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"Research Note"

INTEGRAL INEQUALITIES FOR SUBMANIFOLDS OF HESSIAN MANIFOLDS WITH CONSTANT HESSIAN SECTIONAL CURVATURE*

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Abstract – In this paper, we obtain two intrinsic integral inequalities of Hessian manifolds.

Keywords – Hessian manifolds, Hessian sectional curvature

1. INTRODUCTION

We will use the same notation and terminologies as in [1] unless otherwise stated. Let *M* be a flat affine manifold with flat affine connection *D.* Among Riemannian metrics on *M* there exists an important class of Riemannian metrics compatible with the flat affine connection *D*. A Riemannian metric *g* on *M* is said to be Hessian metric if *g* is locally expressed by $g = D^2 u$, where *u* is a local smooth function. We call such a pair (*D*, *g*) a Hessian structure on *M* and a triple (*M, D, g*) a Hessian manifold. The geometry of Hessian manifold is deeply related to Kaehlerian geometry and affine differential geometry.

Let *M* be a Hessian manifold with Hessian structure (*D, g*). We express various geometric concepts for the Hessian structure (*D, g*) in terms of the affine coordinate system $\{x^1, \cdots, x^{n+1}\}$ with respect to *D,* i.e $D\,dx^i = 0$.

i) The Hessian metric;

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

ii) Let γ be a tensor field of type (1, 2) defined by

$$\gamma(X, Y) = \nabla_X Y - D_X Y$$

where ∇ is the Riemannian connection for *g*. Then we have

$$\gamma^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k},$$

$$\gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k},$$

$$\gamma_{ijk} = \gamma_{jik} = \gamma_{kji}$$

where Γ^i_{jk} are the Christoffel 's symbols of ∇.

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iii) Define a tensor field *S* of type (1, 3) by

$$S = D\gamma$$

and call it the Hessian curvature tensor for (*D, g*). Then we have

$$S^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k},$$

$$S^i_{jkl} = \frac{1}{2}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^r} - \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s},$$

$$S_{ijkl} = S_{ilkj} = S_{kjil} = S_{jilk} = S_{klij}.$$

iv) The Riemannian curvature tensor for ∇ ;

$$R^i_{jkl} = \gamma^i_{rk}\gamma^r_{jl} - \gamma^i_{rl}\gamma^r_{jk},$$

$$R_{ijkl} = \frac{1}{2}(S_{jikl} - S_{ijkl}).. \quad (1)$$

Definition 1. 1. Let ς be an endomorphism of the space of contravariant symmetric tensor fields of degree 2 defined by

$$\varsigma(\xi)^{ik} = S^{i\ k}_{\ j\ l}\xi^{jl}$$

Then ς is a symmetric operator.

Definition 1. 2. For a non-zero contravariant symmetric tensor ξ_x of degree at *x* we set

$$h(\xi_x) = \frac{\langle \varsigma(\xi_x), \xi_x \rangle}{\langle \xi_x, \xi_x \rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x.

Theorem 1. 1. Let *(M, D, g)* be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h(\xi_x)$ depends only on *x*, then (*M, D, g*) is of constant Hessian sectional curvature. *(M, D, g)* is of constant Hessian sectional curvature *c* if and only if

$$S_{ijkl} = \frac{c}{2}(g_{ij}g_{kl} + g_{il}g_{kj}) \quad (2)$$

Corollary 1. 1. If a Hessian manifold (*M, D, g)* is a space of constant Hessian sectional curvature *c*, then the Riemannian manifold *(M, g)* is a space of constant sectional curvature $-\frac{c}{4}$.

2. LOCAL FORMULAS

Let M' be an n-dimensional Riemannian manifold immersed in *M*. M' is called a hypersurface.

We choose a local field of Riemannian orthonormal frames $e_1, \cdots, e_{n+1}$ in *M* such that, restricted to M', $e_1, \cdots, e_n$ are tangent to M'. Let $w_1, \cdots, w_{n+1}$ be its dual frame field such that the Riemannian metric of *M* is given by

$$ds^2 = \sum (w_A)^2$$

Then the structure equations of *M* are given by [2].

$$dw_A = -\sum w_{AB} \wedge w_B \qquad w_{AB} + w_{BA} = 0 \tag{3}$$

$$dw_{AB} = -\sum w_{AC} \wedge w_{CB} + \frac{1}{2}\sum K_{ABCD} w_C \wedge w_D \tag{4}$$

$$K_{ABCD} = -\frac{c}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) \tag{5}$$

We restrict these forms to M', then

$$w_{n+1} = 0 \tag{6}$$

and the Riemannian metric of M' is written as $ds^2 = \sum (w_i)^2$. Since $0 = dw_{n+1} = -\sum w_{n+1,i} \wedge w_i$, by Cartan's lemma we may write

$$w_{n+1,i} = \sum h_{ij} w_j \,,\ h_{ij} = h_{ji} \tag{7}$$

From these formulas we obtain the structure equation of M'

$$dw_i = -\sum w_{ij} \wedge w_j \,,\ w_{ij} + w_{ji} = 0 \,, \tag{8}$$

$$dw_{ij} = -\sum w_{ik} \wedge w_{kj} + \frac{1}{2}\sum R'_{ijkl} w_k \wedge w_l \,, \tag{9}$$

$$R'_{ijkl} = \frac{c}{4}(g_{il} g_{kj} - g_{jl} g_{ik}) - (h_{ik} h_{jl} - h_{il} h_{jk}) \tag{10}$$

where R'_{ijkl} are the components of the curvature tensor of M'.
We call

$$h = \sum_{i,j} h_{ij} w_i \otimes w_j$$

the second fundamental form of M'. The square length of *h* is defined by

$$S = \sum_{i,j} (h_{ij})^2 \tag{11}$$

The mean curvature *H* of M' is defined by

$$H = \frac{1}{n}\sum_i h_{ii} \tag{12}$$

If M' is minimal, then

$$\sum_i h_{ii} = 0 \tag{13}$$

Let h_{ijk} and h_{ijkl} denote the covariant derivative of h_{ij}, respectively defined by

$$\sum h_{ijk} w_k = dh_{ij} - \sum h_{ik} w_{kj} - \sum h_{jk} w_{ki} \,, \tag{14}$$

$$\sum h_{ijkl} w_l = dh_{ijk} - \sum h_{ijl} w_{lk} - \sum h_{ilk} w_{lj} - \sum h_{ljk} w_{li} \,. \tag{15}$$

then we have

$$h_{ijk} - h_{ikj} = 0 \,, \tag{16}$$

$$h_{ijkl} - h_{ijlk} = \sum h_{im} R'_{mjkl} + \sum h_{km} R'_{mikl} \text{. [3, 4]} \tag{17}$$

The Laplacian Δh_{ij} of h_{ij} is defined as $\sum h_{ijkk}$ and from (13), (16) and (7) we have

$$\Delta h_{ij} = \sum h_{im} R'_{mkjk} + \sum h_{km} R'_{mijk} \tag{18}$$

We proved the following theorems for Hessian manifolds by using the method of Cao [5].

Theorem 2. 1. Let a Hessian manifold (*M, D, g*) be a space of constant Hessian sectional curvature c and the Riemannian manifold (*M, g*) be a space of constant sectional curvature $-\frac{c}{4}$. If M' is an n- dimensional compact minimal hypersurface in *M*, then

$$\int_{M'} \left\{ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \sum \left(R'_{jm} \right)^2 - \frac{ncR'}{4} \right\} * 1 \leq 0 \tag{19}$$

where $\sum \left(R'_{mijk} \right)^2$ is the square length of the Riemannian curvature tensor, $\sum \left(R'_{jm} \right)^2$ is the square length of Ricci tensor, and R' the scalar curvature of M, and $*1$ is the volume element of M'.

Proof: From (13) and (18)

$$\sum h_{ij} \Delta h_{ij} = \sum h_{ij} h_{mk} R'_{mijk} + \sum h_{ij} h_{im} R'_{mkjk}$$

$$= \frac{1}{2} \sum \left(h_{ij} h_{mk} - h_{mj} h_{ik} \right) R'_{mijk} + \sum \left(h_{ij} h_{im} - h_{jm} h_{ii} \right) R'_{mj}$$

$$= -\frac{c}{4} \left[\left(\frac{1}{2} \sum \left(\delta_{mk} \delta_{ij} - \delta_{mj} \delta_{ik} \right) \right) R'_{mijk} + \sum \left(\delta_{ij} \delta_{im} - \delta_{mj} \delta_{ii} \right) R'_{mkjk} \right]$$

$$+ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \sum \left(R'_{jm} \right)^2$$

$$= \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \sum \left(R'_{jm} \right)^2 - \frac{c}{4} n R' .$$

Since $\int_{M'} \left\{ \sum h_{ij} \Delta h_{ij} \right\} * 1 \leq 0$ [4], we have $\int_{M'} \left\{ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \sum \left(R'_{jm} \right)^2 - \frac{ncR'}{4} \right\} * 1 \leq 0$. Theorem 2.1 is proved.

Theorem 2. 2. Let a Hessian manifold (*M, D, g*) be a space of constant Hessian sectional curvature *c* and the Riemannian manifold (*M, g*) be a space of constant sectional curvature $-\frac{c}{4}$. If M' is an n-dimensional compact minimal hypersurface in *M*, then

$$\int_{M'} \left\{ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \frac{1}{n} S^2 + \frac{ncS}{4} - \frac{c^2}{16} n (n-1)^2 - \frac{cS}{2} \right\} * 1 \leq 0 \tag{20}$$

where $\sum \left(R'_{mijk} \right)^2$ is the square length of the Riemann curvature tensor, S is the square length of the second fundamental form of M' and $*1$ is the volume element of M'.

Proof: From (10) and Lemma 1 in [5]

$$R'_{mj} = \frac{c}{4} (n-1) \delta_{mj} + \sum h_{km} h_{kj}$$

Diagonolize the second fundamental form so that $h_{ij} = \lambda_i \delta_{ij}$, then from (19) we have

$$\sum \left(R'_{mj} \right)^2 = \frac{c^2}{16} n(n-1)^2 + 2(n-1)\frac{c}{4}\mathrm{S} + \sum \lambda_k^4$$

and we use Lemma 1 in [5]

$$\sum \left(R'_{mj} \right)^2 = \frac{c}{4}(n-1)\left[\frac{nc}{4} + 2\mathrm{S}\right] + \frac{1}{n}\mathrm{S}^2$$

Therefore, from Theorem 2.1

$$\int_{M'} \left\{ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \frac{1}{n}\mathrm{S}^2 + \frac{nc\mathrm{S}}{4} - \frac{c^2}{16} n(n-1)^2 - \frac{cS}{2} \right\} * 1 \leq 0 .$$

Theorem 2. 3. Let a Hessian manifold (*M, D, g*) be a space of constant Hessian sectional curvature *c* and the Riemannian manifold (*M, g*) be a space of constant sectional curvature $-\frac{c}{4}$. If M' is an n-dimensional compact minimal hypersurface in *M*, then M' is totally geodesic if and only if

$$\int_{M'} \left\{ \frac{1}{2} \sum \left(R'_{mijk} \right)^2 + \frac{1}{n}\mathrm{S}^2 + \frac{nc\mathrm{S}}{4} - \frac{c^2}{16} n(n-1)^2 - \frac{cS}{2} \right\} * 1 = 0 .$$

Proof: According to Theorem 2.2 if M' is totally geodesic i.e., S=0, $h_{ij} = 0$ then from (10),

$$\sum \left(R'_{mijk} \right)^2 = -\frac{c^2}{8} n(n-1)$$

In this case (19) becomes an equality, then S=0, M' is totally geodesic.

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