ON THE INTEGRAL INVARIANTS OF KINEMATICALLY GENERATED RULED SURFACES[*]

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Abstract – In this paper, the dual area vector of a closed dual spherical curve is kinematically generated and the dual Steineer vector of a motion are extensively studied by the methods of differential geometry. Jacobi's Theorems, known for real curves, are investigated for closed dual curves. The closed trajectory surfaces generated by an oriented line are fixed in a moving rigid body in IR^3, in which the closed dual curves from E. Study's transference principle is studied. The integral invariants of these closed ruled surfaces are calculated by means of the area vector. Moreover, some theorems, results and examples are given.

Keywords – Mechanism, kinematics, area vectors, ruled surfaces, invariants and motions

1. INTRODUCTION

The kinematic geometry of the infinitesimal positions of a rigid body in spatial motions is not only important, but interesting as well. In a spatial motion, the trajectory of the oriented lines and points embedded in a moving rigid body are generally ruled surfaces and curves, respectively. Thus the spatial geometry of ruled surfaces and curves is important in the study of rational design problems in spatial mechanisms. As an example, some characteristic invariants of ruled surfaces were applied to a mechanism theory by A. T. Yang et al., [1]. Also, using the geometry of curves and developable ruled surfaces, some spatial design problems were investigated by H. Pottmann et al., [2], J. A. Schaaf et al., [3] and Wang et al., [4].

Rather unexpectedly, dual numbers have been applied to study the motion of a line in space; in IR^3, they even seem to be the most appropriate apparatus for this purpose. It was first done by E.Study [5], and since his time dual numbers have had an established place in kinematics as a tool to solve problems dealing with lines in space. Vast literature on the subject can be found in [6-8].

The application of dual numbers to the lines of the Euclidean 3-space is carried out by the principle of transference which was formulated by E.Study. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual number extension, i.e. replacing all ordinary quantities by the corresponding dual number quantities [9].

Jacobi [10] showed that the indicatrix of a tangent vector of any real closed spherical curve divides the surface area of a unit sphere into two equal parts. In the same paper, he also showed that the indicatrix of the principal normal vector of any closed space curve also divides the surface area of

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the unit sphere into two equal parts. Then, Fenchel [11] and Avaqumovic [12], using Jacobi Theorems, showed the unit spherical closed curve is the principal normal indicatrix of a closed space curve if the closed spherical curve divides the surface area of the unit sphere into two equal parts. Also, Yapar [13] showed that the spherical indicatrix of each unit vector lying in the osculating plane of a closed spherical curve which is fixed to the curve divides the surface area of the unit sphere into two equal parts.

The angle and length of the pitch, which are the integral invariants of a closed ruled surface, are very important in the study of the geometry of lines from the perspectives of instantaneous space kinematics and mechanisms. In recent years several authors have used these invariants in their investigations concerning the generalization of some of the theorems of plane kinematics to spatial kinematics [7-9, 14-19].

In this study, the integral invariants of closed ruled surfaces kinematically generated are calculated and Jacobi's Theorems are stated by means of the area vector and some relations and theorems are given.

2. PRELIMINARIES

A dual number has the form $a + \varepsilon\, a^*$, where a and a^* are real numbers and ε is the dual unit with the property $\varepsilon^2 = 0$. The set of all dual numbers is a commutative ring over the real numbers field and denoted by ID, [8]. The set

$$ID^3 = \{A = (A_1, A_2, A_3) : A_i \in ID; 1 \le i \le 3\}$$

is a module over the ring ID which is called an ID-module or dual space. We call elements of ID^3 dual vectors. A dual vector **A** may be written as **A**=**a**+ε **a***, **a***=**p**$\wedge$**a**, where **a**, **p** and **a*** are real vectors in IR^3. The inner product of two dual vectors **A** and **B** is defined as

$$\langle A, B \rangle = \langle a, b \rangle + \varepsilon \big(\langle a, b^* \rangle + \langle a^*, b \rangle \big),$$

Where

$$\langle a, b \rangle = \cos\varphi \ \text{and} \ \langle a, b^* \rangle + \langle a^*, b \rangle = -\varphi^* \sin\varphi, \ 0 \le \varphi \le \pi.$$

The cross-product of two dual vectors, **A** and **B,** is given by

$$A \wedge B = a \wedge b + \big(a \wedge b^* + a^* \wedge b \big).$$

Let Φ be the dual angle between the unit dual vectors **A** and **B**, then

$$\langle A, B \rangle = \cos\Phi = \cos\varphi - \varepsilon\varphi^* \sin\varphi$$

where $\Phi = \varphi + \varepsilon\varphi^*$, $0 \le \varphi \le \pi$, $\varphi^* \in IR$, is a dual number. Here, the real numbers φ and φ^* are the angle and the minimal distance between the two oriented lines **A** and **B**, respectively. The geometric place of the points satisfying the equality ||**A**||=(1,0), when **A** $\ne$ (**0**,a*) is called a unit dual sphere in ID-module.

E. Study established a theorem which states "there is a one to one mapping between the dual points of a unit dual sphere and the oriented lines in IR^3". According to E.Study's Theorem; a unit dual vector **A**=**a**+ε **a*** corresponds to only one oriented line in IR^3, where the real part **a** shows the

direction of this line and the dual part $\mathbf{a}^*$ shows the vectorial moment of the unit vector $\mathbf{a}$ with respect to the origin.

Let a moving orthonormal trihedron $\{e_1, e_2, e_3\}$ be made a closed spatial motion along a closed curve $c(x) = x(t)$ in IR^3. During a closed spatial motion, an oriented line fixed in IR^3 generates a closed trajectory surface.

The parametric equation of a closed trajectory surface formed with e_1-axis can be expressed as follows:

$$\Psi(t,u) = x(t) + ue_1(t), \quad \Psi(t,u) = \Psi(t + 2\pi, u) \tag{1}$$

for all t, u $\in IR$.
If we take the moving orthonormal trihedron as

$$\left\{ e_1(t), e_2(t) = \frac{\dot{e}_1(t)}{\|\dot{e}_1(t)\|}, e_3(t) = e_1(t) \wedge e_2(t) \right\},$$

then the axes intersect at the striction point of e_1-generator of the closed ruled surface given by equation (1). In this case, $x(t)$ is the striction point; e_2 and e_3 are called central normal and central tangent, respectively.

The structural equations of closed spatial motion described above are

$$de_i = \sum_{j=1}^{3} w_i^j e_i, \quad w_i^j(t) = -w_j^i(t), \ 1 \le i, j \le 3, \ w_1^3(t) = 0, \tag{2}$$

where the differential forms w_1^2 and w_2^3 are the natural curvature and the natural torsion of e_1-closed trajectory surface, respectively. Equation (2) can be written in the following form

$$de_i(t) = w \wedge e_i, \quad (i = 1,2,3), \tag{3}$$

where $w = w_2^3 e_1 + w_1^2 e_3$ is the Darboux vector of the motion. If $w \ne 0$, then the Pole vector and the Steiner vector are given by:

$$P = \frac{w}{\|w\|}, \quad s = \oint w \tag{4}$$

respectively, where $\|w\|$ is the instantaneous angular velocity of the motion and integration is taken along the closed curve c(x) on fixed space in R'.

The length of the pitch (Öffnungsctracke) of an e_1-closed trajectory surface is defined by:

$$\ell_{e_1} := \oint du = -\oint \langle dx, e_1 \rangle. \tag{5}$$

The orthogonal trajectory of an e_1-closed trajectory surface starting from point P_O on the e_1-generator intersects the same generator at point P_1 which is generally different from P_O. Thus, $\ell_{e_1} = |P_O P_1|$.

Let us consider a unit vector

$$m = \cos\theta \, e_2 + \sin\theta \, e_3$$

on the (e_2, e_3)-plane, such that an m-oriented line generates a developable ruled surface (torse) along the orthogonal trajectory of an e_1-closed trajectory surface during the closed motion. Then the total

change of θ is called the angle of pitch (Öffnungswinkel) of the e_1-closed trajectory surface and given by one of the following forms

$$\lambda_{e_1} := \oint d\theta = -\oint \langle d\,e_2, e_3 \rangle = -\langle e_1, s \rangle. \tag{6}$$

The length of the pitch and the angle of the pitch are well-known integral invariants of a closed trajectory surface [7-19].

There is a one-to-one correspondence between spherical curves and space curves. Hence, the structural equation (2) is also valid for the spherical curves.

Then, the spherical area bounded by a closed spherical curve c(x) is given by

$$f_x = 2\pi(1-\nu) - \langle s, x \rangle, \tag{7}$$

where the vector **x** is the position vector of the point x and ν is the rotation number around point x of the Pole curve c(P), [20-22].

The area vector of a closed space curve c(x) in R' is defined by

$$v_x := \oint x \wedge d\,x, \tag{8}$$

where the integration is taken along the closed curve c(x). The projection area of a closed space curve c(x) in the direction of a unit vector **n**, which is normal to the projection plane, is given as follows [20]:

$$f_{x^n} = \frac{1}{2}\langle v_x, n \rangle. \tag{9}$$

3. THE INTEGRAL INVARIANTS AND THE AREA VECTORS

Let K be a moving dual unit sphere generated by a dual orthonormal trihedron

$$\left\{ E_1 = E_1(t), \ E_2(t) = \frac{\dot{E}_1(t)}{\|\dot{E}_1(t)\|}, \ E_3(t) = E_1(t) \wedge E_2(t) \right\}, \ E_i(t) = (e_i, e_i{}^*), \ i = 1,2,3, \tag{10}$$

and K' be a fixed dual unit sphere with the same center in ID^3. Then the differential equations of the dual spherical closed motion, denoted by K/K' are:

$$dE_i = \sum_{j=1}^{3} \Omega_i^j E_j, \ \Omega_i^j(t) = w_i^j(t) + \varepsilon w_i^{*j}(t), \ \Omega_i^j(t) = -\Omega_j^i(t), \ (i = 1,2,3), \ \Omega_1^3(t) = 0, \tag{11}$$

where the differential forms $\Omega_1^2(t) = w_1^2(t) + \varepsilon w_1^{*2}(t)$ and $\Omega_2^3(t) = w_2^3(t) + \varepsilon w_2^{*3}(t)$ are the dual natural curvature and torsion, respectively. The dual Steiner vector of the closed motion is defined by

$$S = \oint W, \ P = \frac{W}{\|W\|}, \ W = w + \varepsilon w^*, \tag{12}$$

where $W = \Omega_2^3 E_1 + \Omega_1^2 E_3$ and **P** are instantaneous Darboux vector and the dual pole vector of the motion, respectively. As known from the E. Study's transference principle, the dual equation (11)

correspond to the real equation (2) of a closed spatial motion in IR^3. In this sense, the differentiable dual closed curve, $E_1 = E_1(t)$, $t \in IR$, is considered as a closed trajectory surface in IR^3.

Let us consider a differentiable unit dual spherical closed curve

$$c(X) = X(t),\ X(t + 2\pi) = X(t),\ \|X\| = 1,\ t \in IR. \tag{13}$$

We know from E.Study's transference principle that the dual curve defined by (13), which shows a unit dual spherical closed curve, corresponds to an x-closed trajectory surface generated by an x-oriented line fixed in a moving rigid body in IR^3. Thus the curve (13) is called the unit dual spherical image (or indicatrix) of an x-closed trajectory surface. The dual angle of the pitch, Λ_X, of the closed ruled surface $\mathbf{X} = \mathbf{X}(t)$ is equal to the dual projection of the generator on to the dual Steiner vector of the motion K / K', that is [16]:

$$\Lambda_X = -\langle X, S \rangle = 2\pi - A_X = \lambda_x - \varepsilon\ell_x, \tag{14}$$

where $A_X = a_x + \varepsilon a_x{}^*$, the dual spherical surface area of the dual spherical image of X-closed trajectory surface.

Let c(X) be the dual spherical indicatrix on K' of an arbitrary fixed dual point X on K. The dual spherical area F_X surrounded by the dual closed curve c(X) is

$$F_X = 2\pi(1 - \nu) - \langle X, S \rangle. \tag{15}$$

Here ν is the rotation number of the rotation of the centrode c(P) at the point X, and $\mathbf{X}$ denotes the dual position vector of an arbitrary point of the dual closed curve c(X) on K' [16].

The dual area vector of an X(t)-closed spherical curve can be defined by

$$V_X := \oint X \wedge d\,X \tag{16}$$

as an analogue to the definition in [22], where

$$dX = W \wedge X$$

is the differential velocity of an X-dual point fixed of the moving sphere K.

From equations (13) and (16), the dual area vector may be developed as

$$V_X = S - \langle X, S \rangle X \tag{17}$$

or

$$V_X = S + \Lambda_X X \tag{18}$$

This statement shows that there is a relationship between the dual angle of the pitch of an X-closed trajectory surface and its dual area vector. On the other hand, if a scalar product is made with the vector **S** on both sides of equation (17), then we may write

$$\|S\|^2 = \Lambda_X^2 - \Lambda_{V_X} \|V_X\|, \tag{19}$$

where Λ_{V_X} is the dual angle of the pitch of the V_X-trajectory surface generated by the area vector of c(X)-closed spherical indicatrix of X-closed trajectory surface.

It follows from (18) that

$$\left\|V_X\right\| = \sqrt{\left\|S\right\|^2 - \Lambda_X^2} \qquad (20)$$

Thus, with the aid of (19) and (20) the dual angle of the pitch of the V_X-unit area vector trajectory surface is obtained as

$$\Lambda_{V_X} = -\sqrt{\left\|S\right\|^2 - \Lambda_X^2} \qquad (21)$$

So, we may give the following theorem.

Theorem 3. 1. There is the relationship

$$\left\|S\right\|^2 = \Lambda_{V_X}^2 + \Lambda_X^2 \qquad (22)$$

between the dual angle of pitches of V_X and X-closed trajectory surfaces.

By separating equation (22) into real and dual parts, we have

$$\left\|s\right\|^2 = \lambda_{v_x}^2 + \lambda_x^2$$

and

$$\left\langle s, s* \right\rangle = -\lambda_{v_x}\ell_{v_x} - \lambda_x\ell_x.$$

In the case of the axes of the unit area vector $\mathbf{V}_X$ and the Steiner vector $\mathbf{S}$ are perpendicular to each other, we get $\lambda_{v_x} = 0$ and $\ell_{v_x} = 0$. Thus the following result may be given.

Result 3. 2. During the closed spherical motion, the axes of $\mathbf{V}_X \neq \mathbf{0}$ dual area vector and the dual Steiner vector $\mathbf{S}$ are perpendicular to each other if and only if $\Lambda_{V_X} = 0$.
Also, from (14) we have $\Lambda_{V_X} = 0$ if and only if $a_{v_x} = 2\pi$, and $a*_{v_x} = 0$. Thus, the following result can be given.

Result 3. 3. The dual spherical indicatrix of the unit dual area vector $\mathbf{V}_X$ divides the measure of the spherical surface area into two equal parts if and only if $\Lambda_{V_X} = 0$.

A ruled surface $\Psi(t,u) = x(t) + uv_x(t)$ is given by

$$\mathbf{V}_X(t) = \mathbf{v}_x(t) + \varepsilon \, \mathbf{v}*_x(t),$$

where $\mathbf{v}_x$ is the unit area vector and $\mathbf{v}*_x = \mathbf{x} \wedge \mathbf{v}_x$ is the vectorial area vector of $\mathbf{v}_x$ with respect to the origin point. Since the spherical image of $\mathbf{v}_x(t)$ is the unit area vector, the dual area vector $\mathbf{V}_X(t)$ also has unit magnitude. Thus, the ruled surface can be represented by a dual curve on the surface of a unit dual sphere. The dual arc-length of the ruled surface $\mathbf{V}_X(t)$ is given by

$$\left\|\dot{V}_X\right\| = \left\|\dot{v}_x\right\|(1 + \varepsilon d),$$

where $d = \dfrac{\left\langle \dot{v}_x, \dot{v}_x^* \right\rangle}{\left\|\dot{v}_x\right\|^2}$ is the distribution parameter (drall) of this ruled surface.

According to E. Study's transference principle, the following theorem can be given.

Theorem 3. 4. In the lines space, the trajectory surface of the closed spherical indicatrix generated by the unit dual area vector $\mathbf{V}_X$, formed along a unit dual closed curve in a closed spherical motion, is a developable ruled surface.

On the other hand, the oriented dual projection area of a planar region which occured by taking orthogonal projection onto a plane in the direction of a fixed unit vector $\mathbf{N}$ of the curve c(X) is given by

$$2F_{X^n} = \langle \mathrm{N}, \mathrm{V_X} \rangle.$$

The position vector of the point X fixed in the moving sphere K, in terms of the dual orthonormal vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ can be written as

$$\mathrm{X(t)} = \mathrm{X_1 E_1}(t) + \mathrm{X_2 E_2}(t) + \mathrm{X_3 E_3}(t), \tag{23}$$

where X_1, X_2 and X_3 are constant coordinates of X.

Let $c(E_1)$, $c(E_2)$ and $c(E_3)$ be the closed dual spherical indicatrix of the dual orthonormal vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$, respectively. Thus, we can give the following result.

Result 3. 5. The dual area vector of the closed dual curve c(X) drawn on a fixed unit sphere K', by a fixed point X of moving dual unit sphere K, during the closed spherical motion is

$$\mathrm{V_X} = \sum_{i=1}^{3} X_i^2 \mathrm{V_{E_i}} + 2\sum_{\substack{i,k=1 \\ i<k}} X_i X_k \mathrm{V_{E_{i,k}}} \,,$$

where

$$\mathrm{V_{E_i}} = \oint E_i(t) \wedge \dot{E}_i(t)\,dt \text{ and } \mathrm{V_{E_{i,k}}} = \frac{1}{2}\oint \big(E_i(t) \wedge \dot{E}_k(t) + E_k(t) \wedge \dot{E}_i(t)\big)dt \,.$$

Since the motion is closed, we get

$$\oint \mathrm{E}_i(t)\,dt = 0\,.$$

Thus we have

Result 3. 6. The dual Steiner vector $\mathbf{S}$ of the motion in terms of the dual area vectors $\mathrm{V_{E_1}}$, $\mathrm{V_{E_2}}$ and $\mathrm{V_{E_3}}$ is

$$\mathrm{S} = \frac{1}{2}\sum_{i=1}^{3} \mathrm{V_{E_i}}\,.$$

The relation between the orthogonal projection area and the parallel projection area can be given by the following proposition.

Proposition 3. 7. Let F_{X^n} be the oriented dual projection area of the planar region formed by taking the orthogonal projection of closed dual spherical curve c(X) onto the plane, and F_{X^p} be the oriented dual projection area of the planar region formed by parallel projecting of the closed dual spherical curve c(X) onto the same planar region in the direction of a unit dual vector $\mathbf{P}$. Then

$$F_{X^n} = \cos\Theta F_{X^p} \, ,$$

where Θ is the dual angle between two image planes [19].

Thus we can give the following theorem.

Theorem 3. 8. The oriented dual projection area, F_{X^p}, of the planar region formed by parallel projection of the dual closed curve c(X) drawn by a fixed point X, of the moving sphere K, is

$$F_{X^p} = \sum_{i=1}^{3} X_i^2 F_{E_i^p} + 2\sum_{\substack{i,k=1 \\ i<k}}^{3} X_i X_k F_{E_{i,k}^p} \, .$$

Example 3. 9. Let us consider the dual point $X = X(t) = \left(\cos\Phi(t), \sin\Phi(t), 0\right)$, where $\Phi(t) = \theta(t) + \varepsilon\theta^*(t)$. Now, Let's calculate the oriented dual projection area of the dual closed spherical curve c(X) formed during the closed spherical motion. Since

$$X = X(t) = \left(\cos\Phi(t),\, \sin\Phi(t),\, 0\right) = \left(\cos\theta(t) - \varepsilon\theta^*(t)\sin\theta(t), \sin\theta(t) + \varepsilon\theta^*(t)\cos\theta(t), 0\right),$$

we have

$$dX = \left(-\sin\theta(t) - \varepsilon(\frac{d\theta^*}{dt}\sin\theta(t) + \theta^*(t)\cos\theta(t)), \cos\theta(t) + \varepsilon(\frac{d\theta^*}{dt}\cos\theta(t) - \theta^*(t)\sin\theta(t)), 0 + \varepsilon 0^* \right)$$

and

$$X \wedge dX = \left(0,\, 0, 1 + \varepsilon\frac{d\theta^*}{dt} \right).$$

Thus the dual area vector of closed spherical curve c(X) is

$$V_X = \int_0^{2\pi} X(t) \wedge \dot{X}(t)\, dt = (0,0,2\pi) \, .$$

Using equations (15) and (16), the oriented dual projection area of the closed dual spherical curve c(X), we get

$$2F_{X^n} = \left\langle N, V_X \right\rangle = \left\langle (0,0,1),(0,0,2\pi) \right\rangle = 2\pi, \ \ N = n = e_3 \, .$$

Thus the oriented dual projection area is obtained as

$$F_{X^n} = \pi \, .$$

From the Blaschke area formula and equation (7), the following theorem can be given.

Theorem 3. 10. Let c(X) be the dual spherical indicatrix of a fixed point X, and also c(E₁), c(E₂) and c(E₃) be the closed dual spherical indicatrixies of the dual orthonormal vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ during the closed spherical motion, respectively. Then the dual spherical area bounded by the closed spherical curve c(X) in terms of the dual spherical areas F_{E_1}, F_{E_2} and F_{E_3} bounded by the closed spherical indicatrixies c(E₁), c(E₂) and c(E₃) is

$$F_X = 2\pi(1-\nu)\left[1 - \sum_{i=1}^{3} X_i\right] + \sum_{i=1}^{3} X_i F_{E_i} \ ,$$

where ν is the rotation number of motion.

4. THE JACOBI THEOREMS

Let us have a closed dual curve c(X) of class C^2 on a unit dual sphere K' in ID^3. At the initial time, assume that the unit dual sphere K corresponding with K' to be $K = K'$, where K' is a fixed sphere and K is a moving sphere with respect to K'. The curve c(X) describes a closed dual spherical motion. Let us consider the dual moving frame $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ be firmly linked to any point X(t) of the curve c(X). Here, $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are tangent, principal normal and binormal unit dual vectors, respectively. While drawing the closed dual spherical curve c(X) during the dual closed spherical motion, the end points of vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ on K also draw closed spherical curves c(E_1), c(E_2) and c(E_3) on K', respectively. Now let us carry these vectors to the origin point of the unit dual sphere K. Thus, from equations (10), (11), and (16) we have the following theorem.

Theorem 4. 1. Let c(E_1), c(E_2) and c(E_3) be the spherical indicatrixies of the unit dual vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ during the closed dual spherical motion, respectively. The dual area vectors of these closed spherical indicatrixies are

$$\begin{aligned}
V_{E_1} &= S - \langle S, E_1 \rangle E_1, \\
V_{E_2} &= S, \\
V_{E_3} &= \langle S, E_1 \rangle E_1,
\end{aligned} \qquad (24)$$

where $S = \oint W$ is the dual Steiner vector of motion K/K'.

If the expression (24) is separated into its real and dual parts, we have the following equalities:

$$\begin{aligned}
v_{e_1} &= s + e_1 \lambda_{e_1}, & v_{e_1}^* &= s^* + e_1^* \lambda_{e_1} - e_1 \ell_{e_1} \\
v_{e_2} &= s &, & v_{e_2}^* &= s^* \\
v_{e_3} &= -e_1 \lambda_{e_1} &, & v_{e_3}^* &= -e_1^* \lambda_{e_1} + e_1 \ell_{e_1}
\end{aligned} \qquad (25)$$

where v_{e_1}, v_{e_2}, v_{e_3} and $v_{e_1}^*$, $v_{e_2}^*$, $v_{e_3}^*$ are real and dual area vectors, respectively. From theorem (4.1) we can give the following results.

Result 4. 2. The dual area vector V_{E_2} is equal to the sum of the dual area vectors V_{E_1} and V_{E_3}, i.e. $V_{E_2} = V_{E_1} + V_{E_3}$.

Result 4. 3. The unit dual vector $\mathbf{E}_2$ is perpendicular to the dual area vectors V_{E_1} and V_{E_3}.
As a special case of equation (21) we have

$$\Lambda_{V_{E_1}} = -\sqrt{\|S\|^2 - \Lambda_{E_1}^2} \ .$$

Thus, we can give the following theorem.

Theorem 4. 4. There is the relationship

$$\Lambda_{V_{E_1}} = -\sqrt{\|\mathbf{S}\|^2 - \Lambda_{E_1}^2}$$

between the dual angle of the pitches of V_{E_1} and E₁-closed trajectory surfaces, where **S** is the dual Steiner vector of the motion.

Also, from equations (14) and (24) we have

$$\Lambda_{V_{E_2}} = -\|\mathbf{S}\| \tag{26}$$

and

$$\Lambda_{V_{E_3}} = \Lambda_{E_1} \tag{27}$$

on the other hand, since

$$\mathbf{V}_{E_2} = \mathbf{V}_{E_1} + \mathbf{V}_{E_3},$$

from equations (10), (24), (26) and (27) we have

$$\frac{\Lambda_{V_{E_1}}^2}{\|\mathbf{S}\|\Lambda_{V_{E_2}}} - \frac{\Lambda_{V_{E_3}}^2}{\|\mathbf{S}\|\Lambda_{V_{E_2}}} = 1 \tag{28}$$

and

$$\frac{\Lambda_{V_{E_3}}^2}{\|\mathbf{S}\|^2} - \frac{\Lambda_{V_{E_1}}^2}{\|\mathbf{S}\|^2} = 1. \tag{29}$$

Thus we can give the following theorem.

Theorem 4. 5. There are the relations (28) and (29) between the dual angle of pitches of V_{E_1}, V_{E_2} and V_{E_3} -closed trajectory surfaces.
On the other hand, the dual angles of pitch of ruled surfaces corresponding to the closed dual spherical curves c(E₁), c(E₂) and c(E₃), respectively, are:

$$\begin{aligned}
\Lambda_{E_1} &= -\langle \mathbf{E}_1, \mathbf{V}_{E_2} \rangle = \lambda_{e_1} - \varepsilon \ell_{e_1}, \\
\Lambda_{E_2} &= -\langle \mathbf{E}_2, \mathbf{V}_{E_2} \rangle = 0, \\
\Lambda_{E_3} &= -\langle \mathbf{E}_3, \mathbf{V}_{E_2} \rangle = \lambda_{e_3} - \varepsilon \ell_{e_3}
\end{aligned} \tag{30}$$

Now, let us consider the spherical indicatrix c(E₁) of the unit dual vector **E**₁ formed during the closed motion. If the area of the region surrounded by the curve c(E₁) denoted by F_{E_1}, then from equations (14) and (25),

$$F_{E_1} = 2\pi(1 - \nu) + \lambda_{e_1} - \varepsilon \ell_{e_1}. \tag{31}$$

Since the above area should be $F_{E_1} = 2\pi$ according to the Jacobi Theorem, we obtain

$$-2\pi v + \lambda_{e_1} - \varepsilon \ell_{e_1} = 0, \tag{32}$$

where 0 is a dual number.

From equation (31), according to the equality of two dual numbers we have

$$\lambda_{e_1} = 2\pi v, \ \ell_{e_1} = 0. \tag{33}$$

Thus we can give the following theorem.

Theorem 4. 6. Let c(X) be a closed dual spherical curve on the unit dual sphere. Let A be the ruled surface corresponding to the spherical indicatrix of the tangent vector $\mathbf{E}_1$ of the closed dual curve c(X). Let the real angle of the pitch and length of the pitch of the closed ruled surface A be λ_{e_1} and ℓ_{e_1}, respectively, then we have

$$\lambda_{e_1} = -\langle e_1, v_{e_2} \rangle = 2\pi v, \ \ell_1 = \langle e_1, v_{e_2}^* \rangle + \langle e_2^*, v_{e_2} \rangle = 0.$$

From theorem (4.6), since $\ell_1 = \langle e_1, v_{e_2}^* \rangle + \langle e_2^*, v_{e_2} \rangle = 0$, we have the following result.

Result 4. 7. The oriented lines $E_1 = (e_1; e_1^*)$ and $V = V_{E_2} \|V_{E_2}\|^{-1} = (v_{e_2}; v_{e_2}^*)$ are intersected.
From equation (14) we obtain the integral invariants of the closed ruled surface corresponding to the spherical indicatrix c(E$_2$) of the unit dual vector $\mathbf{E}_2$, in the lines space, as the following

$$\lambda_{e_2} = 0, \ \ell_{e_2} = 0. \tag{34}$$

If the area of the region surrounded by the curve c(E$_2$), denoted by F_{E_2}, then from equations (14) and (25) we obtain

$$F_{E_2} = 2\pi(1 - v) \tag{35}$$

Since the above area should be $F_{E_2} = 2\pi$ according to the Jacobi Theorem, we obtain $v = 0$. Thus we can give the following theorem.

Theorem 4. 8. In the Euclidean 3-space IR^3, the closed ruled surface corresponding to the spherical indicatrix of the principal normal vector $\mathbf{E}_2$ of the closed dual curve c(X) is a cone, that is:

$$\lambda_{e_2} = 0, \ \ell_{e_2} = 0.$$

Let C be the closed ruled surface corresponding to the spherical indicatrix of the binormal vector $\mathbf{E}_3$ of the closed dual curve c(X). The area of the spherical region surrounded by c(E$_3$) is

$$F_{E_3} = 2\pi(1 - v) + \lambda_{e_3} - \varepsilon \ell_{e_3}. \tag{36}$$

In addition, the length of the pitch is $\ell_{e_3} = \langle e_3, v_{e_2}^* \rangle + \langle e_3^*, v_{e_2} \rangle$. Thus we can give the following theorem.

Theorem 4. 9. In the lines space, the spherical indicatrix of a binormal vector of any closed dual spherical curve c(X), on the unit dual sphere, corresponds to a closed ruled surface. The length of the pitch of this ruled surface only depends on the curve c(X), and

$$\ell_{e_3} = \left\langle \mathrm{e}_3, \mathrm{v}_{e_2}^* \right\rangle + \left\langle \mathrm{e}_3^*, \mathrm{v}_{e_2} \right\rangle .$$

Now, let us consider all the unit dual vectors firmly attached to the curve which lies in the osculating plane of the closed spherical curve c(X). Let **U** be one of these vectors and $\Theta = \theta + \varepsilon\theta^*$ be the angle between the unit dual vector **U** and the unit dual tangent vector $\mathbf{E}_1$. Thus the vector **U** can be written as follows:

$$\mathrm{U} = \cos\Theta \mathrm{E}_1 + \sin\Theta \mathrm{E}_2 . \tag{37}$$

If the unit dual vector **U** is separated into its real and dual parts, then we obtain

$$\mathrm{u} = \cos\theta\,\mathrm{e}_1 + \sin\theta\,\mathrm{e}_2, \;\; \mathrm{u}^* = \cos\theta\,\mathrm{e}_1^* + \sin\theta\,\mathrm{e}_2^* - \theta^*\sin\theta\,\mathrm{e}_1 + \theta^*\cos\theta\,\mathrm{e}_2 . \tag{38}$$

In the lines space, let U be the ruled surface corresponding to the unit dual spherical indicatrix of the unit dual vector **U**. The dual angle of pitch of this ruled surface, from equations (33), (37) and (38), is obtained as follows:

$$\Lambda_U = -\left\langle \mathrm{U},\mathrm{S} \right\rangle = \lambda_{e_1}\cos\theta - \varepsilon\lambda_{e_1}\theta^*\sin\theta = \lambda_{e_1}\cos\Theta .$$

Thus, the real angle of the pitch and the length of the pitch of the ruled surface U corresponds to closed spherical curve c(U) drawn by the unit dual vector **U** during the motion, in the lines space, are

$$\lambda_u = \lambda_{e_1}\cos\theta, \;\; \ell_u = \lambda_{e_1}\theta^*\sin\theta . \tag{39}$$

On the other hand, from equations (14), (15) and (37), the area of the spherical region surrounded by the closed spherical curve c(U) is obtained as

$$F_U = 2\pi(1-\nu) + \lambda_{e_1}\cos\Theta .$$

Since this area should be 2π, [13], we have

$$\lambda_{e_1}\cos\theta = 2\pi\nu, \;\; \lambda_{e_1}\theta^*\sin\theta = 0 .$$

By taking $0 < \theta < \dfrac{\pi}{2}$ and $\theta^* \neq 0$ we get

$$\lambda_{e_1} = 0 .$$

So, we can give the following theorems:

Theorem 4. 10. The ruled surface corresponding to the spherical indicatrix of the tangent vector $\mathbf{E}_1$ of closed dual curve c(X) is a cone, that is:

$$\lambda_{e_1} = 0, \;\; \ell_{e_1} = 0 .$$

Thus we have

$$\begin{aligned}
\Lambda_{E_1} &= -\left\langle \mathrm{E}_1, \mathrm{V}_{E_2} \right\rangle = 0, \\
\Lambda_{E_2} &= -\left\langle \mathrm{E}_2, \mathrm{V}_{E_2} \right\rangle = 0, \\
\Lambda_{E_3} &= -\left\langle \mathrm{E}_3, \mathrm{V}_{E_2} \right\rangle = \lambda_{e_3} - \varepsilon\ell_{e_3}
\end{aligned} \tag{40}$$

Theorem 4. 11. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the tangent and the principal normal vectors of the closed curve $c(X)$, respectively. The unit dual vector $\mathbf{E}_1$ is perpendicular to the area vector V_{E_2}.
Substituting equalities in the theorem (4.10) into equation (39), we can give the following theorem:

Theorem 4. 12. Let $\mathbf{U}$ be the unit dual vector which lies in the osculating plane of closed unit dual curve $c(X)$. Then the ruled surface corresponding to the spherical dual curve $c(U)$ is a cone, that is

$$\lambda_u = 0, \quad \ell_u = 0.$$

Example 4. 13. Let us consider a unit closed spherical curve $c(x)$ is given by

$$c(x) = x(t) = \left(\cos\theta\cos(t\sec\theta), \cos\theta\sin(t\sec\theta), \sin\theta\right), \quad \theta \neq (2k+1)\frac{\pi}{2}, \, k \in IR.$$

The differential equations of this curve in matrix form can be written as

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & \sec\theta & 0 \\ -\sec\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}.$$

Thus, we get

$$s = (0, 0, t\sec\theta).$$

Using equation (8), the unit area vector of the unit closed curve $c(x)$ is obtained as

$$v_x = -\left(\sin\theta\cos(t\sec\theta), \sin\theta\sin(t\sec\theta), -\cos\theta\right)$$

Let the direction $\mathbf{v}_x$ of a line L be given by

$$v_x = -\left(\sin\theta\cos(t\sec\theta), \sin\theta\sin(t\sec\theta), -\cos\theta\right).$$

Then we have the parametric equation of the ruled surface generated by L:

$$\begin{aligned} \Psi(t,u) &= x(t) + u\,v_x(t) \\ &= \left((\cos\theta - u\sin\theta)\cos(t\sec\theta), (\cos\theta - u\sin\theta)\sin(t\sec\theta), \sin\theta + u\cos\theta\right), \quad u \in IR \end{aligned}$$

The unit dual area vector function representing $\Psi(t.u)$ is given by

$$\begin{aligned} \widetilde{X} &= V_X(t) = v_x + \varepsilon(x \wedge v_x) = v_x + \varepsilon v_x^* \\ &= -\left(\sin\theta\cos(t\sec\theta), \sin\theta\sin(t\sec\theta), -\cos\theta\right) + \varepsilon\left(\sin(t\sec\theta), -\cos(t\sec\theta), 0\right), \end{aligned}$$

where $v_x^* = x \wedge v_x$ is the area vectorial moment of the unit area vector v_x.

The differential equations of the unit dual closed spherical curve $c(\mathbf{V}_X) = \mathbf{V}_X(t)$ in matrix form are obtained as

$$\begin{bmatrix} \dot{E}_1(\bar{t}) \\ \dot{E}_2(\bar{t}) \\ \dot{E}_3(\bar{t}) \end{bmatrix} = \begin{bmatrix} 0 & \csc\theta & 0 \\ -\csc\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1(\bar{t}) \\ E_2(\bar{t}) \\ E_3(\bar{t}) \end{bmatrix}, \quad \theta \neq 2k\pi, \, k \in IR,$$

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where $\bar{t} = t \tan \theta$ is the dual arc-length of the dual closed curve $c(\widetilde{X})$.

Thus, we get

$$S = (0, 0, \bar{t} \csc \theta).$$

From equation (14) the dual angle of the pitch of a V_X-closed trajectory surface is obtained as

$$\Lambda_{V_X} = -t.$$

From equations (14) and (24), the dual angles of the pitch of ruled surfaces corresponding to the closed dual spherical curves $c(E_1)$, $c(E_2)$ and $c(E_3)$, respectively, are obtained as

$$\Lambda_{E_1} = -\langle E_1, V_{E_2} \rangle = 0,$$
$$\Lambda_{E_2} = -\langle E_2, V_{E_2} \rangle = 0,$$
$$\Lambda_{E_3} = -\langle E_3, V_{E_2} \rangle = -\bar{t} \csc \theta.$$

Also, from theorem (4.4) and the equations (26) and (27) the dual angles of the pitch of V_{E_1}, V_{E_2} and V_{E_3}-area vectors trajectory surfaces are found as

$$\Lambda_{V_{E_1}} = -t^2 \csc^2 \theta,$$
$$\Lambda_{V_{E_2}} = -t^2 \csc^2 \theta,$$
$$\Lambda_{V_{E_3}} = 0,$$

respectively. If the distribution parameter of the closed ruled surface $\Psi(t, u)$ is denoted by d, then the distribution parameter d is obtained as

$$d = \frac{|\dot{x}, v_x, \dot{v}_x|}{\|\dot{v}_x\|^2} = 0.$$

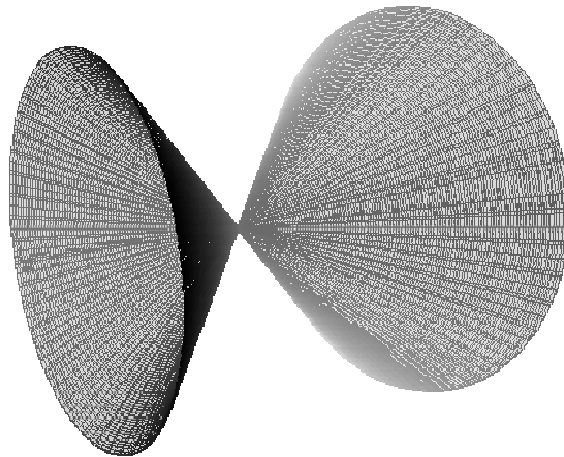Hence, the closed ruled surface $\Psi(t, u)$ is developable. (see Fig. 1)

Fig.1. A developable ruled surface

5. CONCLUSIONS

1. The starting points of this paper are the definitions of the area vector of a given closed space curve, and the projection area of this curve in the direction of a unit vector given in [17].

2. Using the area vector of a closed dual spherical curve, the integral invariants of the ruled surfaces in the lines space corresponding to the closed spherical curve with the E. Study transference principle are investigated and Jacobi's Theorems are given with a different method. These closed curves and ruled surfaces are an important and effective tool in studying spatial kinematics.

It is hoped that this study will bring a different interpretation to the studies in this field and will contribute to the study of rational design problems of space mechanisms.

REFERENCES

1. Yang, A.T., Kirso, Y. & Roth, B. (1975). On a kinematics theory for ruled surface. *Proceeding of the fourth world Congress on the Theory Mach. and Mech. Newcastle Upon Tyne* (737-742) England.

2. Pottmann, H., Lü, W. & Ravani, B. (1995). Rational ruled surfaces. Technical Report No. 23, Institute Für Geometrie, Technische Uiv. Wien.

3. Schaaf, J. A. & Ravani, B. (1998). Geometric continuity of ruled surfaces. *Computer Aided Geometric Design, 15*, 289-310.

4. Wang, D. L., Liu, J. & Xiao, D. Z. (1997). Kinematic differential geometry of a rigid body in spatial motion-(I, II, III). *Mechanism and Machine Theory, 32*, 419-457.

5. Study, E. (1891). Von den bewegungen und umlegungen. *Mathem. Annalen, 39*, 441-564.

6. Blaschke, W. (1958). Anwendung dualer quaternionen auf die kinematik. *Annales Academia Scientiarum Fennicae*, 1-13.

7. Yang, A. T. & Freudenstein, F. (1964). Application of dual-number quaternion algebra to the analysis of spatial mechanisms. *Journal of Applied Mechanics, 31*, 300-308.

8. Veldkamp, G. R. (1976). On the use of dual numbers, vectors, and matrices in instantaneous, spatial kinematics. *Mechanism and Machine Theory, 11*, 141-156.

9. Bottema, O. & Roth, B. (1979). *Theoretical kinematics*. Amsterdam, North-Holland.

10. Jacobi, J. (1842). Über einique merkwürdige kurven theorem. *Schumaster Astr. Nch., 20*, 115-120.

11. Fenchel, W. (1934). Über einem Jacobischen satz der theorie. *Tohoku, Math. Journal, 39*, 95-97.

12. Avakumovic, V. G. (1951). Über geschlossene kurven auf der kugel. *Sbornik Radova, Sýrpska Academia Nauka, 7*, 101-108.

13. Yapar, Z. (1979). Uzay kinematiğinde Jacobi Teoremleri. *K.T.Ü., Tem.Bil. Fak. 20*.

14. Hoschek, J. (1971). Liniengeometrie, *Zürich*, 20-22.

15. Hoschek, J. (1973). Integralinvarianten von regelflachen. *Arch.Math. Vol. XXIV*, 218-224.

16. Hacısalihoğlu, H. H. (1972). On the pitch of a closed ruled surfaces. *Mechanism and Machine Theory, 7*, 291-305.

17. Müller, H. R. (1980). Über öffnungsma β e kinematisch erzeugter geschlossener regelflachen. *Abhandl. Braunschw. Wiss. Ges., 40*, 7-16.

18. Güneş, R. & Keleş, S. (1994). Closed spherical motions and Holditch's Theorem. *Mech. and Mach.Theory, 29*(5), 755-758.

19. Karadağ, H. B. (1994). On the closed spherical curves and Jacobi's Theorems. PhD Thesis, University of Inonu.

20. Müller, H. R. (1980). Erweiterung des satzes von Hlditch für geschlossene raumkurven. *Abhandl. Braunschw Wiss. Ges. 31*, 129-135.

21. Pottmann, H. (1987). Über geschlossene spharische kurven. *El. Math. 42*, 62-68.

22. Pottman, H. (1989). Die flachenvektoren der bahnkurven geschlossener raumlicher zwanglaufe. *Beitrage Zur Algebra und Geometrie, 29*, 143-160.