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HYPERRULED SURFACES IN MINKOWSKI 4-SPACE*

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Abstract – In this paper, the time-like hyperruled surfaces in the Minkowski 4-space and their algebraic invariants are worked. Also some characteristic results are found about these algebraic invariants.

Keywords – Ruled surfaces, rulings, main curvature, scalar curvature, time-like vector

1. INTRODUCTION

The Minkowski space is the space R^4 with the Lorentzian inner product

$$g_0 = -dt^2 + dx^2 + dy^2 + dz^2$$

which is denoted by R_1^4. The representation of g_0 in the matrix form with respect to the standard basis of R_1^4 is $\eta = diag(-1,1,1,1)$. Suppose that R_1^4 is a 4-dimensional vector space over the field of real numbers. A symmetric bilinear form $\beta : R_1^4 \times R_1^4 \to R$ is called

i) positive (resp. negative), definite if and only if $\vec{\omega} \neq \vec{0}$ implies $\beta\left(\vec{\omega}, \vec{\omega}\right) > 0$ (resp. $\beta\left(\vec{\omega}, \vec{\omega}\right) < 0$) for all $\vec{\omega}$ in R_1^4,

ii) non-degenerate if and only if $\beta\left(\vec{\omega}, \vec{z}\right) = 0$ for all $\vec{z}$ in R_1^4, implying that $\vec{\omega} = \vec{0}$, and

iii) indefinite if and only if there exists $\vec{\omega}$ and $\vec{z}$ in R_1^4 such that $\beta\left(\vec{\omega}, \vec{\omega}\right) > 0$ and $\beta\left(\vec{z}, \vec{z}\right) < 0$, [1].

A non-degenerate, symmetric bilinear form β is called a *scalar product*. A scalar product may be positive definite, negative definite or indefinite.
For an indefinite scalar product β in R_1^4, a nonzero vector $\vec{\omega}$ is said to be

i) space-like if and only if $\beta\left(\vec{\omega}, \vec{\omega}\right) > 0$,

ii) time-like if and only if $\beta\left(\vec{\omega}, \vec{\omega}\right) < 0$,

iii) null if and only if $\beta\left(\vec{\omega}, \vec{\omega}\right) = 0$.

The vector $\vec{0}$ is taken to be *space-like*. The label space-like, time-like or null is called the *causal character* of a vector. A curve is called *time-like* (or *space-like*) *curve* if the tangent vector at every point of the curve is a time-like (or space-like) vector. A surface is called *time-like surface* if each

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tangential bundle of the surface is a time-like subspace of R_1^4, [1]. A ruled surface is a surface swept out by a straight line ℓ moving along a curve α. Such a surface has a parametrization in the ruled form

$$\varphi(t,v)=\alpha(t)+ve_1(t),$$

where α is the *base curve* and e_1 is the *director vector* of ℓ. The various positions of the generating line ℓ are called the *rulings* of the surface. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a *developable* or *cylindrical* surface. All other ruled surfaces are called *skew* surfaces [2].

2. TIME-LIKE RULED SURFACES

Let

$$\begin{aligned} \alpha : I &\to R_1^4 \\ t &\to \alpha(t)=(\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t)) \end{aligned} \tag{1}$$

be a differentiable time-like curve in the Minkowski space, where $0 \in I$.
A space-like straight line,

$$\begin{aligned} \ell : R &\to R_1^4 \\ v &\to \ell(v)=\alpha(t)+ve_1(t); \end{aligned} \tag{2}$$

where $e_1(t)$ is the director vector of ℓ at the point $\alpha(t)$ such that $e_1(t)$ and the tangent vector of α are linearly independent at every point of the curve α. Since ℓ is a space-like straight line $<e_1,e_1>=1$, and $\dot{e}_1$ denotes the derivative of the vector field e_1 along the curve α, we have $<\dot{e}_1,e_1>=0$.

When ℓ moves along α, it generates a ruled surface given by the chart $(I\times R,\varphi)$, where

$$\begin{aligned} \varphi : I\times R &\to R_1^4 \\ (t,v) &\to \varphi(t,v)=\alpha(t)+ve_1(t). \end{aligned} \tag{3}$$

This ruled surface will be denoted by *M*. Taking the derivatives of φ with respect to *t* and *v*, we have

$$\varphi_t=\dot{\alpha}(t)+v\dot{e}_1(t) \text{ and } \varphi_v=e_1(t).$$

Note that $rank[\varphi_t,\varphi_v]=rank[\dot{\alpha}+v\dot{e}_1,e_1]=2$
So *M* is 2-manifold in the Minkowski space R_1^4.

3. TIME-LIKE HYPERRULED SURFACES IN THE MINKOWSKI SPACE R_1^4

Throughout this section we assume that

$$1\le i,j\le 2 \text{ and } 0\le m,n\le 2.$$

Let *M* be a time-like ruled surface in R_1^4, with a base curve α and the generating line ℓ. If we take the space-like plane $E_2(t)$ with spanning by the vectors $e_i(t)$, instead of the generating line ℓ, then M is a 3-manifold in R_1^4. In this case M is called a *hyperruled surface* and can be (locally) represented by the chart (U,φ), where $U = I \times R^2$ and

$$\begin{aligned} &\varphi : I \times R^2 \to R_1^4 \\ &\quad (t,v) \to \varphi(t,v) = \alpha(t) + v^i e_i(t), \qquad v = (v^1, v^2). \end{aligned} \tag{4}$$

Suppose that the base curve α is an orthogonal trajectory of the generating plane $E_2(t)$. If

$$rank[e_0, e_1, e_2, \dot{e}_1, \dot{e}_2] = 4 - k \tag{5}$$

Then

i) if $k = 0$ in (5), then M is called non-developable,
ii) if $k = 1$ in (5), then M is called developable,
where e_0 is the unit tangent vector field of the base curve α, which is a time-like curve, and $\dot{e}_i$ is the derivative of the vector fields e_i along α.

We begin with some properties of a general pseudo-Riemann manifold M. Suppose that $\overline{D}$ is the Levi-Civita connection on R_1^4, while D is the Levi-Civita connection of M. Then, for any vector fields X, Y on M, we have the Gauss equation:

$$\overline{D}_X Y = D_X Y + V(X,Y) \tag{6}$$

where V is the second fundamental form of M.

If the ξ is the unit normal vector field on M, we have the Weingarten equation giving the tangential and normal components of $\overline{D}_X \xi$;

$$\overline{D}_X \xi = -A_\xi(X) + D_X^\perp \xi, \tag{7}$$

where A_ξ is determined at each point of a self-adjoint linear map on $\chi(M)$, and $D^\perp$ is a metric connection in the normal bundle of M [3].

Let $X, Y \in \chi(M)$ and $\xi \in \chi(M^\perp)$. Then, by combining (6), (7) and the Minkowski inner product on R_1^4, denoted by $<.,.>$, yield that

$$< V(X,Y), \xi > = < Y, A_\xi(X) >. \tag{8}$$

Assume that $\{e_0, e_1, e_2\}$ is an orthonormal base field of the tangential bundle of M and ξ is the unit normal vector field of M. Then we have the following Weingarten equation

$$\overline{D}_{e_m} \xi = a_m^n e_n + b_m \xi, \tag{9}$$

where the Einstein summation is used. a_m^n's are coefficients of the matrix A_ξ, and

$$a_m^n = < \overline{D}_{e_m} \xi, e_n > = - < \xi, \overline{D}_{e_m} e_n >.$$

Since the generating space $E_2(t)$ of M is a space-like subspace in R_1^4, we have that $< e_i, e_j > = \delta_{ij}$ and $\overline{D}_{e_i} e_j = 0$, which imply that $a_i^j = 0$ and

$$a_0^n = < \overline{D}_{e_0} \xi, e_n > = - < \xi, \overline{D}_{e_0} e_n > = - < \xi, \dot{e}_n > = -a_n,$$

so we may write the matrix A_ξ as

$$A_\xi = \begin{bmatrix} a_0 & -a_1 & -a_2 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{bmatrix}.$$

Lemma 3. 1. Consider the orthonormal base fields e_0, e_1, e_2 of *M*. Then the Riemannian curvature $\kappa_\sigma(e_i, e_0)$ in the two-dimensional direction σ of $\chi(M)$, spanned by the vector fields e_i and e_0, is given by

$$\kappa_\sigma(e_i, e_0) = -< \overline{D}_{e_i} e_0, \overline{D}_{e_i} e_0 > . \tag{10}$$

Proof: Suppose that R is the curvature tensor of *M*, then

$$\kappa_\sigma(e_i, e_0) =< e_i, R(e_i, e_0)e_0 > .$$

But we see from the Gauss equation that

$$< e_i, R(e_i, e_0)e_0 > =< V(e_i, e_i), V(e_0, e_0) > - < V(e_i, e_0), V(e_i, e_0) >$$

and we know that $V(e_i, e_i) = 0$. Moreover, we have

$$< \overline{D}_{e_i} e_0, e_j > =< e_0, \overline{D}_{e_i} e_j > = 0 \Rightarrow \overline{D}_{e_i} e_0 \perp e_j$$

and

$$< \overline{D}_{e_i} e_0, e_0 > =< e_0, \overline{D}_{e_i} e_0 > = 0 \Rightarrow \overline{D}_{e_i} e_0 \perp e_0 .$$

This means that $\overline{D}_{e_i} e_0$ is a normal vector field or

$$\overline{D}_{e_i} e_0 = V(e_i, e_0) \tag{11}$$

which completes the proof.

4. THE ALGEBRAIC INVARIANTS OF THE HYPERRULED SURFACES IN THE SPACE R_1^4

Let *M* be a time-like hyperruled surface in the Minkowski 4-space R_1^4. Then the space of tangent vector fields of *M* denoted by $\chi(M)$, is a time-like vector subspace of R_1^4 over the field of real numbers. Let *A* be linear operator on $\chi(M)$. A characteristic value of *A* is a scalar λ in R such that there exists a non-zero vector field X in $\chi(M)$, with $A(X) = \lambda X$, where X is called the *characteristic vector* of A corresponding to λ. The set of all X's is called the *characteristic space* of A.

The function $f(\lambda) = \det(A - \lambda \in)$ is called the characteristic polynomial of *A*, where $\in = diag(-1,1,1)$ is the matrix of the induced metric on $\chi(M)$. In order to find the roots of the characteristic polynomial we must solve the characteristic equation $\det(A - \lambda \in) = 0$

$$\begin{vmatrix} a_0 + \lambda & -a_1 & -a_2 \\ a_1 & -\lambda & 0 \\ a_2 & 0 & -\lambda \end{vmatrix} = (a_0 + \lambda)\lambda^2 - a_1^2\lambda - a_2^2\lambda = 0$$

or

$$\lambda\left(\lambda^2 + a_0\lambda - a_1^2 - a_2^2\right) = 0.$$

This implies that
$\lambda = 0$ and $\lambda^2 + a_0\lambda - \left(a_1^2 + a_2^2\right) = 0$
Since $\Delta = a_0^2 + 4\left(a_1^2 + a_2^2\right) > 0,$ the solution of the characteristic equation are

$$\lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}\left(a_0 + \sqrt{\Delta}\right) \text{ and } \lambda_3 = \frac{1}{2}\left(-a_0 + \sqrt{\Delta}\right).$$

Thus we may give the following result:

Result 4. 1. Let *M* be a time-like hyperruled surface in R_1^4. If $\lambda_2 = \lambda_3$, then *M* is minimal and developable.

Proof: Let $\lambda_2 = \lambda_3$, then $\Delta = 0$, which implies that $a_0 = a_1 = a_2 = 0.$
Thus $a_0 = 0$ implies that $trA_\xi = 0$, and so *M* is minimal.
By lemma 1, $a_i = 0$ implies that $\kappa(e_i, e_0) = 0$ and so *M* is developable.

Let us find the characteristic vector corresponding to characteristic values $\lambda_1, \lambda_2, \lambda_3$ of the matrix *A*. The vector field X_1 corresponding to λ_1 is obtained by the solution of the equation

$$AX_1 = 0 \Leftrightarrow X_1(t) = \left(0, t, -\frac{a_1}{a_2}t\right).$$

Similarly, the vector fields X_2 and X_3, corresponding to λ_2 and λ_3, are obtained by the solutions of the equations

$$AX_2 = \lambda_2 X_2 \Leftrightarrow X_2(t) = \left(t, -\frac{a_1}{\lambda_2}t, -\frac{a_2}{\lambda_2}t\right)$$

and

$$AX_3 = \lambda_3 X_3 \Leftrightarrow X_3(t) = \left(t, -\frac{a_1}{\lambda_3}t, -\frac{a_2}{\lambda_3}t\right).$$

Since the vector fields $X_k(t),\ k = 1,2,3$ have one arbitrary parameter, the dimension of the characteristic space is equal to 1. Therefore, we can choose an orthonormal base field $\phi = \left\{\overline{X}_1, \overline{X}_2, \overline{X}_3\right\}$ of $\chi(M)$ corresponding to characteristic values $\lambda_1, \lambda_2, \lambda_3$.
If we denote the matrix of the linear map *A* with respect to the orthogonal base ϕ by S, then we observe that

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

S is called as the *Weingarten* (or *Shape*) *operator* of *M* with respect to the base ϕ.
Thus we can state the following results:

Result 4. 2. Let *M* be a time-like hyperruled surface in R_1^4, and S be the shape operator of *M*. Then

i) The main curvature of *M* is $\|H\| = -\frac{a_0}{3}$.

ii) The Gauss curvature of *M* is $\kappa = 0$.

Definition 4. 1. Let *M* be a time-like hyperruled surface with curvature tensor R in R_1^4. If $\{e_0, e_1, e_2\}$ is an orthonormal base field of $\chi(M)$, then the *Ricci curvature tensor* S is defined by

$$S : \chi(M) \times \chi(M) \to R$$
$$(X,Y) \to S(X,Y) = \sum_m \varepsilon_m < R(e_m, X)Y, e_m >$$

where

$$\varepsilon_m = \begin{cases} -1, & m = 0 \\ 1, & m = 1,2 \end{cases}.$$

The *scalar curvature* of *M* is defined by

$$r = \sum_m S(e_m, e_m),$$

and the *scalar normal curvature* of *M* is defined by

$$r_n = \sum_{\sigma,\nu} M\left(A\xi_\sigma A\xi_\nu - A\xi_\nu A\xi_\sigma\right); \ \sigma,\nu \in \{1,2\}, \text{[4]}.$$

Thus we can find the following results for the time-like hyperruled surfaces in the Minkowski 4-space R_1^4 :

Result 4. 3. Let *M* be a time-like hyperruled surface with a base curve α and the generating space $E_2(t)$ spanning by the vectors $e_i(t)$ in the Minkowski 4-space R_1^4. Then the scalar curvature of *M* is

$$r = -2\sum_i a_i^2,$$

where $a_i =< \xi, \dot{e}_i >$ and $\xi \in \chi(M^\perp)$.

Proof: Let $\{e_0, e_1, e_2\}$ be an orthonormal base field of *M*. Then

$$r = \sum_m S(e_m, e_m) = S(e_0, e_0) + \sum_i S(e_i, e_i)$$

$$S(e_0, e_0) = \sum_m < R(e_m, e_0)e_0, e_m >$$

$$= \sum_i < R(e_i, e_0)e_0, e_i >$$

$$= \sum_i \kappa(e_i, e_0) = -\sum_i a_i^2$$

$$S(e_i, e_i) = \sum_m < R(e_m, e_i)e_i, e_m > = \kappa_\sigma(e_i, e_0) = -a_i^2$$

which implies that $S(e_0, e_0) = -\sum_i S(e_i, e_i)$.

So we have $r = -2\sum_i S(e_i, e_i) = -2\sum_i a_i^2$.

Thus we derive the following results for a time-like hyperruled surface in R_1^4 :

i) i) $r = 0$ if *M* is developable,

ii) ii) $r = 0$ and *M* is minimal if *M* is hyperplane,

iii) The scalar normal curvature of *M* is always zero.

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