \overset{\lambda_3}{\to} & (Y_4,S) & \overset{\lambda_4}{\to} & (Y_5,S)\end{array}$$

be a commutative diagram such that each row is exact, then we have (see [4]):

a. Let $\phi_1:(X_1,S)\to(Y_1,S)$ be an epimorphism and $\phi_2:(X_2,S)\to(Y_2,S)$, $\phi_4:(X_4,S)\to(Y_4,S)$ be monomorphisms, then $\phi_3:(X_3,S)\to(Y_3,S)$ is a monomorphism.

b. Let $\phi_5:(X_5,S)\to(Y_5,S)$ be a monomorphism and $\phi_2:(X_2,S)\to(Y_2,S)$, $\phi_4:(X_4,S)\to(Y_4,S)$ be epimorphisms, then $\phi_3:(X_3,S)\to(Y_3,S)$ is an epimorphism.

c. Let $\phi_1:(X_1,S)\to(Y_1,S)$ be an epimorphism, $\phi_5:(X_5,S)\to(Y_5,S)$ be a monomorphism and $\phi_2:(X_2,S)\to(Y_2,S)$, $\phi_4:(X_4,S)\to(Y_4,S)$ be isomorphisms, then $\phi_3:(X_3,S)\to(Y_3,S)$ is an isomorphism.

Proof:

a. Let $a,b\in X_3$ be such that $\phi_3(a)=\phi_3(b)$, then:

$$\begin{aligned}\phi_3(a)=\phi_3(b)\quad&\Rightarrow\quad\lambda_3\phi_3(a)=\lambda_3\phi_3(b)\\&\Rightarrow\quad\phi_4\mu_3(a)=\phi_4\mu_3(b)\\&\Rightarrow\quad\mu_3(a)=\mu_3(b)\\&\Rightarrow\quad(a,b)\in\mathrm{R}(\mu_3)\end{aligned}$$

$$\Rightarrow \quad a = b \vee (\exists c, d \in X_2 \quad (a,b) = (\mu_2(c), \mu_2(d)))$$

if $c, d \in X_2$ are such that $a = \mu_2(c)$ and $b = \mu_2(d)$, then:

$\phi_3 \mu_2(c) = \phi_3 \mu_2(d)$

$$\Rightarrow \quad \lambda_2 \phi_2(c) = \lambda_2 \phi_2(d)$$
$$\Rightarrow \quad (\phi_2(c), \phi_2(d)) \in \mathrm{R}(\lambda_2)$$
$$\Rightarrow \quad \phi_2(c) = \phi_2(d) \vee (\exists l, k \in Y_1 \quad (\phi_2(c), \phi_2(d)) = (\lambda_1(l), \lambda_1(k)))$$
$$\Rightarrow \quad c = d \vee (\exists l', k' \in X_1 \quad (\phi_2(c), \phi_2(d)) = (\lambda_1 \phi_1(l'), \lambda_1 \phi_1(k')))$$
$$\Rightarrow \quad a = b \vee (\exists l', k' \in X_1 \quad (\phi_2(c), \phi_2(d)) = (\phi_2 \mu_1(l'), \phi_2 \mu_1(k')))$$
$$\Rightarrow \quad a = b \vee (\exists l', k' \in X_1 \quad (c,d) = (\mu_1(l'), \mu_1(k')))$$
$$\Rightarrow \quad a = b \vee \mu_2(c) = \mu_2(d) \text{ (By the definition of exact sequences)}$$
$$\Rightarrow \quad a = b$$

thus $\phi_3 : (X_3, S) \to (Y_3, S)$ is a monomorphism.

b. Let $a \in Y_3$ and $u \in X_3$. Choose $b \in X_4$ such that $\lambda_3(a) = \phi_4(b)$, we have:

$\lambda_4 \lambda_3(a) = \lambda_4 \phi_4(b)$

$$\Rightarrow \quad \lambda_4 \lambda_3 \phi_3(u) = \phi_5 \mu_4(b) \qquad \text{(use: } \lambda_4 \lambda_3(Y_3) = \{\lambda_4 \lambda_3(a)\}\text{)}$$
$$\Rightarrow \quad \lambda_4 \phi_4 \mu_3(u) = \phi_5 \mu_4(b)$$
$$\Rightarrow \quad \phi_5 \mu_4 \mu_3(u) = \phi_5 \mu_4(b)$$
$$\Rightarrow \quad \mu_4 \mu_3(u) = \mu_4(b)$$
$$\Rightarrow \quad \mu_3(u) = b \vee (\exists v, w \in X_3 \quad (\mu_3(u), b) = (\mu_3(v), \mu_3(w)))$$
$$\Rightarrow \quad \exists c \in X_3 \quad b = \mu_3(c)$$
$$\Rightarrow \quad \exists c \in X_3 \quad \lambda_3(a) = \phi_4 \mu_3(c)$$
$$\Rightarrow \quad \exists c \in X_3 \quad \lambda_3(a) = \lambda_3 \phi_3(c)$$
$$\Rightarrow \quad \exists c \in X_3 \quad (a = \phi_3(c) \vee (\exists v, w \in Y_2 \quad (a, \phi_3(c)) = (\lambda_2(v), \lambda_2(w))))$$
$$\Rightarrow \quad a \in \phi_3(X_3) \vee (\exists c \in Y_2 \quad a = \lambda_2(c))$$
$$\Rightarrow \quad a \in \phi_3(X_3) \vee (\exists c \in X_2 \quad a = \lambda_2 \phi_2(c))$$
$$\Rightarrow \quad a \in \phi_3(X_3) \vee (\exists c \in X_2 \quad a = \phi_3 \mu_2(c))$$
$$\Rightarrow \quad a \in \phi_3(X_3)$$

thus $\phi_3 : (X_3, S) \to (Y_3, S)$ is an epimorphism.

c. Use (a) and (b) Lemma 1.3, (The Short Five Lemma is a corollary of (c)).

Remark 1. 7. If $(\{t\}, S) \to (X, S) \to (Y, S) \overset{f}{\to} (Z, S) \to (\{t\}, S)$ and $(\{t\}, S) \to (Z, S) \overset{g}{\to} (V, S) \to (W, S) \to (\{t\}, S)$ are exact sequences, then $(\{t\}, S) \to (X, S) \to (Y, S) \overset{g \circ f}{\to} (V, S) \to (W, S) \to (\{t\}, S)$ is an exact sequence (see [4]).

Theorem 1. 8. In the following diagram suppose that each row is an exact sequence:

$$\begin{array}{ccccccc} & & & (X_3, S) & & & \\ & & \overset{\phi_2}{\nearrow} & & \overset{\phi_3}{\searrow} & & \\ (X_1, S) \overset{\phi_1}{\to} (X_2, S) & & & & & & (X_4, S) \overset{\phi_4}{\to} (X_5, S), \\ & & \underset{\phi_2'}{\searrow} & & \underset{\phi_3'}{\nearrow} & & \\ & & & (X_3', S) & & & \end{array}$$

then we have:

a. If $\psi : (X_3, S) \to (X_3', S)$ is a homomorphism, such that the following diagram commutes:

$$\begin{array}{ccccc} & \phi_2 & (X_3,S) & \phi_3 & \\ (X_2,S) & \nearrow & \downarrow \psi & \searrow & (X_4,S) \\ & \searrow & (X'_3,S) & \nearrow & \\ & \phi'_2 & & \phi'_3 & \end{array} \qquad (1.8.1)$$

then $\psi:(X_3,S)\to(X'_3,S)$ is an isomorphism.

b. $\mathrm{R}(\phi_2)=\mathrm{R}(\phi'_2)$. Moreover, $(\phi_2(X_2),S)$ and $(\phi'_2(X_2),S)$ are isomorph.

c. If $\mathrm{card}(\phi_3(X_3))>1$, then $\phi_3(X_3)=\phi'_3(X'_3)$.

d. If $\mathrm{card}(\phi_3(X_3))=1$, then:

1. $\mathrm{card}(\phi'_3(X'_3))=1$,
2. ϕ_4 is one to one,
3. $\phi_2:(X_2,S)\to(X_3,S)$ and $\phi'_2:(X_2,S)\to(X'_3,S)$ are onto,
4. if $\psi=\phi'_2\phi_2^{-1}$, then (1.8.1) is a commutative diagram and $\phi'_2\phi_2^{-1}:(X_3,S)\to(X'_3,S)$ is an isomorphism.

e.

1. $\mathrm{card}(\phi_3(X_3))=\mathrm{card}(\phi'_3(X'_3))$,
2. $(\frac{X_3}{\mathrm{R}(\phi_3)},S)$ and $(\frac{X'_3}{\mathrm{R}(\phi'_3)},S)$ are isomorph.

f. $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2\times\phi'_2}{\to}(X_3\times X'_3,S)$ is an exact sequence.

Proof:

a. Use Lemma 1.6 (The Five Lemma).

b. $\mathrm{R}(\phi_2)=\phi_1\times\phi_1(X_1\times X_1)\cup\Delta_{X_2}=\mathrm{R}(\phi'_2)$. In addition, $(\phi_2(X_2),S)$, $(\frac{X_2}{\mathrm{R}(\phi_2)},S)$ $(=(\frac{X'_2}{\mathrm{R}(\phi'_2)},S))$ and $(\phi'_2(X_2),S)$ are isomorph.

c. Since $\mathrm{card}(\phi_3(X_3))>1$, for each $u\in\phi_3(X_3)$ there exists $v\in\phi_3(X_3)$ such that $u\neq v$, moreover, $\phi_4(u)=\phi_4(v)$, so $(u,v)\in\phi'_3\times\phi'_3(X'_3\times X'_3)$, therefore $u,v\in\phi'_3(X'_3)$, $\phi_3(X_3)\subseteq\phi'_3(X'_3)$ and $\mathrm{card}(\phi'_3(X'_3))>1$. By a similar method we have $\phi_3(X_3)\supseteq\phi'_3(X'_3)$.

d. 1. Use (c).

2. $\phi_3\times\phi_3(X_3\times X_3)$ is a singleton set, so $\phi_3\times\phi_3(X_3\times X_3)\subseteq\Delta_{X_4}$, $\Delta_{X_4}=\mathrm{R}(\phi_4)$ and ϕ_4 is one to one.

3. Using $\phi_2\times\phi_2(X_2\times X_2)\cup\Delta_{X_3}=\mathrm{R}(\phi_3)=X_3\times X_3$ we have $\phi_2(X_2)=X_3$ (by a similar method $\phi'_2(X_2)=X'_3$).

e. Use (c) and (d).

f. For each $a,b\in X_1$ we have $\phi_2\phi_1(a)=\phi_2\phi_1(b)$ and $\phi'_2\phi_1(a)=\phi'_2\phi_1(b)$, so $\phi_2\times\phi'_2(\phi_1(a))=\phi_2\times\phi'_2(\phi_1(b))$, therefore $\phi_1\times\phi_1(X_1\times X_1)\subseteq\mathrm{R}(\phi_2\times\phi'_2)$. In order to complete the proof use $\mathrm{R}(\phi_2\times\phi'_2)-\Delta_{X_2}\subseteq\mathrm{R}(\phi_2)-\Delta_{X_2}\subseteq\phi_1\times\phi_1(X_1\times X_1)$.

2. P- EXACT SEQUENCES OF TRANSFORMATION SEMIGROUPS

As a matter of fact, in the definition of an exact sequence of transformation semigroups, with the exception of the choice of functions, which should be homomorphisms, you will not find any other role of the phase semigroup, and to some extent this problem will be solved by the definition of $\mathrm{P}-$ exact, $\mathrm{P}_\Gamma-$exact, $\overline{\mathrm{P}}_\Gamma$ $-$exact and $\overline{\overline{\mathrm{P}}}_\Gamma$ $-$exact sequences (of transformation semigroups).

Definition 2. 1.

- $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)$ $(n\geq 2)$ is called a $\mathrm{P}-$exact sequence (of transformation semigroups) if for each $i\in\{1,\ldots,n-1\}$, $\phi_i\times\phi_i(\mathrm{P}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$.
- For each $i\in\{1,\ldots,n+1\}$ let A_i be a nonempty subset of X_i and $\Gamma=(A_i)_{i=1}^{n+1}$. $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)$ $(n\geq 2)$ is called a $\mathrm{P}_\Gamma-$exact (resp. $\overline{\mathrm{P}}_\Gamma-$exact, $\overline{\overline{\mathrm{P}}}_\Gamma-$exact) sequence (of transformation semigroups) if for each $i\in\{1,\ldots,n-1\}$, $\phi_i\times\phi_i(\mathrm{P}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$ (resp. $\phi_i\times\phi_i(\overline{\mathrm{P}}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$, $\phi_i\times\phi_i(\overline{\overline{\mathrm{P}}}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$) (as a matter of fact, A_n and A_{n+1} are extra).
- An infinite sequence of transformation semigroup homomorphisms $\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)\overset{\phi_{n+1}}{\to}(X_{n+2},S)\overset{\phi_{n+2}}{\to}\cdots$ is $\mathrm{P}-$exact if for each $i\in\mathbf{Z}$, $\phi_i\times\phi_i(\mathrm{P}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$.
- For each $i\in\mathbf{Z}$, let A_i be a nonempty subset of X_i and $\Gamma=(A_i)_{i\in\mathbf{Z}}$. An infinite sequence of transformation semigroup homomorphisms $\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)\overset{\phi_{n+1}}{\to}(X_{n+2},S)\overset{\phi_{n+2}}{\to}\cdots$ is $\mathrm{P}_\Gamma-$exact (resp. $\overline{\mathrm{P}}_\Gamma-$exact, $\overline{\overline{\mathrm{P}}}_\Gamma-$exact) sequence (of transformation semigroups) if for each $i\in\mathbf{Z}$, $\phi_i\times\phi_i(\mathrm{P}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$ (resp. $\phi_i\times\phi_i(\overline{\mathrm{P}}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$, $\phi_i\times\phi_i(\overline{\overline{\mathrm{P}}}_{A_i}(X_i))\cup\Delta_{X_{i+1}}=\mathrm{R}(\phi_{i+1})$).

Note 2. 2. Let (X_1,S), (X_2,S) and (X_3,S) be transformation semigroups. Let $\varnothing\neq A_i\subseteq X_i$ $(i=1,2,3)$ and $tS=\{t\}$, then:

- $(\{t\},S)\overset{\phi_0}{\to}(X_1,S)\overset{\phi_1}{\to}(X_2,S)$ is $\mathrm{P}-$exact (resp. for $\Gamma=(\{t\},A_1,A_2)$: $\mathrm{P}_\Gamma-$exact, $\overline{\mathrm{P}}_\Gamma-$exact, $\overline{\overline{\mathrm{P}}}_\Gamma-$exact) if and only if $\phi_1:(X_1,S)\to(X_2,S)$ is a monomorphism (ϕ_0 exists if and only if there exists $u\in X_1$ such that $uS=\{u\}$).
- If $(X_2,S)\overset{\phi_2}{\to}(X_3,S)\to(\{t\},S)$ is $\mathrm{P}-$exact (resp. for $\Gamma=(A_2,A_3,\{t\})$: $\mathrm{P}_\Gamma-$exact, $\overline{\mathrm{P}}_\Gamma-$exact, $\overline{\overline{\mathrm{P}}}_\Gamma-$exact), then $\phi_2:(X_2,S)\to(X_3,S)$ is an epimorphism.

Lemma 2. 3. (The Short Five Lemma) Let

$$\begin{array}{ccccccccc}(\{t\},S) & \to & (X_1,S) & \overset{\mu_1}{\to} & (X_2,S) & \overset{\mu_2}{\to} & (X_3,S) & \to & (\{t\},S)\\ & & \downarrow\phi_1 & & \downarrow\phi_2 & & \downarrow\phi_3 & & \\ (\{t\},S) & \to & (Y_1,S) & \overset{\lambda_1}{\to} & (Y_2,S) & \overset{\lambda_2}{\to} & (Y_3,S) & \to & (\{t\},S)\end{array}$$

be a commutative diagram of transformation semigroup homomorphisms such that the first row is $\mathrm{Q}_\Gamma-$exact ($\mathrm{P}-$exact) and the second row is $\mathrm{Q}_\Lambda-$exact ($\mathrm{P}-$exact), where $\mathrm{Q}\in\{\mathrm{P},\overline{\mathrm{P}},\overline{\overline{\mathrm{P}}}\}$, $\Gamma=(\{t\},A_1,A_2,A_3,\{t\})$ (for $i=1,2,3$, A_i is a nonempty subset of X_i) and $\Lambda=(\{t\},B_1,B_2,B_3,\{t\})$ (for $i=1,2,3$, B_i is a nonempty subset of Y_i), then we have:

- If $\phi_1 : (X_1, S) \to (Y_1, S)$ and $\phi_3 : (X_3, S) \to (Y_3, S)$ are monomorphisms, then $\phi_2 : (X_2, S) \to (Y_2, S)$ is a monomorphism.
- If $\phi_1 : (X_1, S) \to (Y_1, S)$ and $\phi_3 : (X_3, S) \to (Y_3, S)$ are epimorphisms, then $\phi_2 : (X_2, S) \to (Y_2, S)$ is an epimorphism.
- If $\phi_1 : (X_1, S) \to (Y_1, S)$ and $\phi_3 : (X_3, S) \to (Y_3, S)$ are isomorphisms, then $\phi_2 : (X_2, S) \to (Y_2, S)$ is an isomorphism.

Proof: Use Note 2.2 and a similar method described in Lemma 1.3.

Definition 2. 4. $P-$exact (resp. $P_\Gamma-$exact, $\overline{P}_\Gamma-$exact and $\overline{\overline{P}}_\Gamma-$exact where $\Gamma = (A_i)_{i\in\mathbf{Z}}$ and for each $i \in \mathbf{Z}$, A_i is a nonempty subset of X_i) sequence $\cdots \overset{\mu_{n-2}}{\to} (X_{n-1}, S) \overset{\mu_{n-1}}{\to} (X_n, S) \overset{\mu_n}{\to} (X_{n+1}, S) \overset{\mu_{n+1}}{\to} \cdots$ and $P-$exact (resp. $P_\Lambda-$exact, $\overline{P}_\Lambda-$exact and $\overline{\overline{P}}_\Lambda-$exact where $\Lambda = (B_i)_{i\in\mathbf{Z}}$ and for each $i \in \mathbf{Z}$, B_i is a nonempty subset of Y_i) sequence $\cdots \overset{\lambda_{n-2}}{\to} (Y_{n-1}, S) \overset{\lambda_{n-1}}{\to} (Y_n, S) \overset{\lambda_n}{\to} (Y_{n+1}, S) \overset{\lambda_{n+1}}{\to} \cdots$ are isomorph if there are isomorphisms $\phi_i : (X_i, S) \to (Y_i, S)$ $(i \in \mathbf{Z})$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \overset{\mu_{n-2}}{\to} & (X_{n-1}, S) & \overset{\mu_{n-1}}{\to} & (X_n, S) & \overset{\mu_n}{\to} & (X_{n+1}, S) & \overset{\mu_{n+1}}{\to} & \cdots \\ & & \downarrow \phi_{n-1} & & \downarrow \phi_n & & \downarrow \phi_{n+1} & & \\ \cdots & \overset{\lambda_{n-2}}{\to} & (Y_{n-1}, S) & \overset{\lambda_{n-1}}{\to} & (Y_n, S) & \overset{\lambda_n}{\to} & (Y_{n+1}, S) & \overset{\lambda_{n+1}}{\to} & \cdots \end{array},$$

(resp. moreover $\phi_i(A_i) = B_i$).

Lemma 2. 5. Let

$$\begin{array}{ccccccc} (X_1, S) & \overset{\mu_1}{\to} & (X_2, S) & \overset{\mu_2}{\to} & (X_3, S) & \overset{\mu_3}{\to} & (X_4, S) \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \downarrow \phi_3 & & \downarrow \phi_4 \\ (Y_1, S) & \overset{\lambda_1}{\to} & (Y_2, S) & \overset{\lambda_2}{\to} & (Y_3, S) & \overset{\lambda_3}{\to} & (Y_4, S) \end{array}$$

be a commutative diagram such that the first row is $Q_\Gamma-$exact ($P-$exact) and the second row is $Q_\Lambda-$exact ($P-$exact), where $Q \in \{P, \overline{P}, \overline{\overline{P}}\}$, $\Gamma = (A_1, A_2, A_3, A_4)$ (for $i = 1,2,3,4$, A_i is a nonempty subset of X_i) and $\Lambda = (\phi_1(A_1), \phi_2(A_2), \phi_3(A_3), \phi_4(A_4))$, then we have:

a. Let $\phi_1 : (X_1, S) \to (Y_1, S)$ be an isomorphism and $\phi_2 : (X_2, S) \to (Y_2, S)$, $\phi_4 : (X_4, S) \to (Y_4, S)$ be monomorphisms, then $\phi_3 : (X_3, S) \to (Y_3, S)$ is a monomorphism.

b. Let $X_1 = Y_1 = \{t\}$ and $\phi_2 : (X_2, S) \to (Y_2, S)$, $\phi_4 : (X_4, S) \to (Y_4, S)$ be monomorphisms and $\phi_3(X_3) = Y_3$, then $\phi_3 : (X_3, S) \to (Y_3, S)$ is an isomorphism.

Proof:

a. Use a similar method described in Lemma 1.6 (a).

b. Use (a).

Theorem 2. 6. In the following diagram suppose each row be a $P-$exact (resp. the first row is $P_\Gamma-$exact, $\overline{P}_\Gamma-$exact, $\overline{\overline{P}}_\Gamma-$exact, where $\Gamma = (A_1, A_2, A_3, A_4)$ (A_i is a nonempty subset of X_i, for

$i = 1,2,3,4$) and the second row is $\mathrm{P}_\Lambda -$ exact, $\overline{\mathrm{P}}_\Lambda -$ exact, $\overline{\overline{\mathrm{P}}}_\Lambda -$ exact, where $\Lambda = (A_1, A_2, A'_3, A_4)$ (A'_3 is a nonempty subset of X'_3) sequence:

$$(X_1,S)\overset{\phi_1}{\to}(X_2,S) \begin{array}{c} \overset{\phi_2}{\nearrow} \\ \underset{\phi'_2}{\searrow} \end{array} \begin{array}{c} (X_3,S) \\ \\ (X'_3,S) \end{array} \begin{array}{c} \overset{\phi_3}{\searrow} \\ \underset{\phi'_3}{\nearrow} \end{array} (X_4,S),$$

then we have:

a. If $\psi : (X_3, S) \to (X'_3, S)$ is an epimorphism, such that the following diagram commutes:

$$(X_2,S) \begin{array}{c} \overset{\phi_2}{\nearrow} \\ \underset{\phi'_2}{\searrow} \end{array} \begin{array}{c} (X_3,S) \\ \downarrow\psi \\ (X'_3,S) \end{array} \begin{array}{c} \overset{\phi_3}{\searrow} \\ \underset{\phi'_3}{\nearrow} \end{array} (X_4,S)$$

(and $\psi(A_3) = A'_3$), then $\psi : (X_3, S) \to (X'_3, S)$ is an isomorphism.

b. $\mathrm{R}(\phi_2) = \mathrm{R}(\phi'_2)$, moreover $(\phi_2(X_2), S)$ and $(\phi'_2(X_2), S)$ are isomorph.

Proof: Use Lemma 2.5 and a similar method described in Theorem 1.8.

Theorem 2. 7. For each $i \in \mathbf{Z}$, let X_i and Y_i be nonempty closed subsets of compact Hausdorff topological space Z_i, such that $X_i \cap Y_i = \varnothing$ and $X_i \cup Y_i = Z_i$. Consider:

$$\cdots \overset{\phi_{n-1}}{\to} (X_n, S_1) \overset{\phi_n}{\to} (X_{n+1}, S_1) \overset{\phi_{n+1}}{\to} \cdots \quad (2.7.1)$$

$$\cdots \overset{\mu_{n-1}}{\to} (Y_n, S_2) \overset{\mu_n}{\to} (Y_{n+1}, S_2) \overset{\mu_{n+1}}{\to} \cdots \quad (2.7.2)$$

and

$$\cdots \overset{\phi_{n-1}\cup\mu_{n-1}}{\to} (Z_n, S) \overset{\phi_n\cup\mu_n}{\to} (Z_{n+1}, S) \overset{\phi_{n+1}\cup\mu_{n+1}}{\to} \cdots \quad (2.7.3)$$

where $S = S_1 \times S_2$ and:

$$z(s_1, s_2) = \begin{cases} zs_1 & z \in X_i \\ zs_2 & z \in Y_i \end{cases} \quad (\forall z \in Z_i, \forall i \in \mathbf{Z}, \forall (s_1, s_2) \in S_1 \times S_2),$$

then for $\Gamma = (A_i)_{i\in\mathbf{Z}}$ (for $i \in \mathbf{Z}$, A_i is a nonempty subset of X_i), $\Lambda = (B_i)_{i\in\mathbf{Z}}$ (for $i \in \mathbf{Z}$, B_i is a nonempty subset of Y_i) and $\Theta = (A_i \cup B_i)_{i\in\mathbf{Z}}$, we have:

a. (2.7.1) and (2.7.2) are $\mathrm{P}-$ exact, if and only if (2.7.3) is $\mathrm{P}-$ exact (if (2.7.1) and (2.7.2) are $\mathrm{P}-$ exact, then (2.7.3) is $\mathrm{P}_\Theta -$ exact).

b. (2.7.1) is $\mathrm{P}_\Gamma -$ exact and (2.7.2) is $\mathrm{P}-$ exact, if and only if (2.7.3) is $\mathrm{P}_\Gamma -$ exact.

c. (2.7.1) is $\overline{\mathrm{P}}_\Gamma -$ exact and (2.7.2) is $\mathrm{P}-$ exact, if and only if (2.7.3) is $\overline{\mathrm{P}}_\Gamma -$ exact.

d. (2.7.1) is $\overline{\mathrm{P}}_\Gamma -$ exact and (2.7.2) is $\overline{\mathrm{P}}_\Lambda -$ exact, if and only if (2.7.3) is $\overline{\mathrm{P}}_\Theta -$ exact.

Proof: b. Using [5], for each $n \in \mathbf{Z}$, we have $\mathrm{P}_{A_n}(Z_n, S) = \mathrm{P}_{A_n}(X_n, S_1) \cup \mathrm{P}(Y_n, S_2)$.

Suppose (2.7.1) is $\mathrm{P}_\Gamma-$exact and (2.7.2) is $\mathrm{P}-$exact, so for each $n\in\mathbf{Z}$, we have $\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\Delta_{X_{n+1}}=\mathrm{R}(\phi_{n+1})$ and $\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{Y_{n+1}}=\mathrm{R}(\mu_{n+1})$, therefore:

$$\begin{aligned}
&(\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}_{A_n}(Z_n,S))\cup\Delta_{Z_{n+1}}\\
&\quad=(\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}_{A_n}(X_n,S_1)\cup\mathrm{P}(Y_n,S_2))\cup\Delta_{Z_{n+1}}\\
&\quad=(\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}_{A_n}(X_n,S_1))\cup(\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}(Y_n,S_2))\cup\Delta_{Z_{n+1}}\\
&\quad=\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{Z_{n+1}}\\
&\quad=\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{X_{n+1}}\cup\Delta_{Y_{n+1}}\\
&\quad=\mathrm{R}(\phi_{n+1})\cup\mathrm{R}(\mu_{n+1})\\
&\quad=\mathrm{R}(\phi_{n+1}\cup\mu_{n+1})\qquad(\text{since } X_n\cap Y_n=\varnothing)
\end{aligned}$$

thus (2.7.3) is $\mathrm{P}_\Gamma-$exact.

On the other hand, if (2.7.3) is $\mathrm{P}_\Gamma-$exact, for each $n\in\mathbf{Z}$, using a similar method as the above we have:

$$\begin{aligned}
\mathrm{R}(\varphi_{n+1})&=(\mathrm{R}(\phi_{n+1})\cup\mathrm{R}(\mu_{n+1}))\cap(X_{n+1}\times X_{n+1})\\
&=\mathrm{R}(\phi_{n+1}\cup\mu_{n+1})\cap(X_{n+1}\times X_{n+1})\\
&=((\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}_{A_n}(Z_n,S))\cup\Delta_{Z_{n+1}})\cap(X_{n+1}\times X_{n+1})\\
&=(\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{X_{n+1}}\cup\Delta_{Y_{n+1}})\cap(X_{n+1}\times X_{n+1})\\
&=\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\Delta_{X_{n+1}}
\end{aligned}$$

thus (2.7.1) is $\mathrm{P}_\Gamma-$exact. And for each $n\in\mathbf{Z}$, we have:

$$\begin{aligned}
\mathrm{R}(\mu_{n+1})&=(\mathrm{R}(\phi_{n+1})\cup\mathrm{R}(\mu_{n+1}))\cap(Y_{n+1}\times Y_{n+1})\\
&=\mathrm{R}(\phi_{n+1}\cup\mu_{n+1})\cap(Y_{n+1}\times Y_{n+1})\\
&=((\phi_n\cup\mu_n)\times(\phi_n\cup\mu_n)(\mathrm{P}_{A_n}(Z_n,S))\cup\Delta_{Z_{n+1}})\cap(Y_{n+1}\times Y_{n+1})\\
&=(\phi_n\times\phi_n(\mathrm{P}_{A_n}(X_n,S_1))\cup\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{X_{n+1}}\cup\Delta_{Y_{n+1}})\cap(Y_{n+1}\times Y_{n+1})\\
&=\mu_n\times\mu_n(\mathrm{P}(Y_n,S_2))\cup\Delta_{Y_{n+1}}
\end{aligned}$$

which leads to the desired result.

For the other items using [5], for each $n\in\mathbf{Z}$, we have:

$\underline{\mathrm{P}}(Z_n,S)=\mathrm{P}(X_n,S_1)\cup\mathrm{P}(Y_n,S_2)$,
$\overline{\mathrm{P}}_{A_n}(Z_n,S)=\overline{\mathrm{P}}_{A_n}(X_n,S_1)\cup\mathrm{P}(Y_n,S_2)$,
$\underline{\mathrm{P}}_{(A_n\cup B_n)}(Z_n,S)=\mathrm{P}(X_n,S_1)\cup\mathrm{P}(Y_n,S_2)\ (=\mathrm{P}(Z_n,S))$,
$\overline{\mathrm{P}}_{(A_n\cup B_n)}(Z_n,S)=\overline{\mathrm{P}}_{A_n}(X_n,S_1)\cup\overline{\mathrm{P}}_{B_n}(Y_n,S_2)$,

now use a similar method.

Theorem 2. 8. Consider:

$$\cdots\xrightarrow{\phi_{n-1}}(X_n,S_1)\xrightarrow{\phi_n}(X_{n+1},S_1)\xrightarrow{\phi_{n+1}}\cdots\qquad(2.8.1)$$

$$\cdots\xrightarrow{\mu_{n-1}}(Y_n,S_2)\xrightarrow{\mu_n}(Y_{n+1},S_2)\xrightarrow{\mu_{n+1}}\cdots\qquad(2.8.2)$$

and

$$\cdots\xrightarrow{\phi_{n-1}\times\mu_{n-1}}(Z_n,S)\xrightarrow{\phi_n\times\mu_n}(Z_{n+1},S)\xrightarrow{\phi_{n+1}\times\mu_{n+1}}\cdots\qquad(2.8.3)$$

where $Z_i=X_i\times Y_i$, $S=S_1\times S_2$ and:

$$(x,y)(s_1,s_2)=(xs_1,ys_2)\quad(\forall(x,y)\in X_i\times Y_i,\forall i\in\mathbf{Z},\forall(s_1,s_2)\in S_1\times S_2),$$

then for $\Theta=(A_i)_{i\in\mathbf{Z}}$ (for $i\in\mathbf{Z}$, A_i is a nonempty subset of Z_i), $\Gamma=(\pi_1(A_i))_{i\in\mathbf{Z}}$ and $\Lambda=(\pi_2(A_i))_{i\in\mathbf{Z}}$ (where π_1 and π_2 are projection maps), we have:
a. (2.8.1) and (2.8.2) are $\mathrm{P}-$ exact, if and only if (2.8.3) is $\mathrm{P}-$ exact.
b. (2.8.1) is $\underline{\mathrm{P}}_\Gamma-$exact and (2.8.2) is $\underline{\mathrm{P}}_\Lambda-$exact, if and only if (2.8.3) is $\underline{\mathrm{P}}_\Theta-$exact.
c. (2.8.1) is $\overline{\mathrm{P}}_\Gamma-$exact and (2.8.2) is $\overline{\mathrm{P}}_\Lambda-$exact, if and only if (2.8.3) is $\overline{\mathrm{P}}_\Theta-$exact.

Proof: Using [5], for each $n\in\mathbf{Z}$, we have:
$\mathrm{P}(Z_n,S)=\{((x_1,x_2),(y_1,y_2))\in Z_n\times Z_n \mid (x_1,y_1)\in \mathrm{P}(X_n,S_1),(x_2,y_2)\in\mathrm{P}(Y_n,S_2)\}$,
$\underline{\mathrm{P}}_{A_n}(Z_n,S)=\{((x_1,x_2),(y_1,y_2))\in Z_n\times Z_n | (x_1,y_1)\in \underline{\mathrm{P}}_{\pi_1(A_n)}(X_n,S_1),(x_2,y_2)\in\underline{\mathrm{P}}_{\pi_2(A_n)}(Y_n,S_2)\}$,
$\overline{\mathrm{P}}_{A_n}(Z_n,S)=\{((x_1,x_2),(y_1,y_2))\in Z_n\times Z_n | (x_1,y_1)\in \overline{\mathrm{P}}_{\pi_1(A_n)}(X_n,S_1),(x_2,y_2)\in\overline{\mathrm{P}}_{\pi_2(A_n)}(Y_n,S_2)\}$,
now use a similar method described in Theorem 2.7.

Theorem 2. 9. Let e be the identity of the semigroup S; $S_1,\ldots,S_m$ be subsemigroups of S, $e\in\bigcap_{i=1}^m S_i$ and $S=\bigcup_{i=1}^m S_i$. If the sequences $\cdots\overset{\phi_{n-1}}{\to}(X_n,S_i)\overset{\phi_n}{\to}(X_{n+1},S_i)\overset{\phi_{n+1}}{\to}\cdots$ for $i=1,\ldots,m$ are $\mathrm{P}-$exact (resp. $\underline{\mathrm{P}}_\Gamma-$exact (where $\Gamma=(A_i)_{i\in\mathbf{Z}}$ and for each $i\in\mathbf{Z}$, A_i is a nonempty subset of X_i), $\overline{\mathrm{P}}_\Lambda-$exact (where $\Lambda=(B_i)_{i\in\mathbf{Z}}$ and for each $i\in\mathbf{Z}$, $B_i\in\bigcap_{j=1}^m\overline{\mathcal{M}}(X_i,S_j)\cap\overline{\mathcal{M}}(X_i,S)$)), then $\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)\overset{\phi_{n+1}}{\to}\cdots$ is $\mathrm{P}-$exact (resp. $\underline{\mathrm{P}}_\Gamma-$exact, $\overline{\mathrm{P}}_\Lambda-$exact).

Proof: Suppose for each $i\in\{1,\ldots,m\}$, $\cdots\overset{\phi_{n-1}}{\to}(X_n,S_i)\overset{\phi_n}{\to}(X_{n+1},S_i)\overset{\phi_{n+1}}{\to}\cdots$ be a $\mathrm{P}-$exact sequence, thus for each $n\in\mathbf{Z}, \phi_n\times\phi_n(\mathrm{P}(X_n,S_i))\cup\Delta_{X_{n+1}}=\mathrm{R}(\phi_{n+1})$. Using [6], for each $n\in\mathbf{Z}$, we have $\mathrm{P}(X_n,S)=\bigcup_{i=1}^m\mathrm{P}(X_n,S_i)$, so:

$$\begin{aligned}\phi_n\times\phi_n(\mathrm{P}(X_n,S_i))\cup\Delta_{X_{n+1}} &=\phi_n\times\phi_n(\bigcup_{i=1}^m\mathrm{P}(X_n,S_i))\cup\Delta_{X_{n+1}}\\ &=\bigcup_{i=1}^m\phi_n\times\phi_n(\mathrm{P}(X_n,S_i))\cup\Delta_{X_{n+1}}\\ &=\mathrm{R}(\phi_{n+1})\end{aligned}$$

so $\cdots\overset{\phi_{n-1}}{\to}(X_n,S)\overset{\phi_n}{\to}(X_{n+1},S)\overset{\phi_{n+1}}{\to}\cdots$ is $\mathrm{P}-$exact (for the rest, use a similar method and note to the fact that by [6], for each $n\in\mathbf{Z}$, we have $\underline{\mathrm{P}}_{A_n}(X_n,S)=\bigcup_{i=1}^m\underline{\mathrm{P}}_{A_n}(X_n,S_i)$ and $\overline{\mathrm{P}}_{B_n}(X_n,S)=\bigcup_{i=1}^m\overline{\mathrm{P}}_{B_n}(X_n,S_i)$).

3. EXAMPLES

Most of the examples given in this section deal with fort spaces.
X is a fort space with the particular point t if X is considered with the topology $\{U\subseteq X \mid t\notin U\vee \mathrm{card}(X-U)<\aleph_0\}$ (where t is a member of X) [7].

Example 3. 1. For $i=1,2,3$, let X_i be an infinite fort space with the particular point t, and $(\{t\},S)\overset{\phi_0}{\to}(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}(X_3,S)\overset{\phi_3}{\to}(\{t\},S)$ be an exact (resp. $\mathrm{P}-$exact, $\underline{\mathrm{P}}_\Gamma-$exact, $\overline{\mathrm{P}}_\Gamma-$exact, $\overline{\overline{\mathrm{P}}}_\Gamma-$exact (where $\Gamma=(\{t\},A_1,A_2,A_3,\{t\})$ and for $i=1,2,3$, A_i is a nonempty subset of X_i)) sequence, then $\phi_i(t)=t$ (for $i=1,2$) (since ϕ_1 is one to one and $\phi_2:X_2\to X_3$ is onto).

Example 3. 2. In this example, for an arbitrary transformation semigroup like (X,S), we will assume that X is an infinite fort space with the particular point t, and that S is a group such that for each $x \in X$, we have $xS = \{x\}$ if and only if $x = t$.

In the sequence $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}(X_3,S)$, let:

a. $\phi_1 \times \phi_1(X_1 \times X_1) \cup \Delta_{X_2} = \mathrm{R}(\phi_2)$ (i.e., $(X_1,S)\overset{\phi_1}{\to}(X_2,S)\overset{\phi_2}{\to}(X_3,S)$ is exact).

b. $\phi_1(X_1) = \phi_2^{-1}(t)$.

c. $\phi_1(X_1) \subseteq \phi_2^{-1}(t)$.

d. $\phi_1 \times \phi_1(X_1 \times X_1) \cup \Delta_{X_2} \subseteq \mathrm{R}(\phi_2)$.

e. $\phi_1 \times \phi_1(X_1 \times X_1) \cup \Delta_{X_2} \supseteq \mathrm{R}(\phi_2)$.

We have "(a)$\Rightarrow$(b)", "(b)$\Rightarrow$(c)", and "(c)$\Rightarrow$(d)".

Thus (a), (b)$\wedge$(e), (c)$\wedge$(e), and (d)$\wedge$(e) are equivalent.

Proof:

- "(a)$\Rightarrow$(b)": Let $x \in X_1$, then $(\phi_1(x),t) = (\phi_1(x),\phi_1(t)) \in \mathrm{R}(\phi_2)$, so $\phi_2\phi_1(x) = \phi_2(t) = t$, and $\phi_1(x) \in \phi_2^{-1}(t)$. On the other hand, if $y \in \phi_2^{-1}(t)$, then $\phi_2(y) = t = \phi_2(t)$ and $(y,t) \in \mathrm{R}(\phi_2) = \phi_1 \times \phi_1(X_1 \times X_1) \cup \Delta_{X_2}$, so $y \in \phi_1(X_1)$ or $y = t = \phi_1(t) \in \phi_1(X_1)$, which completes this part of proof.
- "(c)$\Rightarrow$(d)": Let $x, y \in X_1$, by the assumption $\phi_2(\phi_1(x)) = t = \phi_2(\phi_1(y))$, so $(\phi_1(x),\phi_1(y)) \in \mathrm{R}(\phi_2)$.

Example 3. 3. Suppose $X = \left\{\frac{1}{n} \mid n \in \mathbf{N}\right\} \cup \{0\}$ with the induced topology of $\mathbf{R}$.

- Let $\xi : X \to X$ be such that:

$$x\xi = \begin{cases} \frac{1}{n+1} & x = \frac{1}{n} \wedge n \in \mathbf{N} - 2\mathbf{N} \\ 0 & \text{otherwise} \end{cases},$$

$S = \{\xi^n \mid n \in \mathbf{N} \cup \{0\}\}$, then $\cdots\overset{\xi}{\to}(X,S)\overset{\xi}{\to}(X,S)\overset{\xi}{\to}(X,S)\overset{\xi}{\to}\cdots$ is an exact sequence.

- For $p \geq 2$ and $m \in \{1,\dots,p-1\}$, let $\xi : X \to X$ be such that:

$$x\xi = \begin{cases} \frac{1}{n+1} & x = \frac{1}{n} \wedge n \in \mathbf{N} - p\mathbf{N} \\ 0 & \text{otherwise} \end{cases},$$

$S = \{\xi^n \mid n \in \mathbf{N} \cup \{0\}\}$, then $\cdots\overset{\xi^m}{\to}(X,S)\overset{\xi^m}{\to}(X,S)\overset{\xi^m}{\to}(X,S)\overset{\xi^m}{\to}\cdots$ is an exact sequence if and only if p is even and $m = \frac{p}{2}$, since $\xi^m(X) = X - \bigcup_{t=1}^{m}\left\{\frac{1}{n} \mid n \in p\mathbf{N} + t\right\}$, $\left(\xi^m\right)^{-1}(0) = X - \bigcup_{t=1}^{p-m}\left\{\frac{1}{n} \mid n \in p\mathbf{N} + t\right\}$ and $\mathrm{R}(\xi^m) = (\left(\xi^m\right)^{-1}(0) \times \left(\xi^m\right)^{-1}(0)) \cup \Delta_X$ (moreover for each $n \geq p$, $\xi^n = 0$).

Using a same method, if $\{m_i\}_{i\in\mathbf{Z}}$ is a sequence in $\{1,\dots,p-1\}$, then $\cdots \overset{\xi^{m_{i-1}}}{\to} (X,S)\overset{\xi^{m_i}}{\to}(X,S) \overset{\xi^{m_{i+1}}}{\to} \cdots$ is an exact sequence if and only if for each $i \in \mathbf{Z}$, we have $p = m_i + m_{i+1}$ (so for each $i \in \mathbf{Z}$, $m_{2i} = m_0$ and $m_{2i+1} = p - m_0$ and we can name the above exact sequence, exact sequence related to m_0).

Example 3. 4. Let $X = \left\{(x, \sin\frac{1}{x}) \mid x \in (0,1]\right\} \cup (\{0\} \times [-1,1])$ (with the induced topology of $\mathbf{R}^2$). Let S be the set of all homeomorphisms of X, then for each $s \in S$, there exists u in the set of all homeomorphisms of $[-1,1]$ and v in the set of all homeomorphisms of $(0,1]$, such that $(0,x)s = (0,u(x))$ $(x \in [-1,1])$ and $(x, \sin\frac{1}{x})s = (v(x), \sin\frac{1}{v(x)})$ $(x \in (0,1])$ (use the linear connected components of X). Moreover, in the transformation semigroup (X,S), we have:

b. For all $(x, y) \in X$, we have:

$$(x,y)S = \begin{cases} \{(0,-1),(0,1)\} & x=0, y=-1,1 \\ \{0\}\times(-1,1) & x=0, y\in(-1,1) \\ \{(z,\sin\frac{1}{z}) \mid z\in(0,1)\} & x\in(0,1) \\ \{(1,\sin 1)\} & x=1 \end{cases}.$$

So:

$$\overline{(x,y)S} = \begin{cases} \{(0,-1),(0,1)\} & x=0, y=-1,1 \\ \{0\}\times[-1,1] & x=0, y\in(-1,1) \\ X & x\in(0,1) \\ \{(1,\sin 1)\} & x=1 \end{cases}.$$

b. $\{(0,-1\},(0,1),(1,\sin 1)\}$ is the set of all almost periodic points of X.

c. Let $(Y,S)\overset{\eta}{\rightarrow}(X,S)\overset{\mu}{\rightarrow}(Z,S)$, be an arbitrary sequence named (3.4.1), and $\mu(1,\sin 1)=t$.

- $\eta(Y)=X$: (3.4.1) is exact if and only if $\mu(X)=\{t\}$.
- $\eta(Y)=\{(0,-1),(0,1),(1,\sin 1)\}$: (3.4.1) is exact if and only if $\mu(0,-1)=\mu(0,1)=t$ and $\mu|_{X-\{(0,-1),(0,1),(1,\sin 1)\}}$ is one to one.
- $\eta(Y)=\{(0,-1),(0,1)\}$: (3.4.1) is exact if and only if $\mu(0,-1)=\mu(0,1)$ and $\mu|_{X-\{(0,-1),(0,1)\}}$ is one to one.
- $\eta(Y)=\{(1,\sin 1)\}$: (3.4.1) is exact if and only if μ is one to one (the other cases has been omitted).

Note 3. 5. The consideration of Example 3.4, leads us to the following fact that in the sequence $(X_1,S)\overset{\phi_1}{\rightarrow}(X_2,S)\overset{\phi_2}{\rightarrow}(X_3,S)$, the following statements are equivalent:

- $(X_1,S)\underset{\phi_1}{\rightarrow}(X_2,S)\underset{\phi_2}{\rightarrow}(X_3,S)$ is an exact sequence.
- For all $a\in X_1$, $\phi_2\phi_1(X_1)=\{\phi_2\phi_1(a)\}$ and $\phi_2|_{X_2-\phi_1(X_1)}$ is one to one.
- There exists $a\in X_2$, such that $\phi_2\phi_1(X_1)=\{\phi_2(a)\}$ and $\phi_2|_{X_2-\phi_1(X_1)}$ is one to one.
- $\phi_1\times\phi_1(X_1\times X_1)\cup\Delta_{X_2}\subseteq \mathrm{R}(\phi_2)$ and $\phi_2|_{X_2-\phi_1(X_1)}$ is one to one.

So if for $i=1,2,3$, X_i is an infinite fort space with the particular point b such that $xS=\{x\}$ if and only if $x=b$, then $(X_1,S)\overset{\phi_1}{\rightarrow}(X_2,S)\overset{\phi_2}{\rightarrow}(X_3,S)$ is exact if and only if $\phi_1(X_1)=\phi_2^{-1}(b)$ and $\phi_2|_{X_2-\phi_1(X_1)}$ is one to one (see also Example 3.2).

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