

EXTREMAL ORDERS INSIDE SIMPLE ARTINIAN RINGS*

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Abstract – The aim of this paper is to study orders over a valuation ring V with arbitrary rank in a central simple F -algebra Q . The relation between all of the orders is explained with a diagram. It is then shown that inside Bezout order, extremal V -orders are precisely semi-hereditary. In the last section, the effect of Henselization on maximal and semi-hereditary orders is examined.

Keywords – Dubrovin valuation rings, extremal orders, Henselization

1. INTRODUCTION

In this paper, all rings are associative with a multiplicative unit and all modules are unitary. If A is a ring, $J(A)$ will denote its Jacobson radical, $U(A)$ its group of units, $Z(A)$ its center, A^* its set of nonzero divisors, and $M_n(A)$ the ring of $n \times n$ matrices over A . The residue ring $A/J(A)$ will be denoted by \bar{A} . And Q denotes a simple artinian ring with finite dimension over its center $Z(Q)$, while D denotes a division ring.

In the second section we briefly discuss some of the ring theoretic properties and definitions.

In the third section we will see that semihereditary V -orders are extremal V -orders and obtain a diagram of maximal V -orders when V is a Henselian valuation ring.

In the fourth section we show that inside Bezout orders, extremal V -orders are precisely semihereditary, which is a generalization of Proposition 2.1 of [1].

In the last section we will examine the effect of Henselization on maximal and semihereditary orders.

2. DEFINITION AND PRELIMINARIES

In this paper F denotes a field and Q is a central simple F -Algebra, i.e., Q is a F -Algebra with $[Q:F] < \infty$ and $F=Z(Q)$.

The most successful extension of the classical valuation theory on F to Q is the one introduced by Dubrovin in [2] and [3].

Definition 2. 1. A subring B of a central simple F -algebra Q is called a Dubrovin valuation ring in Q if

- (1) B has an ideal M such that B/M is a simple artinian ring and
- (2) For each $q \in Q \setminus B$ there exist $b, a \in B$ such that $bq, qa \in B \setminus M$.

*Received by the editor July 25, 2003 and in final revised form September 10, 2005

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The following properties of Dubrovin valuation rings were proved by Dubrovin in [2, 3].

- i) The two sided ideals of B are totally ordered by inclusion, where two sided ideals are a B -bimodule of Q . Therefore we have $M=J(B)$
- ii) Each finitely generated left (resp, right) ideal of B is principal.
- iii)(a) Let V be a valuation ring of F , then there exists a Dubrovin valuation ring of B in Q such that $B \cap F=V$, [2-4].
- (b) If B , and B' are two Dubrovin valuation rings of Q extending V , then $B'=dBd^{-1}$ for some $d \in Q^*$ [5, 6].

Therefore, for every valuation ring V of $F=Z(Q)$, there is a unique (up to conjugate) associated Dubrovin valuation ring B of Q . It is reasonable to expect that B will carry much information about the arithmetic of Q in relation to V , (see [7] Theorem 3.4 and [8] Theorem 3.7).

Definition 2. 2. Let Q be a finite-dimensional F -Algebra and V a ring with quotient field F . A subring R of Q is said to be an order in Q if $RF=Q$. If $V=Z(R)$, then R is said to be a V -order if, in addition, R is integral over V . If R is maximal with respect to inclusion among V -order of Q , then R is said to be a maximal order over V .

- a) In the case V is a discrete valuation ring, then by ([9], 18.6 and 18.2) any V -order in a central simple F -algebra is a finite V -module, so for such V , Definition 2.2 agrees with the usual one, as in [10].
- b) In this paper we assume V is a commutative valuation ring in F of arbitrary Krull-dimension. The integrality hypothesis in the above definition is used to guarantee the existence of maximal orders for any Q and V . But finitely generated maximal V -orders need not exist, (see [7] Proposition 2.3).
- c) Let V be a valuation ring of a field F , and Q a central simple F -Algebra. If B is an integral Dubrovin extension of V to Q (i.e., B is a Dubrovin valuation ring of Q such that B is integral over V and $V=B \cap F$) then B is a maximal V -order (by Example 2.2 [7]).

Definition 2. 3. A ring R is said to be *extremal* if for every overring S such that $J(R) \subseteq J(S)$ we have $S=R$. If S is an overring of R , we say that R is extremal in S if R is extremal among all subrings of S . A V -order R is said to be an extremal V -order (or just extremal when the context is clear) if it is extremal among all V -orders in Q .

Definition 2. 4. A ring R is said to right (resp left) Bezout if every finitely generated right (left) ideal is principal. It is called Bezout if it is both right and left Bezout.

If V is a valuation ring, then there exists a Bezout V -order B in Q and each Bezout V -order is a maximal order by ([7] Theorem 3.4), and if B , and B' are two Bezout V -orders, then B , and B' are conjugate (by Theorem 6.12 [4]).

Definition 2. 5. A ring R is said to be right semihereditary (resp right hereditary) if every finitely generated right ideal (resp every right ideal) is projective as a right R -module. A ring is said to be semihereditary (resp hereditary) if it is both left and right semihereditary (resp hereditary).

a) If V be Dedekind domain with quotient field F and Q is a central simple F -Algebra, where $Q \cong M_n(D)$ and D is a division ring with center F , then R is a hereditary V -order if and only if R is an extremal (see 39.14 [10]).

b) Let V be a valuation ring of $F=Z(Q)$ and Q a central simple F -Algebra. J.S. Kauta proved that every semihereditary V -Order is extremal (see Theorem 1.5 [11]), but the converse is not true. If F is

a field, $Q=M_2(F)$, V_n is a discrete valuation ring of dimension n , and R is a maximal V_n -order in Q , then there are three possibilities for the isomorphism class of R .

(1) $R \cong M_2(V_m)$, where V_m is the overring of V_n of dimension m . In this case R is a Bezout.

(2) $R \cong \begin{bmatrix} V_m & J(V_p) \\ V_p & V_m \end{bmatrix}$, where $m < p$. In this case R is semihereditary, but not Bezout.

(3) R is primary (i.e., $J(R)$ is a maximal ideal of R) but not Bezout (see [7], Theorem 5.7). Let R be maximal V -order in $M_2(F)$ which is primary, but not Bezout. Such an order cannot be semihereditary, since any primary semihereditary order is a Dubrovin valuation ring ([3]: Theorem 4), and hence Bezout.

3. MAXIMAL ORDERS OVER HENSELIAN VALUATION RINGS

In this section D always means a finite dimensional algebra with center F . A subring B of D is said to be a total valuation ring in D if $d \in B$ or $d^{-1} \in B$ for all nonzero $d \in D$.

We recall that a valuation ring V in a field F is Henselian when Hensel's Lemma holds for V , i.e., for every monic polynomial $f \in V[x]$, if its image $\bar{f} \in \bar{V}[x]$, where $\bar{V} = V/J(V)$ has a factorization $\bar{f} = \bar{g}\bar{h}$ on $\bar{V}[x]$ with \bar{g}, \bar{h} monic and $\gcd(\bar{g}, \bar{h}) = 1$, then there exist monic $g, h \in V[x]$ with $f = gh, \bar{g} = \bar{g}$ and $\bar{h} = \bar{h}$, where \bar{g} and \bar{h} are images g and h respectively.

There are several other equivalent characterizations of the Henselian valuation ring, but the most relevant here is the following.

A valuation ring V in a field F is Henselian if V has a unique extension to each field $F \subset K$ with K algebraic over F (see [9] Coro.16.6 for a proof).

Now let D be a division algebra finite dimensional over its center $Z(D)=F$, and V a Henselian valuation ring of F . Schilling ([12] P.53, Theorem 9) proved that the integral closure V in D forms a ring B . The ring B is a total valuation ring of V and by ([13], Theorem 1) and B is the unique extension V to D . Therefore B is an invariant valuation ring of D (i.e., $dBd^{-1} = B$ for any $d \in D^*$).

Theorem 3. 1. Let D be a division algebra admitting a total valuation ring extending V . Then the integral closure of V in D is the unique extremal V -order (and hence the unique semihereditary V -order) in D .

Proof: By ([14]: Lemma 2) V has only a finite number of extensions to D . If B_1, \dots, B_n are all the extensions of V , then B_i and B_j are conjugate for all i, j by ([14]: Theorem 2). Let $T = \text{Int}_D(V)$ be the integral closure of V in D . Then $T = \bigcap_{i=1}^n B_i$ by ([14]: Theorem 3). Let R be an extremal V -order.

Then $R \subseteq T$, because R is integral over V . But both R and $J(B_i)$ contain $J(V)$. Hence for each i , $R / (J(B_i) \cap R)$ is finite dimensional over $V/J(V)$. But one has the embedding $R / (J(B_i) \cap R) \rightarrow B_i / J(B_i)$ and $[B_i / J(B_i) : V/J(V)] \leq [D:F] < \infty$ by ([14]: Lemma 3). It follows that $R / (J(B_i) \cap R)$ is division algebra, and hence $J(B_i) \cap R$ is a maximal ideal of R . Hence, $J(R) \subseteq J(B_i) \cap R$.

Let $x \in \bigcap_i J(B_i)$ and $a, b \in J(T)$. Then $1-axb \in U(B_i)$ for all i , and thus $1-axb \in U(T)$. Therefore $x \in J(T)$. Hence $J(R) \subseteq \bigcap_i J(B_i) \subseteq J(T)$. Since R is extremal, we must have $R=T$.

On the other hand, T is a Bezout V -order by ([7]: Theorem 3.4) and every such T is a semihereditary V -order in D .

Corollary 3. 2. Let V be a valuation ring of F , and D suppose admits and invariant valuation ring B extending V . Then B is the unique extremal (and hence the unique semihereditary) V -order in D .

Proof: Since the extensions of V to D are conjugate, B is the unique extension of V to D . So the corollary follows from Theorem 3.1.

In the rest of the section we assume V to be a Henselian valuation ring of F , and D be a finite dimensional division algebra over its center $Z(D)=F$.

Let B be the unique extension of V to D , and let β be the set of all nonzero B -submodules of D . Then β is totally ordered. For if I and J are two B -submodules of D such that $I \not\subseteq J$, there exists an $a \in I \setminus J$. Then if $b \in J$, then $ab^{-1} \notin B$; thus $ba^{-1} \in B$, and hence $b \in Ba \subset I \Rightarrow J \subseteq I$.

Definition 3. 3. Let I be a B -submodule of D . We define Γ^I to be $\{d \in D: dI \subseteq B\}$.

$$\begin{bmatrix} B, B_{1,2}, \dots, B_{1,n} \\ B_{2,1}, B, B_{2,3}, \dots, B_{2,n} \\ \dots \\ B_{n,1}, B_{n,2}, \dots, B_{n,n-1}, B \end{bmatrix}$$

Definition 3. 4. Let $Q=M_n(D)$. An order $R=$ is said to be of type Φ H if

- i) $B_{i,j} \in \beta$.
 - ii) If $d \notin B_{i,j}$, then $d^l \in B_{j,i}$ for all $d \neq 0 \in D$. (Morandi's condition).
 - iii) $B_{r,j}B_{j,s} \subseteq B_{r,s}$, for all $1 \leq r, s, j \leq n$.
- We denote R by $(B_{i,j})$

Lemma 3. 5. (a) R is a ring and $RF=RD=Q$, i.e., R is an order.
 (b), $B_{i,j} \subseteq B \subseteq B_{j,i}$ or $B_{j,i} \subseteq B \subseteq B_{i,j}$ for all i, j .

Proof: (a) by (iii) R is a ring, because $B_{i,j} \neq 0$ for all i, j , therefore $RF=RD=Q$.
 For (b) since β is totally ordered, we have $B_{i,j} \subset B$ or $B \subseteq B_{i,j}$. If $B_{i,j} \subset B$, then $1 \notin B_{i,j}$, and hence, $1 \in B_{j,i}$ by (ii). Thus $B=BI \subseteq B_{j,i}$, and so $B_{i,j} \subseteq B \subseteq B_{j,i}$.

If $B \subseteq B_{i,j}$, then $B_{i,j}B_{j,i} \subseteq B_{i,i}=B \Rightarrow B_{j,i}=B_{j,i}I \subseteq B$, and hence $B_{j,i} \subseteq B \subseteq B_{i,j}$.

Lemma 3. 6. (Morandi) Let $Q=M_n(D)$ and $R=(B_{i,j})$. Then xR is projective as a R -module for all $x \in Q$.

Proof: We first suppose xR is projective for all $x \in e_{i,i}R$ for any i . We prove xR is projective for any x (where $e_{i,i}$ is matrix $n \times n$ with 1 in (i,i) entry and zero in the others). We do this by showing that $e_{i,x}R$ is projective, where $e_i=e_{1,1}+e_{2,2}+\dots+e_{i,i}$. We use induction on i , the case $i=1$ is true by assumption (because if $x=(d_{i,j})$ then $xe_{1,1}R=(xe_{1,1})R$, and since $xe_{1,1}=d_{1,1}e_{1,1}$ and $d_{1,1} \in B_{i,j}$ or $d_{1,1} \in B_{j,i}$, therefore $xe_{1,1} \in e_{1,1}R$). So suppose $e_{i-1,x}R$ is projective for all $x \in e_{i,i}R$. We have the exact sequence of R -modules.

$0 \rightarrow e_i x R \cap (1 - e_{i-1}) R \rightarrow e_i x R \rightarrow e_{i-1} e_i x R \rightarrow 0$, where $I = e_{1,1} + e_{2,2} + \dots + e_{n,n} = e_n$. Now $e_{i-1} e_i x R = e_{i-1} x R$ and $e_i x R \cap (1 - e_{i-1}) R \subseteq e_i R \cap (1 - e_{i-1}) R = e_i R$ (because $1 - e_{i-1} = e_{i,i} + \dots + e_{n,n}$). Since $e_{i-1} x R$ is projective by the induction of the sequence splits. So $e_i x R \cong e_{i-1} x R \oplus (e_i x R \cap (1 - e_{i-1}) R)$.

Thus $e_{i-1} x R \oplus (e_i x R \cap (1 - e_{i-1}) R)$ is a cyclic right R -module and is a submodule of $e_i R$. Hence it is projective by assumption. Therefore we obtain $e_i x R$ as a sum of two projective modules, thus it is projective. Thus by induction, $e_i x R$ is projective for all i . Setting $i = n$, then $e_n x R = x R$ is a projective.

We now show that $x R$ is projective for all $x \in e_{ii} M_n(D)$. Recall that $x R$ is projective if and only if the annihilator $ann_R(x) = e R$ for some idempotent $e \in R$. This holds for $x \in Q$, not just for $x \in R$ as $R F = Q$ and $ann_R(x) = ann_R(x \alpha)$ for any $\alpha \in F^*$.

Say $x = \sum_{j=1}^i x_j e_{i,j} \in e_{ii} M_n(D)$ with $x_j \in D$. If $x = 0$ then $ann_R(x) = R$ and we are done.

Also, by Lemma 2.5 of [7] there is an i_0 with $x_j x_{i_0}^{-1} \in B_{i_0,j}$ for all j , and so $x_{i_0}^{-1} x_j \in B_{i_0,j}$ for all j . Let e be the permutation matrix which switches the i_0 th and i th rows. Let

$$e = I_n - x_{i_0}^{-1} (Ex) = \begin{bmatrix} 1, & 0, & 0, & \dots, & 0, & 0 \\ 0, & 1, & 0, & \dots, & 0, & 0 \\ \cdot & & & & & \\ -x_1 x_{i_0}^{-1}, & \dots, & -x_{i_0-1} x_{i_0}^{-1}, & 0, & -x_{i_0+1} x_{i_0}^{-1}, & \dots, & -x_n x_{i_0}^{-1} \\ 0, & & & 0, & 1, & 0, & \dots, & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ 0, & 0, & 0, & \dots, & 0, & 1 \end{bmatrix}.$$

We have $e \in R$ since $x_j x_{i_0}^{-1} \in B_{i_0,j}$. Also $x e = x I_n - x x_{i_0}^{-1} (Ex) = x - x = 0$ $x e = x I_n - x x_{i_0}^{-1} (Ex) = x - x = 0$, and so $e \in ann_R(x)$.

Let $a \in ann_R(x)$, then $ea = (I_n - x_{i_0}^{-1} (Ex))a = a - 0 = a$. Thus $e^2 = e$, and $ann_R(x) = e R$ is generated by an idempotent. Therefore $x R$ is projective.

Theorem 3. 7. (J.S. KAUTA) R is a semihereditary V -order if and only if R is conjugate to an order of type ΦH . Therefore orders of type ΦH are extremal. (See Theorem 4.7 [7] and 39.14 (ii) [10] for special cases of this theorem.)

Proof: Suppose R is a semihereditary V -order. Then R contains a full set of primitive orthogonal idempotents. After a conjugation, if necessary, we may assume all the standard idempotents $e_{1,1}, e_{2,2}, \dots, e_{n,n} \in R$. Since R is integral over V , $e_{i,i} R e_{i,i}$ is integral over V . Also $e_{i,i} R e_{i,i} F = e_{i,i} R F e_{i,i} = e_{i,i} D e_{i,i} = D$, therefore $e_{i,i} R e_{i,i}$ is a V -order; indeed, $e_{i,i} R e_{i,i}$ is a semihereditary V -order in D . Hence $e_{i,i} R e_{i,i} = B$ (because B is an invariant valuation ring extending V ; therefore B is the unique extremal and hence the unique semihereditary V -order in D). Set $B_{i,j} = e_{i,i} R e_{j,j}$. Then $B_{i,j} \neq 0$, since R is an order in Q . Since $B \subseteq R$, we have $B e_{i,i} R e_{j,j} = e_{i,i} B R e_{j,j} = e_{i,i} R e_{j,j} = e_{i,i} R e_{j,j} B$, therefore $B B_{i,j} = B_{i,j} B = B_{i,j}$ and so $B_{i,j}$ is a B -bisubmodule of D . Now R is a ring and $R e_{j,j} e_{j,j} R = R e_{j,j} R \subseteq R$; so $B_{k,j} B_{j,l} \subseteq B_{k,l}$, where $B_{k,j} = e_{k,k} R e_{j,j}$ and $B_{j,l} = e_{j,j} R e_{l,l}$ holds. We only have to show Morandi's condition holds.

Suppose $\exists i_0, j_0$ and an $0 \neq \alpha \in D$ such that $\alpha \notin B_{i_0, j_0}$ and $\alpha^{-1} \notin B_{j_0, i_0}$. Since B is an invariant valuation ring, $i_0 \neq j_0$. Let $\Gamma = (e_{i_0, i_0} + e_{j_0, j_0})R(e_{i_0, i_0} + e_{j_0, j_0}) \cong \begin{bmatrix} B & B_{j_0, i_0} \\ B_{i_0, j_0} & B \end{bmatrix}$. Then Γ is a semihereditary order in $M_2(D)$ by [15]. Consider $x = \begin{bmatrix} \alpha & 1 \\ 0 & 0 \end{bmatrix} \in M_2(D)$.

Then $\text{ann}_\Gamma(x) = \left\{ \begin{bmatrix} t & r \\ -\alpha t & -\alpha r \end{bmatrix} \text{ such that } t, \alpha r \in B, r \in B_{j_0, i_0}, \alpha t \in B_{i_0, j_0} \right\}$ (see the proof of Theorem 1.5 [11]). We have $\alpha t \in B_{i_0, j_0}$ and $t \in B$. But $\alpha \notin B_{i_0, j_0}$. So $t \in J(B)$. Since Γ is a semihereditary order in $M_2(D)$, $\text{ann}_\Gamma(x)$ is generated by an idempotent $\begin{bmatrix} a & b \\ -\alpha a & -\alpha b \end{bmatrix} = \begin{bmatrix} a & b \\ -\alpha a & -\alpha b \end{bmatrix}^2$. So $1 = a - b\alpha$.

But $a \in J(B)$, so $b\alpha$ is a unit in B . Hence αb is also a unit in B . But $b \in B_{j_0, i_0} \supseteq \alpha b B = B$ since αb is a unit in B , hence $\alpha^{-1} \in B_{j_0, i_0}$, a contradiction, and so we have Morandi's condition.

On the other hand, let $R = (B_{i,j})$ be of type ΦH . We want to show that R is a semihereditary V -order in $Q = M_2(D)$. By Lemma 2.5, R is a ring with the identity element of Q , and $FR = Q$. By the proof of ([7], Proposition 4.3), R is a V -order. But $M_r(R)$ is of type ΦH whenever R is. Hence Lemma 2.6 shows that for each r , every principal right ideal of $M_r(R)$ is projective. So R is right semihereditary by [12]. Similarly, R is left semihereditary and hence it is semihereditary.

Proposition 3. 8. Every Bezout V -order is a semihereditary V -order, but the converse does not hold.

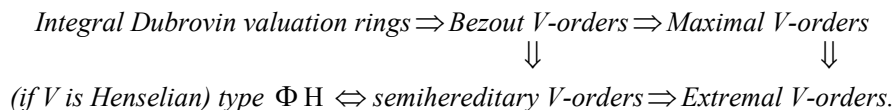
Proof: Suppose

$$R = \begin{bmatrix} B \supset J(B_{1,2}) \supset, \dots, \supset J(B_{1,n}) \\ \cap \quad \cap \quad \quad \quad \cap \\ B_{2,1} \supset B \supset, \dots, \supset J(B_{2,n}) \\ \cap \quad \cap \quad \dots, \quad \cap \\ \dots \\ \cap, \dots, \dots, \cap \\ B_{n,1} \supset B_{n,2} \supset, \dots, \supset B \end{bmatrix},$$

where $B_{i,j}$ is an overring B for all i, j and $B_{i,j} \neq B$ for some i, j . By Theorem 2.7 and Theorem 2.6 of [11] R is semihereditary maximal V -order. But $B_{n,1} \supset B$ by assumption. Let $W = B_{n,1} \cap F$, then $RW \subset M_n(B_{n,1})$, since $WB \subset WB_{n,1} = B_{n,1}$. If R is a Bezout, then $R \cong M_n(B)$ by Corollary 3.5 of [7]. But RW would be a Dubrovin valuation ring over W and $RW \subset M_n(B_{n,1})$. Therefore $RW = M_n(B_{n,1})$, a contradiction.

If R is a Bezout V -order, by Proposition 1.8 and Example 1.15 of [16], then R is semihereditary and also more examples of semihereditary orders can be found in [17].

Therefore we have the following diagram in general.



4. SEMIHEREDITARY ORDERS INSIDE BEZOUT ORDERS

Let V be a discrete valuation ring of F and Q a central simple F -algebra. By Wedderburn structure theorem $Q \cong M_n(D)$, where D is a division algebra with center F .

By (10-4) Corollary of [10] every V -order in Q is contained in a maximal V -order in Q . If V be complete valuation ring, then the integral closure V in D , i.e., $\Delta = \text{int}_D(D)$ is the unique maximal V -order in D . let R be an V -order in Q . Then by Theorem (39-14) of [10], R is a hereditary order if R is an Extremal V -order.

In this case R is precisely,

$$R = \begin{bmatrix} (\Delta)(P)(P), \dots, (P) \\ (\Delta)(\Delta)(P), \dots, (P) \\ \vdots \\ (\Delta)(\Delta), \dots, (\Delta) \end{bmatrix}^{(n_1, n_2, \dots, n_r)}$$

where $P = J(\Delta)$ and $n_1 + n_2 + \dots + n_r = n$.

Now we assume V is a Henselian valuation ring of F , not necessarily discrete. Let R be an Extremal V -order inside an integral Dubrovin valuation ring of B with $B \cap F = V$. We know the integral closure V in D i.e., $\Delta = \text{int}_D(V)$ is a unique maximal V -order in D , and so $B \cong M_n(\Delta)$ is a Dubrovin valuation ring and we can consider $R \subset M_n(\Delta)$. By (Proposition [1]) R is semihereditary. So in this case we have

$$R = \begin{bmatrix} (\Delta), (J(\Delta)), \dots, (J(\Delta)) \\ (\Delta), (\Delta), (J(\Delta)), \dots, (J(\Delta)) \\ \vdots \\ (\Delta), (\Delta), \dots, (\Delta) \end{bmatrix}^{(n_1, n_2, \dots, n_r)}$$

where $n_1 + n_2 + \dots + n_r = n$ and $R=B$ if $J(R) = J(\Delta)R$ if $J^{-1}(\Delta) = \Delta$.

If V isn't Henselian, then $B_h = B \otimes_v V_h$ is a Dubrovin valuation ring. Therefore

$$B / J(B) \cong B_h / J(B_h)$$

$J(B) \otimes_v V_h \subseteq R \otimes_v V_h = R_h$. Hence we have \cup \cup , thus R_h is semihereditary

$$R / J(B) \cong R_h / J(B_h)$$

and so R is semihereditary by ([11] Proposition 3.3). Thus inside an integral Dubrovin valuation ring, extremal V -orders are precisely the semihereditary V -orders.

Corollary 4. 1. Let R be an extremal V -order inside a Dubrovin valuation ring of B , and if $R \subseteq R' \subseteq B$, then R' is extremal V -order in B .

Proof: Since R is semihereditary, R' is a semihereditary V -order (by Lemma 4.10 of [7]), and so R' is an extremal V -order.

Corollary 4. 2. Let R be an extremal V -order inside an integral Dubrovin valuation ring with $J(B)$ a non-principal ideal of B . Then $R=B$ if $J(R)=J(V)R$.

Now the generalization of Proposition 2.1 of [1] is given.

Theorem 4. 3. Let R be an Extremal V -order sitting inside a Bezout V -order B . Then R is a semihereditary V -order.

Proof: By induction on $[Q: F]$. If $[Q: F]=1$, then B is an integral Dubrovin valuation ring and so R is a semihereditary.

Now we assume B is not a Dubrovin valuation ring. Then there exists an integral Dubrovin valuation ring T of Q , with center $W \supset V$ such that

$$i) T \supset B \quad ii) J(T) \subseteq J(B) \subseteq J(R) \quad iii) \tilde{R} = R/J(T), \tilde{B} = B/J(T)$$

are $V/J(W)$ -orders in $\bar{T} = T/J(T)$, and (iv) $[\bar{T} : Z(\bar{T})] < [Q : F]$. By induction, \tilde{R} is semihereditary and so R is semihereditary (by Lemma 4.11 of [7]).

5. THE HENSELIZATION

We now consider V to be a valuation ring of a field F of arbitrary rank which need not be Henselian. One aim of this section is to examine the effect of Henselization on Bezout and maximal semihereditary V -orders.

Let (V_h, F_h) be the Henselization of (V, F) (see [9] for definition).

Let Q be a central simple F -algebra, then $Q \otimes_F F_h$ is a central simple F_h -algebra and by ([10] Corollary 7.8) and also by Wedderburn's Theorem $Q \otimes_F F_h \cong M_n(D)$ for some n , where D is a division algebra finite dimension over F_h .

Let R be a V -order in Q . Clearly if $R \otimes_V V_h$ is a maximal V_h -order, then R is a maximal V -order. Thus the difficulty lies in proving the converse.

If V be a discrete valuation ring, then a V -order R of Q is a maximal order if R is a Dubrovin valuation ring ([6]: Example 1.15). Therefore, in this case $R \otimes_V V_h$ is a Dubrovin valuation ring of $Q \otimes_F F_h$, which is integral over V_h . Thus $R \otimes_V V_h$ is a maximal V_h -order.

On the other hand, there exists a Bezout maximal V -order R such that $R \otimes_V V_h$ is a semihereditary maximal order, but is not Bezout, (see [7] Example 4.14).

P. Morandi [7] mentioned two questions.

- (1) Suppose R is a maximal V -order in a central simple F -algebra Q . Let (F_h, V_h) be the Henselization of (V, F) . Then $R \otimes_V V_h$ is a V_h -order in $Q \otimes_F F_h$. Is $R \otimes_V V_h$ a maximal order?
- (2) If R is semihereditary, then $R \otimes_V V_h$ is a V_h -order in $Q \otimes_F F_h$. Is $R \otimes_V V_h$ semihereditary?

Now we assume that B is an invariant valuation ring extension of V_h to D and $R \cong (B_{i,j})$, an order of type ΦH in $Q \otimes_F F_h$.

Theorem 5. 1. Suppose Q is a central simple F -algebra and V is a valuation ring in F . If T is a Bezout V -order in Q , then $T \otimes_V V_h$ is conjugate to an order type ΦH such that $B_{i,j}^{-1} = B_{j,i}$ for all i, j and $J(T) \otimes_V V_h = J(B)(T \otimes_V V_h)$.

Moreover, $T \otimes_V V_h$ is a Dubrovin valuation ring if T is a Dubrovin valuation ring. In this case $T \otimes_V V_h$ is conjugate to $M_n(B)$.

Proof: By Theorem 17 of [18], $T \otimes_V V_h$ is a semihereditary maximal V_h -order in $Q \otimes_F F_h$. Therefore $T \otimes_V V_h$ is conjugate to an order type ΦH . And by Theorem 2.7 of [11] $B_{i,j}^{-1} = B_{j,i}$ for all i, j and $J(T) \otimes_V V_h = J(B)(T \otimes_V V_h)$. Also, $T \otimes_V V_h$ is Bezout if T is Dubrovin valuation ring (see Theorem 17 in [18]). Since V_h is Henselian, $T \otimes_V V_h$ is a Dubrovin valuation ring, and so $T \otimes_V V_h$ is conjugate to $M_n(B)$.

J. S. Kauta ([11]: Theorem 3.4) proved that a V -order R is semihereditary if its Henselization $R \otimes_V V_h$ is a semihereditary. So the answer (2) is yes.

Theorem 5. 2. If R is a maximal V -order in a central simple F -algebra Q , then $R \otimes_V V_h$ is a maximal V_h -order in $Q \otimes_F F_h$ if one of the following conditions holds.

- (1) R is a Bezout ring.
- (2) R is a semihereditary ring.
- (3) R is a finitely generated V -module.
- (4) $\text{Rank} V = 1$

Proof: If R is a Bezout ring, then by Theorem 17 of [18] $R \otimes_V V_h$ is a maximal V_h -order. And if R is a semihereditary ring, it follows from Theorem 1 of [19].

Now we suppose that R is a finitely generated V -module. Then R is contained in a Bezout V -order T by ([7], Prop.3). Since $[T/J(T):V/J(V)] < \infty$, there exists $t_1, \dots, t_n \in T$ such that $T = t_1V + \dots + t_nV + J(T)$. But by ([11]: Prop. 1.4) $J(T) \subseteq R$ (since maximal orders are extremal). Hence T is a finitely generated Bezout V -order. By the maximality of R , we have $T = R$. Therefore R is a Bezout V -order.

(4) Let (V_h, F_h) be the Henselization of (V, F) . Then $(V, F) \subseteq (V_h, F_h) \subseteq (V, F)$, where (V, F) is the complement of (V_h, F_h) with respect to the metric induced by the valuation corresponding of V . Hence V is dense in V_h and by (Proposition of [19]) we have $R \otimes_V V_h$ as a maximal V_h -order in $Q \otimes_F F_h$.

Let B be a unique extension valuation ring V_h to D , where $Q \otimes_F F_h \cong M_n(D)$ and $R = (B_{i,j})$ is order type ΦH . Then we have the following theorem.

Theorem 5. 3. Suppose Q is a central simple F -algebra and V is a valuation ring in F . If T is a maximal semihereditary V -order in Q , then $T \otimes_V V_h$ is conjugate to an order type ΦH such that $B_{i,j}^{-1} = B_{j,i}$ for all i, j .

Proof: By Theorem 5.2, (2) $T \otimes_V V_h$ is a semihereditary maximal V_h -order, and by Theorem 3.7 $T \otimes_V V_h$ is conjugate to an order $R = (B_{i,j})$. On the other hand, R is a semihereditary maximal order, and by Theorem 2.6 of [11] we have $B_{i,j} = B_{j,i}^{-1}$ for all i, j .

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