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THE ADOMIAN DECOMPOSITION METHOD FOR THE SOLUTION OF THE TRANSIENT ENERGY EQUATION IN ROCKS SUBJECTEDTO LASER IRRADIATION*

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Abstract – A 2-D heat conduction model has been solved by using the Adomian Decomposition Method to predict the transient temperature and heat flux distribution in a thick solid that is irradiated by a laser source. The laser source may operate in a continuous wave (CW) mode or repeated pulse (RP) mode and may have arbitrary, spatial and temporal profiles.

A generalized solution containing five-terms approximation of a rapidly convergent series is obtained. The solution is then applied to some special cases of practical interest, such as laser irradiation of sandstones and limestones. Laser drilling of geologic formations is being considered by the petroleum industry in the foreseeable future. The 2-D transient temperature distribution is presented in a graphical form and discussed. A comparison between the results obtained from the Adomian method and those obtained numerically by using the Crank-Nicholson method is also presented.

Keywords – Adomian Decomposition Method, Crank-Nicholson method, laser irradiation, energy equation

1. INTRODUCTION

During the past two decades considerable attention has been paid to the development and advancement of high energy laser power for their possible use in many industrial applications. This is mainly due to the great potential of laser machining of materials. Laser applications in engineering and industry include, among others, welding, drilling, cutting, scribing, and heat treatment.

One of the principal advantages of laser machining is its ability to cut very hard materials easily with great precision. Lasers may provide a cheaper alternative to conventional machining and have found widespread use in the industry. However, the physical phenomena involved in many laser applications are not fully understood. A better qualitative understanding of the physical mechanisms governing these phenomena will diminish the need for extensive trial and error experiments. An accurate knowledge of the physical processes involving the temperature profile and the heat flux distribution is essential for understanding the laser-drilling system.

Modeling of laser drilling, cutting and scribing has been addressed by a number of investigators. Dabby and Paek [1], Yilbas [2], Wagner [3], and Chen and Muzumder [4] presented a simple one-dimensional drilling model. They reported interesting observations concerning thermally induced effects due to the usage of high-energy intensities concentrated on a small area. Blackwell [5] studied this phenomenon and concluded that a metal explosion below the surface occurs due to the effects of these

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thermal stresses. Blackwell [5] explained this explosive material removal by stating that the maximum temperature (before the phase change occurs at the exposed surface) is located inside the body because of the heat loss to the surroundings. An analytical solution of 1-D heat conduction with a laser energy incident on a metal surface was given by Zubair and Chaudhry [6, 7]. Modest and others [8-10] presented a numerical solution to the problem and studied the effects of thermal stresses in ceramic. They concluded that micro-cracks might occur due to the high temperature gradients within the material.

Another very interesting and promising application is the combination of thermo-mechanical methods. The usage of lasers for the thermal weakening of rocks seems to be very practical in many applications such as the creation of fractures in oil and gas reservoirs and the combination of thermo-mechanical methods for rock destruction.

In most cases, there is no analytical solution for non-linear partial differential equations governing the problem. The difficulties associated with non-linearity of partial differential equations have led many investigators to either linearize the problem or use numerical methods (that eventually linearizes the problem). Both of these methods lack rigor or desired accuracy.

The method of solution used in this study is the Adomian Decomposition Method, well addressed in references [11-13]. This method has received a great deal of attention in recent years. The main advantage of this method lies within the fact that it provides the solution in a rapidly convergent series with elegantly computed terms. There is no linearization involved.

2. MODELING OF THE LASER ABLATION PROCESS

The laser-rock interaction can be divided into three main stages. The first stage is called the heating-up period during which the temperature is below the melting temperature and no melting or vaporization will occur. The solid absorbs the incident laser energy and, as a consequence, the bulk temperature increases with time. The second stage represents the melting stage, which is started as soon as the highest temperature (at the center of the laser beam, i.e. (0,0)) reaches the melting point. All the laser energy absorbed during this stage will result in melting more liquid, as well as increasing the liquid temperature. When the highest temperature reaches the vaporization temperature, the third stage starts and evaporation will occur at the liquid surface.

a) Formulation of the Problem

The origin of the z-axis is taken at the top surface of the work piece and the r-axis is taken from the centerline of the laser beam. Fig. 1 shows a schematic diagram of the physical model under study. Convective and radioactive heat losses from the top surface of the work piece were treated via interfacial heat transfer coefficients. A null heat transfer condition is set-up at the lateral boundaries to the modeled domain.

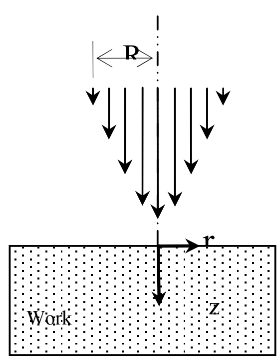

Fig. 1. Physical model

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This paper is concerned with the solution of the Fourier equation describing the heat transfer process within the material before melting starts (i.e. first stage only). By assuming a homogeneous body with an energy source term, the heat conduction equation can be written as [2, 8]:

$$\frac{\partial T}{\partial t} = \frac{k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right] + q_{Laser}}{\rho C_P} \tag{1}$$

The energy source was considered as a Gaussian laser beam which is a function of time and space, being absorbed within the material, and has the effect of an internally distributed heat source. Thus the energy source term may be modeled as;

$$q_{Laser}\,(r,z,t) = \frac{2q_0(t)\,\gamma\,(1-\Re)}{\pi\,R^2}\cdot e^{-\left[2\left(\frac{r}{R}\right)^2 + mZ\right]} \tag{2}$$

Where

$q_0(t)$ is the time dependent radiation intensity,

R is the radius of the laser focus defined by the value of laser beam intensity as:

$$I\,(r=R) = \frac{I(r=0)}{e} \tag{3}$$

γ is the absorption coefficient.

$\Re$ is the surface reflectance

The appropriate initial and boundary conditions are assumed to be the following:

At $t=0$ $\quad T(r,z,t) = Tair$

At $t > 0$ $\quad z=0 \quad -k\frac{\partial T(r,0,t)}{\partial z} = h_C\,(T(r,0,t) - Tair) + h_R\,(T(r,0,t) - T_{Sky})$

at $r \to \infty$ $\quad \frac{\partial T(r,z,t)}{\partial r} = 0$

at $z \to \infty$ $\quad \frac{\partial T(r,z,t)}{\partial z} = 0$

Where T_{air} is the ambient air temperature, T_{sky} is the sky temperature, h_C is the convective heat transfer coefficient, and h_R is the radiative heat transfer coefficient.

b) The Adomian Decomposition Method

In this section, the solution to the partial differential equation (1) is constructed by using the Adomian Decomposition Method. To derive the canonical form of the equation which is the suitable form for applying the Adomian method, equation (1) can be re-written as:

$$L_t T(r,z,t) = \frac{k\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right] + q_{Laser}}{\rho\,C_P} \tag{4}$$

Where $L_t = \frac{\partial}{\partial t}$ is the partial derivative operator with the inverse, $L_t^{-1} = \int_0^t (.)dt.$ applying the inverse operator to equation (4), the following canonical form will be obtained;

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$$T(r,z,t) = T(r,z,0) + \alpha\int_0^t [\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}]\, dt + \beta \int_0^t e^{-\left[2\left(\frac{r}{R}\right)^2 + mZ\right]} dt \tag{5}$$

Where $T(r,z,t)$ is the initial temperature, α is the thermal diffusivity $\left(\alpha = \frac{K}{\rho\, C_P}\right)$ and β is the laser energy parameter $\left(\beta = \frac{2q_0(t)\,\gamma\,(1-\Re)}{\pi\, R^2 \rho\, C_P}\right)$; for the simplicity of the model α, $T(r,z,0)$ and β are considered constants.

The Adomian Decomposition Method considers the solution as the sum of a series, say:

$$T(r,z,t) = \sum_{n=0}^{\infty} T_n \tag{6}$$

And second term on the right hand side in (5), which depends on $T(r,z,t)$ as the sum of the following series:

$$\int_0^t [\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}] dt = \sum_{n=0}^{\infty} A_n(T_0, \cdots, T_n) \tag{7}$$

Where $A_n(T_0,\cdots,T_n)$, called the Adomian polynomial, is a function of $T_0,\cdots T_n$ and must be computed. Using an alternate Algorithm for computing Adomian polynomials [14], the Adomian procedure consists of the following scheme:

$$T_0(r,z,t) \;=\; T(r,z,0) \;+ \beta\, t\, e^{-\left[2\left(\frac{r}{R}\right)^2 + mZ\right]} \tag{8}$$

$$T_{n+1}(r,z,t) = \alpha\int_0^t [\frac{\partial^2 T_n(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T_n(r,z,t)}{\partial r} + \frac{\partial^2 T_n(r,z,t)}{\partial z^2}]dt \qquad n = 0,1,2,\cdots \tag{9}$$

From (8) and (9), the following terms of the series (5) will be derived;

$$T_0(r,z,t) = T(r,z,0) + \beta\, t e^{-\left(2\left(\frac{r}{R}\right)^2 + m\,z\right)} \tag{10}$$

$$T_1(r,z,t) = \frac{t^2}{6R^2}\left[48\alpha\beta\left(\frac{r}{R}\right)^2 - 24\alpha\beta + 3(mR)^2\right] e^{-\left(2\left(\frac{r}{R}\right)^2 + mz\right)} \tag{11}$$

$$T_2(r,z,t) = \frac{t^3}{6R^4}\left[\begin{array}{l} 128\alpha^2\beta - 512\alpha^2\beta\left(\frac{r}{R}\right)^2 + 256\alpha^2\beta\left(\frac{r}{R}\right)^4 - 16\alpha^2\beta(mR)^2 \\ +32\alpha^2\beta(mr)^2 + \alpha^2\beta(mR)^4 \end{array}\right] e^{-\left(2\left(\frac{r}{R}\right)^2 + mz\right)} \tag{12}$$

And the final equation for the temperature distribution is considered as the following approximation:

$$T(r,z,t) \;\approx\; \sum_{n=0}^{4} T_n \tag{13}$$

It should be emphasized that the solution given by equation (13) represents an approximation for the transient temperature distribution with good accuracy due to the general time dependent, especially decaying laser source. This formulation may be used to discuss several heat conduction problems arising in laser induced processing of materials. However, the main drawback of this method is its inability to accommodate for the effect of boundary conditions. Despite this shortcoming, the Adomian Decomposition Method is able to predict the transient temperature distribution with a good accuracy, especially at points away from the top surface.

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c) The Numerical Method

Equation (1), along with its initial and boundary conditions, was also solved numerically by using the Crank-Nicholson Method, in which old and new time temperature values were employed utilizing the iterative implicit method. The Crank-Nicholson Method is numerically stable for all values of the ratio $\lambda = \Delta t / (\Delta r + \Delta z)^2$, and that it converges with discrimination error $\mathbf{O}\left[(\Delta t)^2 + (\Delta r + \Delta z)^2 \right]$. Although this method has a distinct improvement in accuracy as compared with the other implicit or explicit methods, the computation is only slightly more complicated than the implicit method.

First the heat conduction equation was transformed to a finite difference form and then a Fortran Computer code was developed to solve for the transient temperature distribution.

3. RESULTS AND DISCUSSION

In this section, the preceding formulation is used to study the heat conduction phenomena in sandstones and limestones being irradiated with a laser source. Table (1) lists the thermo-physical properties of sandstones and limestones, as used in this study. Studying the temperature and heat flux distribution in such materials is of practical interest in many applications such as laser drilling, thermal weakening of rocks, and laser machining of ceramics.

Table 1. Thermo physical properties of Sandstones and Lime stones

Property	Sandstone	Limestone
Reflectivity (%)	20	30
Emissivity (%)	93	85
Absorptivity (%)	85	75
Melting Temp. ^{o}C	1540	1260
Evaporation Temp. ^{o}C	2200	2000
Density kg/m^3	2640	2710
Thermal Conductivity, W/m- ^{o}C	6.2	4.8
Specific Heat, kJ/kg-^{o}C	0.28	0.18

Fig. 2 shows the maximum temperature (at $r = 0$ and $z = 0$) as a function of time for sandstone as predicted by the Adomian method and that predicted numerically at various laser power levels. It clearly shows the dramatic rise in heating rate with increasing laser pulse intensity. However, it should be noted that the maximum temperature predicted by the numerical method was not at the surface, rather it was below the surface and inside the solid material. This is attributed to the heat loss to the surroundings. This result re-emphasizes the results achieved in previous studies [4, 5, 7].

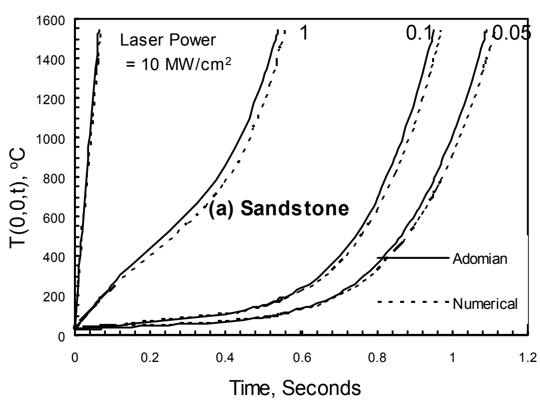

Fig. 2

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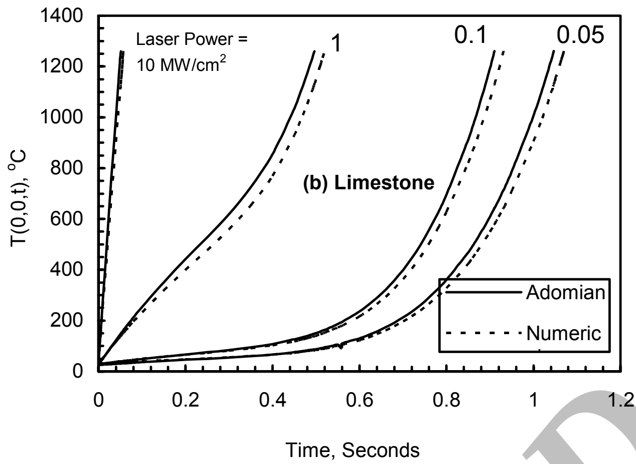

Fig.2. Maximum surface temperature as a function of time for sandstone and limestone (Comparison between Adomian and Numerical Predictions)

This phenomenon can be further manifested by studying the change in centerline temperature as a function of depth as given in Figs. 3 and 4. These Figures clearly show the reduction in temperature with increasing depth. It also shows the location of maximum temperature predicted by both methods. The Adomian method predicted the maximum temperature will be at the surface ($z = 0$) because of its inability to account for the surface heat loss. This is probably the biggest drawback of the Adomian Decomposition Method. It is clearly shown that the effect of this boundary condition diminishes as z increases. In other words, both methods predicted identical temperature distribution within the solid material and away from the top surfaces as depicted in Figs. 3 and 4. The radial graphical representation of the temperature distribution at various depths within the solid material for sandstone and limestone is shown in Figs. 5 and 6, respectively. The temperature distribution results show a very intensive temperature variation with space as a result of the high heat flux incident on the surface. These results show that the rate of surface temperature increase in limestone is higher than that for sandstone for the same laser power intensity. This is owing to the fact that less heat is being conducted through the limestone (due to its lower thermal conductivity) as compared with that of sandstone.

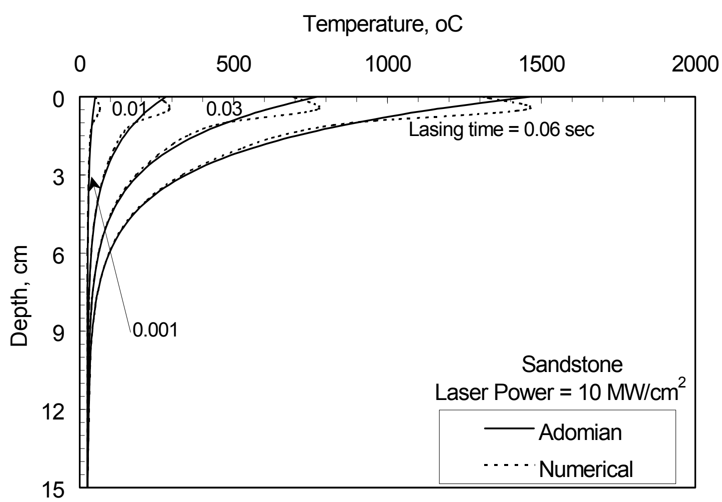

Fig. 3

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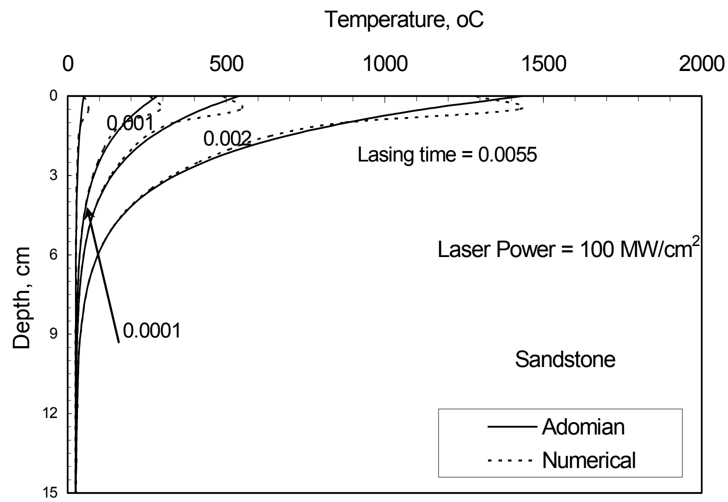

Fig. 3. Variation of centerline (r=0) temperature with depth for various lasing times and different laser power, for sandstone

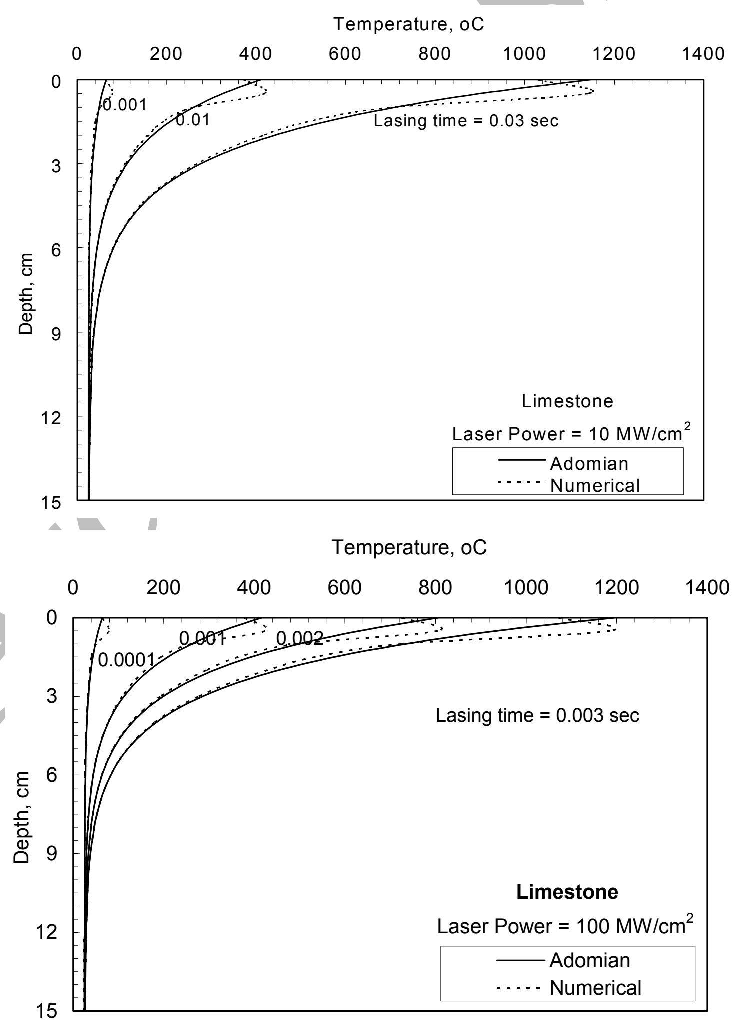

Fig. 4. Variation of centerline (r=0) temperature with depth for various lasing times and different laser power, for limestone

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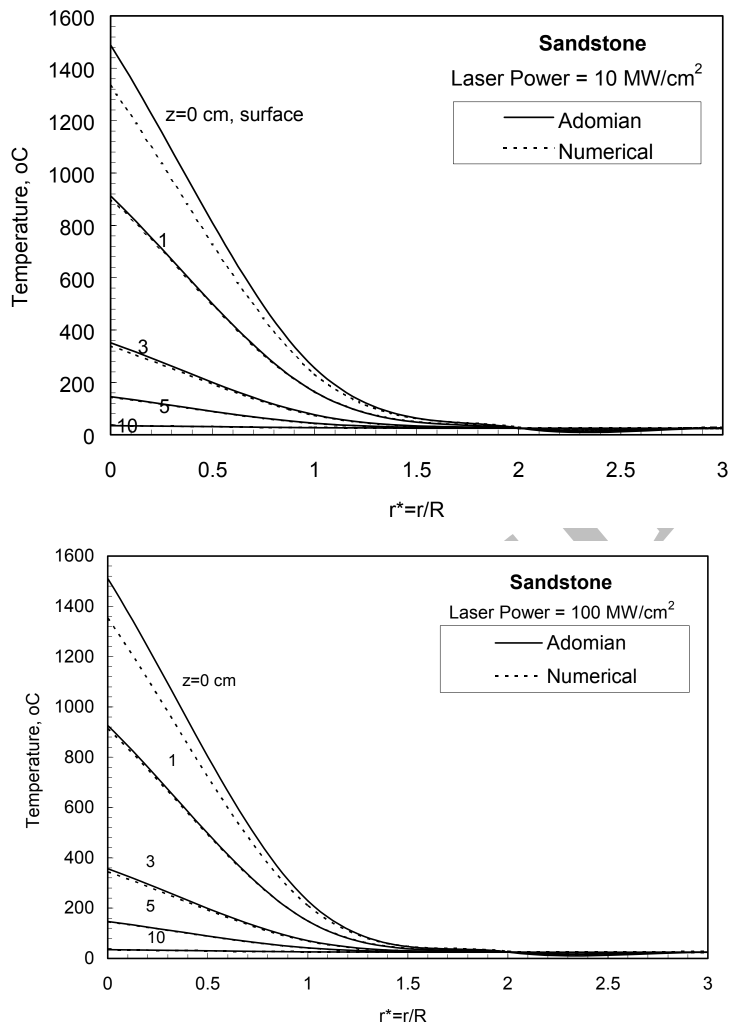

Fig. 5. Radial variation of temperature at different depths and before the start of melting, for sandstone

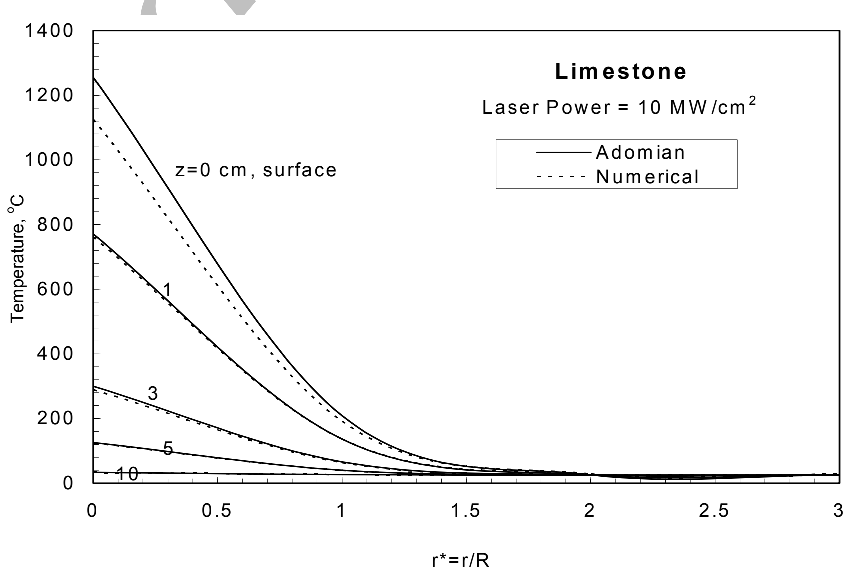

Fig. 6

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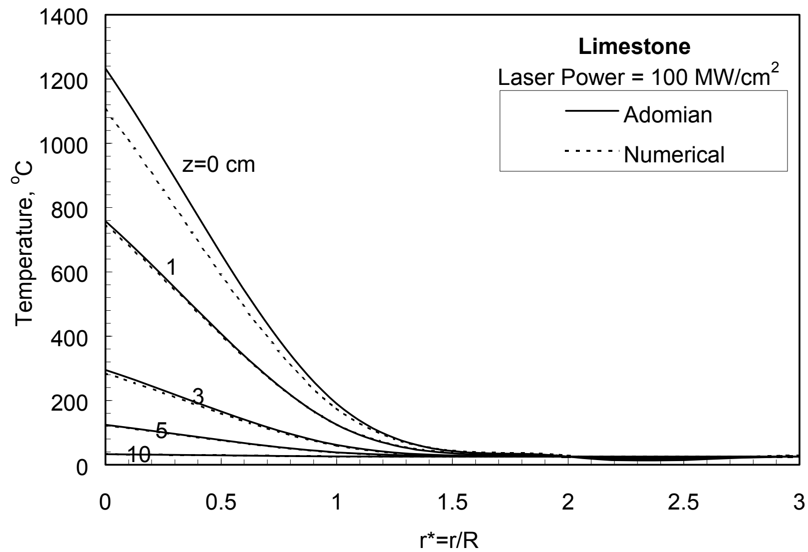

Fig. 6. Radial variation of temperature at different depths and before the start of melting, for limestone

In addition to presenting the temperature distribution, an accurate knowledge of the heat flux distribution within the solid material is also important in the investigation of the thermal behavior of the material under study. Figures 7 and 8 show the non-linear variation in conduction heat flux with depth for sandstones and limestones and under different laser power conditions. Comparing the results obtained from both methods, it can be clearly seen that the results presented in Figs. 7 and 8 re-emphasize those results given in Figs. 3 and 4 concerning the identical temperature distribution away from the surface. Due to the high heat flux encountered in this case, which results in intensive temperature distributions, the heat conduction process is non-linear as illustrated by Figs. 7 and 8. The negative heat flux values at the surface resulting from the numerical solution indicate that the heat flows in the negative z direction (heat loss to the surroundings).

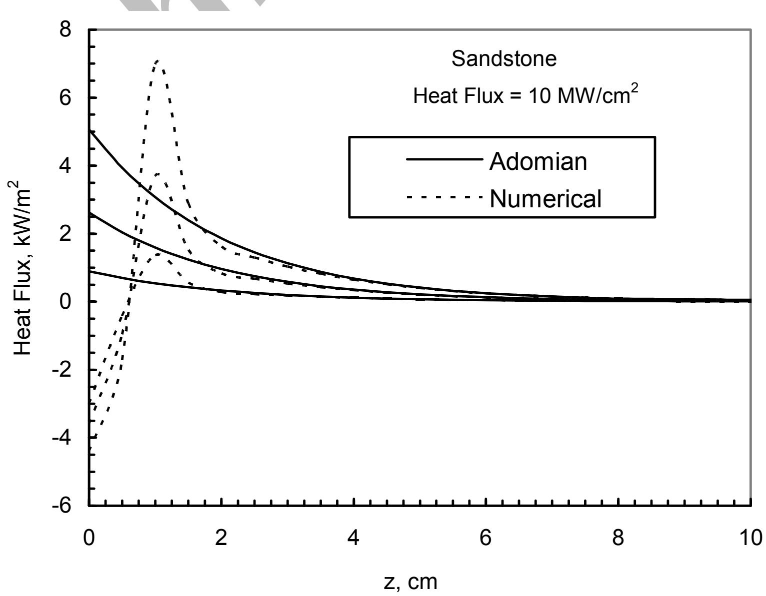

Fig. 7

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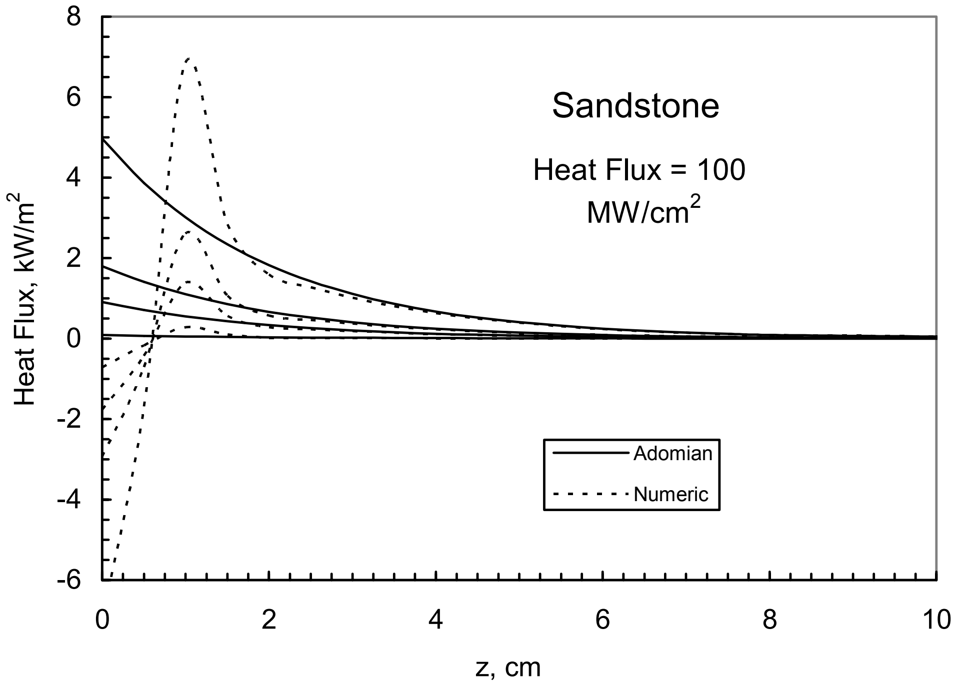

Fig. 7. Variation of conduction heat flux within the solid material with depth for various lasing times and different laser power, for sandstone

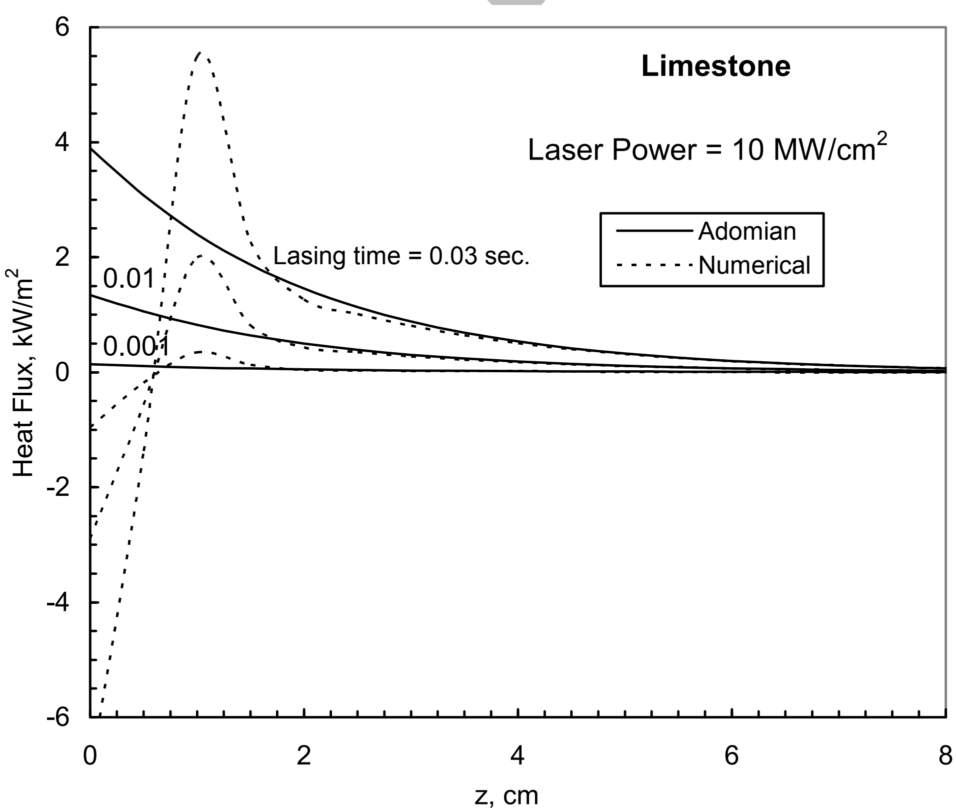

Fig. 8.

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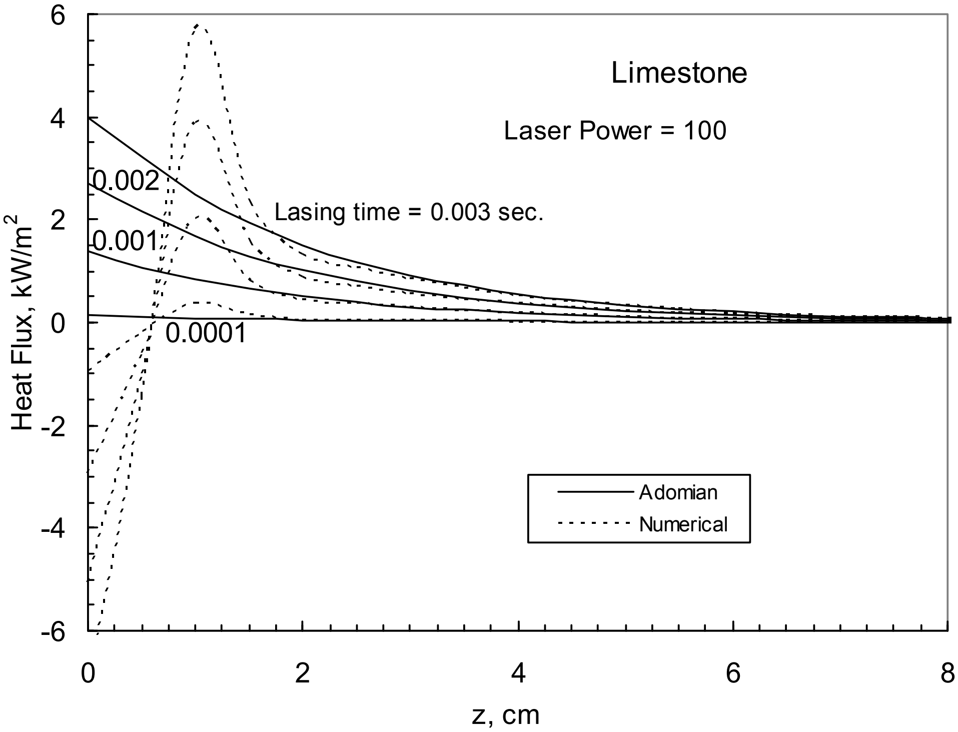

Fig. 8. Variation of conduction heat flux within the solid material with depth for various lasing times and different laser power, for limestone

4. CONCLUDING REMARKS

The objective of the present work was to apply the Adomian Decomposition Method for estimating the temperature distributions in sandstone and limestone rocks, subjected to laser irradiation. The goal has been achieved by formally deriving the solution as a five term approximation with a high degree of accuracy. The computational size was reasonable with a rapid convergence at the 5th term.

Although, the surface boundary condition was not considered in the solution obtained by the Adomian Decomposition Method (due to restrictions pertaining to the method), the results obtained at points away from the surface were in good agreement with those results obtained from the numerical method (Crank Nicholson method). This demonstrates the reliability and efficiency of the Adomian method. However, the comparison between the two methods remains under investigation and the research to modify the Adomian method to accommodate the effect of boundary conditions is underway.

The next step of this work will be the use of parametric representation of the temperature and heat flux distributions and to develop the appropriate thermal stress parameters to explain the thermal behavior of sandstones and limestones.

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