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A SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY INVOLVING QUASI POWER INCREASING SEQUENCES*

E. SAVAS

Department of Mathematics, Yuzuncu Yil University, Van, Turkey
Email: ekremsavas@yahoo.com

Abstract – We obtain sufficient conditions for the series $\sum a_n \lambda_n$ to be absolutely summable of order k by a triangular matrix.

Keywords – absolute summability, weighted mean matrix, cesaro matrix, summability factor

1. INTRODUCTION

Quite recently, Bor and Ozarslan [1] obtained sufficient conditions for $\sum a_n \lambda_n$ to be summable $\left|\bar{N}, p_n\right|_k, k \geq 1$. Unfortunately, they used an incorrect definition of absolute summability (see, e.g. [2]). In this paper we obtain the corresponding result for triangular matrix using the correct definition of absolute summability. We obtain the correct form of [1] as a corollary.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$T_n := \sum_{v=0}^{n} t_{nv} s_v .$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{1}$$

We may associate with T two lower triangular matrices, $\overline{T}$ and $\widehat{T}$, defined as follows:

$$\overline{t}_{nv} = \sum_{r=v}^{n} t_{nr} \quad n, v = 0,\ 1,\ 2\ ,...,$$

and

$$\hat{t}_{nv} = \overline{t}_{nv} - \overline{t}_{n-1,v}, \quad n = 1,\ 2,\ 3,... .$$

We may write

$$T_n = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i .$$

Thus

$$T_n - T_{n-1} = \sum_{i=0}^{n} \overline{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i \lambda_i$$

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$$=\sum_{i=0}^{n}\overline{a}_{ni}a_i\lambda_i-\sum_{i=0}^{n}\overline{a}_{n-1,i}a_i\lambda_i$$

$$=\sum_{i=0}^{n}(\overline{a}_{ni}-\overline{a}_{n-1,i})a_i\lambda_i$$

$$=\sum_{i=0}^{n}\widehat{a}_{ni}a_i\lambda_i \quad =\sum_{i=1}^{n}\widehat{a}_{ni}\lambda_i(s_i-s_{i-1})$$

$$=\sum_{i=1}^{n}\widehat{a}_{ni}\lambda_i s_i-\sum_{i=1}^{n}\widehat{a}_{ni}\lambda_i s_{i-1}$$

$$=\sum_{i=1}^{n-1}\widehat{a}_{ni}\lambda_i s_i+a_{nn}\lambda_n s_n-\sum_{i=1}^{n}\widehat{a}_{ni}\lambda_i s_{i-1}$$

$$=\sum_{i=1}^{n-1}\widehat{a}_{ni}\lambda_i s_i+a_{nn}\lambda_n s_n-\sum_{i=0}^{n-1}\widehat{a}_{n,i+1}\lambda_{i+1}s_i$$

$$=\sum_{i=1}^{n-1}(\widehat{a}_{ni}\lambda_i-\widehat{a}_{n,i+1}\lambda_{i+1})s_i+a_{nn}\lambda_n s_n\,.$$

We may write

$$(\widehat{a}_{ni}\lambda_i-\widehat{a}_{n,i+1}\lambda_{i+1})=\widehat{a}_{ni}\lambda_i-\widehat{a}_{n,i+1}\lambda_{i+1}-\widehat{a}_{n,i+1}\lambda_i+\widehat{a}_{n,i+1}\lambda_i$$

$$=(\widehat{a}_{ni}-\widehat{a}_{n,i+1})\lambda_i+\widehat{a}_{n,i+1}(\lambda_i-\lambda_{i+1})$$

$$=\lambda_i\Delta_i\widehat{a}_{ni}+\widehat{a}_{n,i+1}\Delta\lambda_i\,.$$

Therefore

$$T_n-T_{n-1}=\sum_{i=1}^{n-1}\Delta_i\widehat{a}_{ni}\lambda_i s_i+\sum_{i=1}^{n-1}\widehat{a}_{n,i+1}\Delta\lambda_i s_i+a_{nn}\lambda_n s_n\,.$$

$$=T_{n1}+T_{n2}+T_{n3}\text{, say.}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

A positive sequence $\{\gamma_n\}$ is said to be a quasi $\beta-$power increasing sequence if there exists a constant $K=K(\beta,\gamma)\geq 1$ such that $Kn^{\beta}\gamma_n\geq m^{\beta}\gamma_m$ holds for all $n\geq m\geq 1$.

It should be noted that every almost increasing sequence is a quasi $\beta-$ power increasing sequence for any non-negative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n=n^{-\beta}$ for $\beta>0$.

2. MAIN RESULT

Theorem 1. Let A be a lower triangular matrix with non-negative entries satisfying

(i) $\overline{a}_{n0}=1$, n = 0,1,...,

(ii) $a_{n-1,v}\geq a_{nv}$ for $n\geq v+1$, and

(iii) $na_{nn}=O(1)$.

Let $\{X_n\}$ be a quasi β – power increasing sequence for some $0<\beta<1$, and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that

(iv) $|\Delta\lambda_n| \leq \beta_n$,

(v) $\lim \beta_n = 0$,

(vi) $\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$, and

(vii) $|\lambda_n| X_n = O(1)$

if

(viii) $\sum_{n=1}^{m} \frac{1}{n}|s_k|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ is summable $|A|_k, k \geq 1$.
We need the following lemma for the proof of our theorem.

Lemma ([3]). Under the conditions on $\{X_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ as taken in the statement of the theorem, the following conditions hold when (vi) is satisfied :

(a) $n\beta_n X_n = O(1)$ and

(b) $\sum_{n=1}^{\infty} \beta_n X_n < \infty$.

Proof of Theorem 1. The complete proof is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \text{ for } r=1,\ 2,\ 3.$$

From the definition of $\widehat{A}$ and using (i) and (ii);

$$\begin{aligned}\widehat{a}_{n,i+1} &= \overline{a}_{n,i+1} - \overline{a}_{n-1,i+1}\\ &= \sum_{v=i+1}^{n} a_{nv} - \sum_{v=i+1}^{n-1} a_{n-1,v}\\ &=1- \sum_{v=0}^{i} a_{nv} - 1 + \sum_{v=0}^{i} a_{n-1,v}\\ &= \sum_{v=0}^{i}(a_{n-1,v} - a_{nv}) \geq 0. \qquad (2)\end{aligned}$$

From (vii), it follows that $\lambda_n = O(1)$. Using Hölder's inequality and (iii)

$$I_1 := \sum_{n=1}^{m+1} n^{k-1}|T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1}(\sum_{i=1}^{n-1}|\Delta_i\widehat{a}_{ni}|\ |\lambda_i|\ |s_i|)^k$$

$$\leq \sum_{n=1}^{m+1} n^{k-1}(\sum_{i=1}^{n-1}|\Delta_i\widehat{a}_{ni}|\ |\lambda_i|^k|s_i|^k)x(\sum_{i=1}^{n-1}|\Delta_i\widehat{a}_{ni}|)^{k-1}.$$

$$\Delta_i \widehat{a}_{ni} = \widehat{a}_{n,i} - \overline{a}_{n,i+1}$$
$$= \overline{a}_{ni} - \overline{a}_{n-1,i} - \overline{a}_{n,i+1} + \overline{a}_{n-1,,i+1}$$
$$= a_{ni} - a_{n-1,i} \quad \leq 0.$$

Thus using (ii),

$$\sum_{i=0}^{n-1} |\Delta_i \widehat{a}_{ni}| = \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nn} = a_{nn}. \tag{3}$$

Using (iv), (vii), (viii) and the condition (b) of Lemma.

$$I_1 := O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \widehat{a}_{ni}| \; |\lambda_i|^k |s_i|^k$$
$$:= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=1}^{n-1} |\Delta_i \widehat{a}_{ni}| \; |\lambda_i| \; |\lambda_i|^{k-1} |s_i|^k$$
$$:= O(1) \sum_{i=1}^{m} |\lambda_i| \; |s_i|^k \sum_{n=1+1}^{m+1} (na_{nn})^{k-1} |\Delta_i \widehat{a}_{ni}|$$
$$:= O(1) \sum_{i=1}^{m} |\lambda_i| \; |s_i|^k a_{ii}$$
$$:= O(1) \sum_{i=1}^{m} |\lambda_i| \left(\sum_{r=1}^{i} |s_r|^k a_{rr} - \sum_{r=1}^{i-1} |s_r|^k a_{rr} \right)$$
$$:= O(1) \left[\sum_{i=1}^{m} |\lambda_i| \sum_{r=1}^{i} |s_r|^k a_{rr} - \sum_{j=0}^{m-1} |\lambda_{j+1}| \sum_{r=1}^{j} |s_r|^k a_{rr} \right]$$
$$:= O(1) \left[\sum_{i=1}^{m-1} \Delta |\lambda_i| \sum_{r=1}^{i} |s_r|^k a_{rr} + |\lambda_m| \sum_{r=1}^{m} |s_r|^k a_{rr} \right]$$
$$:= O(1) \left[\sum_{i=1}^{m-1} \Delta |\lambda_i| \sum_{r=1}^{i} \frac{1}{r} |s_r|^k + |\lambda_m| \sum_{r=1}^{m} \frac{1}{r} |s_r|^k \right]$$
$$:= O(1) \left[\sum_{i=1}^{m-1} \beta_i X_i + |\lambda_m| X_m \right]$$
$$:= O(1).$$

Using Hölder's inequality,

$$I_2 = \sum_{n=1}^{m+1} n^{k-1} |T_{n2}| \leq \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=1}^{n-1} \widehat{a}_{n,i+1} s_i \Delta\lambda_i \right|^k$$
$$\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i| \; |\Delta\lambda_i| \right)^k$$
$$\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i| \; \beta_i \right)^k$$

$$\leq \quad \sum_{n=1}^{m+1} n^{k-1} \Big(\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} |s_i|^{\ k} \beta_{\mathrm{i}} \Big) x \Big(\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \beta_{\mathrm{i}} \Big)^{k-1}.$$

It is easy to see that

$$\sum_{i=1}^{n-1} \widehat{a}_{n,i+1} \beta_{\mathrm{i}} \leq \mathrm{M}\, a_{nn} \tag{4}$$

as in [4].
Using (iii)

$$I_2 = \mathrm{O}(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} \widehat{a}_{n,i+1} |s_i|^k \beta_i$$

$$= \mathrm{O}(1) \sum_{i=1}^{m} \beta_i \ |s_i|^k \sum_{n=1+1}^{m+1} (na_{nn})^{k-1} \widehat{a}_{n,i+1}$$

$$= \mathrm{O}(1) \sum_{i=1}^{m} \beta_i \ |s_i|^k \sum_{n=1+1}^{m+1} \widehat{a}_{n,i+1}.$$

From (2)

$$\sum_{n=i+1}^{m+1} \Big(\sum_{v=0}^{i} (a_{n-1,v} - a_{nv}) \Big) = \sum_{v=0}^{i} \sum_{n=i+1}^{m+1} (a_{n-1,v} - a_{nv})$$

$$= \sum_{v=0}^{i} (a_{iv} - a_{m+1,v})$$

$$\leq \sum_{v=0}^{i} a_{iv} = 1. \tag{5}$$

Using (v), (vi) and (viii)

$$I_2 : \ = \mathrm{O}(1) \sum_{i=1}^{m} \beta_i |s_i|^k = O(1) \sum_{i=1}^{m} i\beta_i \frac{1}{i} |s_i|^k$$

$$= \mathrm{O}(1) \sum_{i=1}^{m} i\beta_i \left[\sum_{r=1}^{i} \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right]$$

$$= \mathrm{O}(1) \left[\sum_{i=1}^{m} i\beta_i \sum_{r=1}^{i} \frac{|s_r|^k}{r} - \sum_{j=1}^{m-1} (j+1)\beta_{j+1} \sum_{r=1}^{j} \frac{|s_r|^k}{r} \right]$$

$$= \mathrm{O}(1) \left[\sum_{i=1}^{m-1} \Delta(i\beta_i) \sum_{r=1}^{i} \frac{|s_r|^k}{r} + O(1) m\beta_m \sum_{i=1}^{m} \frac{|s_i|^k}{i} \right]$$

$$= \mathrm{O}(1) \sum_{i=1}^{m-1} |\Delta(i\beta_i)| X_i + O(1) m\beta_m X_m$$

$$= \mathrm{O}(1) \sum_{i=1}^{m-1} i|\Delta(\beta_i)| X_i + O(1) \sum_{i=1}^{m-1} \beta_{i+1} X_{i+1} + O(1) m\beta_m X_m$$

$$= \mathrm{O}(1),$$

again using the conditions of Lemma.
Using (iii) and (vii),

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n3}|^k \le \sum_{n=1}^{m+1} n^{k-1}|a_{nn}\lambda_n s_n|^k$$

$$=O(1)\sum_{n=1}^{m}(na_{nn})^{k-1}a_{nn}|\lambda_n|^k \; |s_n|^k$$

$$=O(1)\sum_{n=1}^{m} a_{nn}|\lambda_n|^{k-1}|\lambda_n| \; |s_n|^k$$

$$=O(1),$$

as in the proof of I_1.

Corollary 1. Let $\{p_n\}$ be a positive sequence such that $P_n = \sum_{k=0}^{n} p_k \to \infty$, and satisfies (i) $np_n = O(P_n)$.

Let $\{X_n\}$ be a quasi β – power increasing sequence for some $0 < \beta < 1$ and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that

(ii) $|\Delta\lambda_n| \le \beta_n$,

(iii) $\lim \beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$, and

(v) $|\lambda_n|X_n = O(1)$.

If

(vi) $\sum_{n=1}^{m}\frac{1}{n}|s_k|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k, k \ge 1$.

Proof: Conditions (ii), (iii), (iv), (v) and (vi) of Corollary 1 are, respectively, conditions (iv), (v), (vi), (vii) and (viii) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are
automotically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 1.

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