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INFORMATION MATRIX FOR A MIXTURE OF TWO PARETO DISTRIBUTIONS*

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Abstract – The Fisher information matrix for a mixture of two Pareto distributions is derived. Numerical tabulations of the matrix are also provided for practical purposes. The work is motivated by a real-life example.

Keywords – Lerch function, pareto distribution

1. INTRODUCTION

Consider the mixture of two Pareto distributions given by the pdf

$$f(x) = \frac{\alpha\theta a^{\theta}}{x^{\theta+1}} + \frac{(1-\alpha)\phi a^{\phi}}{x^{\phi+1}} \tag{1}$$

for $x \geq a, a > 0, \theta > 0, \phi > 0, 0 < \alpha < 1$ and assume without loss of generality that $\theta > \phi$. This distribution is also known as the double Pareto distribution. It has been used for statistical analysis of the airport network of China [1], and as models for distributions for human settlements, income, and size distributions [2-4]. In the airport network application, Li and Cai [1] state:

" …. Through the study of the airport network of China (ANC), composed of 128 airports (nodes) and 1165 flights (edges), we show the topological structure of ANC conveys two characteristics of small worlds, a short average path length (2.067), and a high degree of clustering (0.733). The cumulative degree distributions of both directed and undirected ANC can be modeled two-regime power laws with different exponents, i.e., the so-called double Pareto law ……."

Li and Cai [1] go on to demonstrate that both weekly and daily cumulative distributions of flight weights (frequencies) of ANC obey double power-law tails.

The aim of this note is to calculate the Fisher information matrix corresponding to (1) and to provide useful numerical tabulations of the matrix (we also provide the estimation procedures by the method of maximum likelihood). For a given x, the Fisher information matrix is defined by

$$\left(I_{jk}\right) = \left\{E\left(-\frac{\partial \log L(\theta)}{\partial \theta_j}\frac{\partial \log L(\theta)}{\partial \theta_k}\right)\right\}$$

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for $j=1,2,\ldots,p$ and $k=1,2,\ldots,p$, where $L(\theta)=f(x)$ and $\theta=(\theta_1,\theta_2,\ldots,\theta_p)$ are the parameters of the pdf f. It has the meaning "information about the parameters θ contained in the observation x." The information matrix plays a significant role in statistical inference in connection with estimation, sufficiency and properties of variances of estimators. It is related to the covariance matrix of the estimate of θ (being its inverse under certain conditions) [5].

The explicit form of the Fisher information matrix for (1) is derived in Sections 2. The calculations use properties of the Lerch function defined by

$$\Phi(z,a,v)=\sum_{k=0}^{\infty}\frac{z^k}{(v+k)^a} \tag{2}$$

for $|z|<1$ and $v\neq 0,-1,-2,\ldots$. Some technical results about this Lerch function are noted in the Appendix. Useful numerical tabulations of the matrix are provided in Section 3.

2. INFORMATION MATRIX

If x is a single observation from (1), then the log-likelihood function can be written as

$$\log L(a,\theta,\phi,\alpha)=\log\left\{\alpha\theta a^{\theta}x^{\phi}+(1-\alpha)\phi a^{\phi}x^{\theta}\right\}-(\theta+\phi+1)\log x. \tag{3}$$

The first-order derivatives of (3) are:

$$\frac{\partial\log L}{\partial a}=\alpha\theta^2a^{\theta-1}Dx^{\phi}+(1-\alpha)\phi^2a^{\phi-1}Dx^{\theta},$$

$$\frac{\partial\log L}{\partial\theta}=\alpha a^{\theta}Dx^{\phi}+\alpha\theta a^{\theta}\log aDx^{\phi}+(1-\alpha)\phi a^{\phi}D\log xx^{\theta}-\log x,$$

$$\frac{\partial\log L}{\partial\phi}=\alpha\theta a^{\theta}D\log xx^{\phi}+(1-\alpha)a^{\theta}Dx^{\theta}+(1-\alpha)\phi a^{\phi}\log aDx^{\theta}-\log x$$

and

$$\frac{\partial\log L}{\partial\alpha}=\theta a^{\theta}Dx^{\phi}-\phi a^{\phi}Dx^{\theta},$$

where

$$\frac{1}{D}=\alpha\theta a^{\theta}x^{\phi}+(1-\alpha)\phi a^{\phi}x^{\theta}.$$

Note that the maximum likelihood estimator for α is either 0 or 1: in fact, $\hat{\alpha}=0$ if $x\geq(\theta/\phi)^{1/(\theta-\phi)}a$, and $\hat{\alpha}=1$ if $x<(\theta/\phi)^{1/(\theta-\phi)}a$. In practice, however, $0<\hat{\alpha}<1$ because one will usually have more than one observation. The second-order derivatives of (3) are:

$$\frac{\partial^2\log L}{\partial a^2}=\alpha(1-\alpha)\theta\phi\left\{(\theta-\phi)^2-\theta-\phi\right\}a^{\theta+\phi-2}D^2x^{\theta+\phi}-\phi^3(1-\alpha)^2a^{2\phi-2}D^2x^{2\theta}-\alpha^2\theta^3a^{2\theta-2}D^2x^{2\phi},$$

$$\frac{\partial^2\log L}{\partial a\partial\theta}=\alpha(1-\alpha)\phi\left\{\theta(\phi-\theta)\log a+\phi-2\theta\right\}a^{\theta+\phi-1}D^2x^{\theta+\phi}-\alpha^2\theta^2a^{2\theta-1}D^2x^{2\phi}$$

$$+\alpha(1-\alpha)\theta\phi(\theta-\phi)a^{\theta+\phi-1}D^2x^{\theta+\phi}\log x,$$

$$\frac{\partial^2\log L}{\partial a\partial\phi}=\alpha(1-\alpha)\theta\left\{2\phi-\theta-\theta\phi\log a+\phi^2\log a\right\}a^{\theta+\phi-1}D^2x^{\theta+\phi}+(1-\alpha)^2\phi^2a^{2\phi-1}D^2x^{2\theta}$$

$$+\alpha(1-\alpha)\theta\phi(\theta-\phi)a^{\theta+\phi-1}D^2x^{\theta+\phi}\log x,$$

$$\frac{\partial^2\log L}{\partial a\partial\alpha}=\theta\phi(\theta-\phi)a^{\theta+\phi-1}D^2x^{\theta+\phi},$$

$$\frac{\partial^2\log L}{\partial\theta^2}=\alpha(1-\alpha)\phi\left\{2+\theta\log a\right\}\log a a^{\theta+\phi}D^2x^{\theta+\phi}-2\alpha(1-\alpha)\phi\left\{1+\theta\log a\right\}a^{\theta+\phi}D^2x^{\theta+\phi}\log x$$

$$+\alpha(1-\alpha)\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}(\log x)^2-\alpha^2a^{2\theta}D^2x^{2\phi},$$

$$\frac{\partial^2\log L}{\partial\theta\partial\phi}=-\alpha(1-\alpha)\left\{1+\theta\log a+\phi\log a+\theta\phi(\log a)^2\right\}a^{\theta+\phi}D^2x^{\theta+\phi}$$

$$+\alpha(1-\alpha)\left\{\theta+\phi+2\theta\phi\log a\right\}a^{\theta+\phi}D^2x^{\theta+\phi}\log x-\alpha(1-\alpha)\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}(\log x)^2,$$

$$\frac{\partial^2\log L}{\partial\theta\partial\alpha}=\phi\left\{1+\theta\log a\right\}a^{\theta+\phi}D^2x^{\theta+\phi}-\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}\log x,$$

$$\frac{\partial^2\log L}{\partial\phi^2}=\alpha(1-\alpha)\theta\left\{2+\phi\log a\right\}\log a a^{\theta+\phi}D^2x^{\theta+\phi}-2\alpha(1-\alpha)\theta\left\{1+\phi\log a\right\}a^{\theta+\phi}D^2x^{\theta+\phi}\log x$$

$$+\alpha(1-\alpha)\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}(\log x)^2-(1-\alpha)^2a^{2\phi}D^2x^{2\theta},$$

$$\frac{\partial^2\log L}{\partial\phi\partial\alpha}=-\theta\left\{1+\phi\log a\right\}a^{\theta+\phi}D^2x^{\theta+\phi}+\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}\log x$$

and

$$\frac{\partial^2\log L}{\partial\alpha^2}=-\theta^2a^{2\theta}D^2x^{2\phi}+2\theta\phi a^{\theta+\phi}D^2x^{\theta+\phi}-\phi^2a^{2\phi}D^2x^{2\theta}.$$

Now, by computing the elements of the Fisher information matrix by easy applications of Lemma 2, the following can be obtained:

$$E\left(-\frac{\partial^2\log L}{\partial a^2}\right)=\frac{\alpha\theta}{a^2(\theta-\phi)}\left\{\theta+\phi-(\theta-\phi)^2\right\}\Psi(0,\theta+1)+\frac{\phi^2(1-\alpha)}{a^2(\theta-\phi)}\Psi(0,\phi+1)$$

$$+\frac{\alpha^2\theta^3}{(1-\alpha)\phi a^2(\theta-\phi)}\Psi(0,2\theta-\phi+1),$$

$$E\left(-\frac{\partial^2\log L}{\partial a\partial\theta}\right)=\frac{\alpha\left\{\theta(\phi-\theta)\log a+\phi-2\theta\right\}}{a(\phi-\theta)}\Psi(0,\theta+1)+\frac{\alpha^2\theta^2}{(1-\alpha)\phi a(\theta-\phi)}\Psi(0,2\theta-\phi+1)$$

$$-\frac{\alpha\theta\log a}{a}\Psi(1,\theta+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \phi}\right) = \frac{\alpha\theta\left\{2\phi - \theta - \theta\phi\log a + \phi^2 \log a\right\}}{a\phi(\phi-\theta)}\Psi(0,\theta+1) - \frac{(1-\alpha)\phi}{a(\theta-\phi)}\Psi(0,\phi+1)$$

$$-\frac{\alpha\theta\log a}{a}\Psi(1,\theta+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \alpha}\right) = -\frac{\theta}{a(1-\alpha)}\Psi(0,\theta+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{\alpha\left\{2+\theta\log a\right\}\log a}{\phi-\theta}\Psi(0,\theta+1) + \frac{2\alpha\left\{1+\theta\log a\right\}\log a}{\phi-\theta}\Psi(1,\theta+1)$$

$$+\frac{2\alpha\theta(\log a)^2}{\phi-\theta}\Psi(2,\theta+1) + \frac{\alpha^2}{(1-\alpha)\phi(\theta-\phi)}\Psi(0,2\theta-\phi+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \theta \partial \phi}\right) = \frac{\alpha\left\{1+\theta\log a + \phi\log a + \theta\phi(\log a)^2\right\}}{\phi(\theta-\phi)}\Psi(0,\theta+1) - \frac{\alpha\left\{\theta+\phi+2\theta\phi\log a\right\}\log a}{\phi(\theta-\phi)}\Psi(1,\theta+1)$$

$$-\frac{2\alpha\theta(\log a)^2}{\phi-\theta}\Psi(2,\theta+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = \frac{1+\theta\log a}{(1-\alpha)(\phi-\theta)}\Psi(0,\theta+1) - \frac{\theta\log a}{(1-\alpha)(\phi-\theta)}\Psi(1,\theta+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \phi^2}\right) = \frac{\alpha\theta\left\{2+\phi\log a\right\}\log a}{\phi(\phi-\theta)}\Psi(0,\theta+1) - \frac{2\alpha\theta\left\{1+\phi\log a\right\}\log a}{\phi-\theta}\Psi(1,\theta+1)$$

$$+\frac{2\alpha\theta(\log a)^2}{\phi-\theta}\Psi(2,\theta+1) + \frac{1-\alpha}{\phi(\theta-\phi)}\Psi(0,\phi+1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \phi \partial \alpha}\right) = \frac{\theta\{1+\phi\log a\}}{(1-\alpha)\phi(\theta-\phi)}\Psi(0,\theta+1) + \frac{\theta\log a}{(1-\alpha)(\phi-\theta)}\Psi(1,\theta+1)$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{\theta^2}{(1-\alpha)\phi(\theta-\phi)}\Psi(0,2\theta-\phi+1) + \frac{2\theta}{(1-\alpha)(\phi-\theta)}\Psi(0,\theta+1) - \frac{\phi}{(1-\alpha)(\phi-\theta)}\Psi(0,\phi+1),$$

where

$$\Psi(m,b) = \sum_{k=0}^{m} \frac{1}{(m-k)!(\log a)^k(\theta-\phi)^k}\Phi\left(\frac{\alpha\theta}{(\alpha-1)\phi}, k+1, \frac{b-1}{\theta-\phi}\right).$$

3. TABLES

In this section, we provide useful numerical tabulations of the Fisher information matrix derived in Section 2 when the mixture parameter α is set to take the values $\alpha = 0.1, 0.2, \dots, 0.9$. Without loss of generality, we assume that a is known and that a = 1.

Tables 1-9 displayed here give the numerical values of the elements

$$E_1 = E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right),$$

$$E_2 = E\left(-\frac{\partial^2 \log L}{\partial \theta \partial \phi}\right),$$

$$E_3 = E\left(-\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right),$$

$$E_4 = E\left(-\frac{\partial^2 \log L}{\partial \phi^2}\right),$$

$$E_5 = E\left(-\frac{\partial^2 \log L}{\partial \phi \partial \alpha}\right)$$

and

$$E_6 = E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right)$$

for $(\theta,\phi)=(2,1)$, $(\theta,\phi)=(3,1)$, $(\theta,\phi)=(4,1)$, $(\theta,\phi)=(5,1)$, $(\theta,\phi)=(6,1)$, $(\theta,\phi)=(7,1)$, $(\theta,\phi)=(8,1)$, $(\theta,\phi)=(9,1)$ and $(\theta,\phi)=(10,1)$.

Table 1. Elements of Fisher information matrix for $\theta = 2, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.001796646666	0.02222283553	0.02178928457	0.8569367614	0.4631200675	0.3353477311
0.2	0.007036415074	0.03950890722	0.04318347130	0.7261318137	0.4294909882	0.3415692568
0.3	0.01564058758	0.05254635660	0.06488624590	0.6056538709	0.3980930711	0.3525779334
0.4	0.02772283875	0.06175564118	0.08767933396	0.4940868882	0.3679789219	0.3695145343
0.5	0.04361835830	0.06731166951	0.1125597775	0.3904215707	0.3381340782	0.3944491547
0.6	0.06397057049	0.06911131219	0.1409843110	0.2940166172	0.3072672554	0.4312369470
0.7	0.08993586884	0.06664634745	0.1754074639	0.2046489532	0.2733780889	0.4877566405
0.8	0.1237042932	0.05863400322	0.2207712950	0.1227603540	0.2325705349	0.5832271263
0.9	0.1702061623	0.04172449496	0.2904298396	0.05043434640	0.1742479697	0.7849375714

Table 2. Elements of Fisher information matrix for $\theta = 3, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0009444627428	0.008852585845	0.02190104206	0.8552213495	0.5987677438	0.7666313463
0.2	0.003512307786	0.01433645448	0.04100118709	0.7275082676	0.5452057308	0.7543101444
0.3	0.007489250925	0.01760564848	0.05884595663	0.6122080373	0.5013308157	0.7606421023
0.4	0.01283110470	0.01935198149	0.07663348905	0.5063789391	0.4641817462	0.7865714464
0.5	0.01962955950	0.02000081511	0.09555108569	0.4080576521	0.4316491969	0.8367983046
0.6	0.02813018654	0.01979683574	0.1171227775	0.3158889014	0.4019641429	0.9223734146
0.7	0.03881884110	0.01882022213	0.1438479611	0.2289346619	0.3731793995	1.068852186
0.8	0.05267742777	0.01690481904	0.1809997695	0.1466082278	0.3420651863	1.346266039
0.9	0.07212355031	0.01319747960	0.2446495014	0.06884638846	0.2992688655	2.045804995

Table 3. Elements of Fisher information matrix for $\theta = 4, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0006227139035	.004223193808	.02080136724	0.8518915675	0.6501516843	1.171065903
0.2	0.002220935067	0.005842365743	0.03697249325	0.7246363152	0.5793443258	1.114026508
0.3	0.004593251930	0.006124212873	0.05100022347	0.6110224050	0.5260532202	1.098073374
0.4	0.007687280008	0.005726907814	0.06436562321	0.5071095337	0.4842419913	1.118846586
0.5	0.01154203976	0.005035372767	0.07825379984	0.4105224891	0.4503580698	1.180736695
0.6	0.01628497988	0.004296373796	0.09399247958	0.3197318674	0.4220653266	1.299678854
0.7	0.02217422284	0.003676954001	0.1136433217	0.2337112026	0.3975355138	1.516324854
0.8	0.02973739345	0.003275747929	0.1415571552	0.1517655985	0.3746554468	1.948228317
0.9	0.04030171775	0.003021786890	0.1915036349	0.07348131740	0.3481021587	3.116951206

Table 4. Elements of Fisher information matrix for $\theta = 5, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0004541707574	0.002136071652	0.01957996654	0.8496645148	0.6694362996	1.541440748
0.2	0.001565533666	0.002177899162	0.03325443103	0.7229937701	0.5845963341	1.423040897
0.3	0.003164400626	0.001379524658	0.04438139712	0.6105023578	0.5245126045	1.375630943
0.4	0.005208348132	0.0003072443976	0.05460628891	0.5075974335	0.4796631357	1.384013413
0.5	0.007719189996	-0.0007298023137	0.06505306159	0.4117232218	0.4450329026	1.449570179
0.6	0.01077455909	-0.001529868080	0.07685056767	0.3213295755	0.4177098570	1.590831122
0.7	0.01453187481	-0.001932098453	0.09166799787	0.2354281080	0.3958570860	1.859969739
0.8	0.01931295875	-0.001777443235	0.1130037851	0.1533820136	0.3780892942	2.413335318
0.9	0.02593251059	-0.0008852338173	0.1521211559	0.07481450360	0.3620934752	3.969241234

Table 5. Elements of Fisher information matrix for $\theta = 6, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0003512042533	0.001045347894	0.01841634174	0.8484108650	0.6739171614	1.879997736
0.2	0.001176830133	0.0003713989652	0.03005494843	0.7225766239	0.5778898287	1.690414171
0.3	0.002336794925	-0.0008307728974	0.03901320746	0.6110527016	0.5130175500	1.607443291
0.4	0.003798614960	-0.002089232838	0.04700864495	0.5088325882	0.4662832233	1.600207325
0.5	0.005576441427	-0.003162896966	0.05508165325	0.4132989133	0.4313471215	1.665141881
0.6	0.007722067282	-0.003893773855	0.06419186619	0.3229361344	0.4047881975	1.821766348
0.7	0.01034009237	-0.004143750243	0.07570344179	0.2368206680	0.3847033272	2.131008939
0.8	0.01364394818	-0.003749149357	0.09245567552	0.1543966875	0.3701434641	2.780506453
0.9	0.01817497953	-0.002450832431	0.1236780387	0.07539208860	0.3607832511	4.651627246

Table 6. Elements of Fisher information matrix for $\theta = 7, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0002823117738	0.0004203566713	0.01734636143	0.8478296257	0.6708335613	2.190260261
0.2	0.0009236781893	-0.0005898925174	0.02732805090	0.7229752917	0.5659397975	1.924088605
0.3	0.001808295502	-0.001921644057	0.03464331964	0.6123279555	0.4977427144	1.804204573
0.4	0.002911273296	-0.003186101995	0.04101851253	0.5106314551	0.4499184222	1.780167678
0.5	0.004242536938	-0.004196586911	0.04740461416	0.4152714910	0.4149837810	1.842187489
0.6	0.005838705140	-0.004830319178	0.05462193332	0.3247881231	0.3890940087	2.009714171
0.7	0.007773464916	-0.004971950918	0.06380111271	0.2383242995	0.3702647300	2.350427858
0.8	0.01019693695	-0.004469152737	0.07728252584	0.1553985184	0.3578067088	3.077474331
0.9	0.01349007457	-0.003046982500	0.1027218224	0.07583986251	0.3526848888	5.207690781

Table 7. Elements of Fisher information matrix for $\theta = 8, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0002333333569	0.00003986584524	0.01637207362	0.8476959192	0.6636725704	2.475623568
0.2	0.0007480239649	-0.001122372099	0.02499534382	0.7238756031	0.5517626357	2.130300146
0.3	0.001447570100	-0.002464331640	0.03104360021	0.6140250797	0.4813020512	1.973629108
0.4	0.002312586423	-0.003666986785	0.03621143029	0.5127511173	0.4329327117	1.932646943
0.5	0.003350331724	-0.004585881808	0.04136251467	0.4174781762	0.3982066543	1.990540942
0.6	0.004587767290	-0.005125379178	0.04720291375	0.3268088152	0.3729360189	2.166005083
0.7	0.006079065336	-0.005189521465	0.05468275444	0.2399475209	0.3550585065	2.532020349
0.8	0.007934518503	-0.004641093244	0.06576124815	0.1564754917	0.3440400112	3.322848148
0.9	0.01043403519	-0.003213091859	0.08687691004	0.07630544318	0.3416844277	5.669062378

Table 8. Elements of Fisher information matrix for $\theta = 9, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0001969582261	-0.0002013069657	0.01548645801	0.8478592144	0.6542761573	2.739078420
0.2	0.0006203890354	-0.001420306220	0.02298552881	0.7250726713	0.5368193078	2.313872242
0.3	0.001189073751	-0.002719456017	0.02803974627	0.6159440015	0.4648685066	2.121324836
0.4	0.001887600246	-0.003837278801	0.03228726638	0.5150204000	0.4163291196	2.063777357
0.5	0.002721402165	-0.004661929606	0.03651015126	0.4197845326	0.3819482863	2.116937031
0.6	0.003710939505	-0.005118542817	0.04132036274	0.3289021251	0.3572644580	2.298300320
0.7	0.004897360936	-0.005126969491	0.04752634213	0.2416327707	0.3401474937	2.685080642
0.8	0.006364469408	-0.004564914482	0.05679132387	0.1576094712	0.3301554604	3.529280423
0.9	0.008324955185	-0.003181965919	0.07460152306	0.07681090201	0.3296382989	6.058113337

Table 9. Elements of Fisher information matrix for $\theta = 10, \phi = 1$

α	E_1	E_2	E_3	E_4	E_5	E_6
0.1	0.0001690344007	-0.0003581496733	0.01468043546	0.8482200001	0.6436808592	2.983193893
0.2	0.0005243152318	-0.001582921816	0.02124032350	0.7264369088	0.5218473128	2.478558081
0.3	0.0009967889979	-0.002816748997	0.02550243386	0.6179605772	0.4489744542	2.251445037
0.4	0.001573942318	-0.003845384938	0.02903445657	0.5173321076	0.4005244323	2.177968107
0.5	0.002259896050	-0.004582265700	0.03254386511	0.4221024237	0.3665770755	2.226129153
0.6	0.003070547674	-0.004968493263	0.03656456291	0.3309982921	0.3424546350	2.411948158
0.7	0.004037975058	-0.004936668870	0.04179203075	0.2433281709	0.3259708889	2.816069208
0.8	0.005227582464	-0.004378005621	0.04965571637	0.1587678045	0.3167375079	3.705602609
0.9	0.006805385993	-0.003061134193	0.06488578025	0.07734708050	0.3174574530	6.390824281

We hope these numbers will be of use to the practitioners of the mixed Pareto distributions. Similar tabulations could be easily derived for other values of θ and ϕ by using the LerchPhi(.) function in MAPLE.

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APPENDIX

We need the following technical lemmas to calculate the elements of the Fisher information matrix.

Lemma 1. The Lerch function defined by (2) can be represented as

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}\exp(-vt)}{1-z\exp(-t)}dt. \tag{4}$$

Proof: see Equation (3), page 27 of Erdelyi [6].

Lemma 2. For a random variable X with the pdf (1),

$$E\left[\frac{(\log X)^m X^{2\theta+\phi-b+1}}{\left\{\alpha\theta a^\theta X^\phi+(1-\alpha)\phi a^\phi X^\theta\right\}^2}\right]=\frac{m!(\log a)^m\Psi(m,b)}{(1-\alpha)\phi(\theta-\phi)a^{b+\phi-1}} \tag{5}$$

for $m\geq 0$ and $n\geq 0$, where

$$\Psi(m,b)=\sum_{k=0}^m \frac{1}{(m-k)!(\log a)^k(\theta-\phi)^k}\Phi\left(\frac{\alpha\theta}{(\alpha-1)\phi},k+1,\frac{b-1}{\theta-\phi}\right).$$

Proof: The expectation can be written as

$$E\left[\frac{(\log X)^m X^{2\theta+\phi-b+1}}{\left\{\alpha\theta a^\theta X^\phi+(1-\alpha)\phi a^\phi X^\theta\right\}^2}\right]$$

$$=\int_a^\infty \frac{(\log x)^m x^{-b}dx}{\alpha\theta a^\theta x^\phi+(1-\alpha)\phi a^\phi x^\theta}$$

$$=\frac{a^{1-b-\phi}}{(1-\alpha)\phi(\theta-\phi)}\int_0^\infty\left\{\frac{t}{\theta-\phi}+\log a\right\}^m\left\{1+\frac{\alpha\theta\exp(-t)}{(1-\alpha)\phi}\right\}^{-1}\exp\left\{\frac{(1-b)t}{\theta-\phi}\right\}dt, \tag{6}$$

which follows by setting $t = (\theta - \phi)\log(x/a)$. By using the binomial expansion

$$(c+d)^n = \sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k,$$

the integral in (6) can be rewritten as

$$\int_0^\infty \sum_{k=0}^{m} \binom{m}{k} (\log a)^{m-k} \left(\frac{t}{\theta-\phi}\right)^k \left\{1 + \frac{\alpha\theta\exp(-t)}{(1-\alpha)\phi}\right\}^{-1} \exp\left\{\frac{(1-b)t}{\theta-\phi}\right\} dt$$

$$= \sum_{k=0}^{m} \binom{m}{k} \frac{(\log a)^{m-k}}{(\theta-\phi)^k} \int_0^\infty t^k \left\{1 + \frac{\alpha\theta\exp(-t)}{(1-\alpha)\phi}\right\}^{-1} \exp\left\{\frac{(1-b)t}{\theta-\phi}\right\} dt. \tag{7}$$

The result in (5) is now immediate on applying (4) to the integral of (7).