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# *N*-ARY HYPERGROUPS*

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**Abstract –** In this paper the class of *n*-ary hypergroups is introduced and several properties are found and examples are presented. *n*-ary hypergroups are a generalization of hypergroups in the sense of Marty. On the other hand, we can consider *n*-ary hypergroups as a good generalization of *n*-ary groups. We define the fundamental relation $\beta^*$ on an *n*-ary hypergroup *H* as the smallest equivalence relation such that $H/\beta^*$ is the *n*-ary group, and then some related properties are investigated.

**Keywords –** hypergroup, *n*-ary hypergroup, *n*-ary group, fundamental equivalence relation



## 1. INTRODUCTION

Hypergroup, which is based on the notion of hyperoperation, has been introduced by Marty in [1] and studied extensively by many mathematicians. For example, the connection between hypergraphs and hypergroups is studied by Corsini [2]. In [3], Corsini and Leoreanu described hypergroups associated with trees and in [4] some applications of hyperstructures in rough sets are given. The hypergroup theory both extends some well-known group results and introduces new topics, thus leading to a wide variety of applications, as well as to a broadening of the investigation fields. A comprehensive review of the theory of hyperstructures appears in [5-8].

The notion of an *n*-ary group was introduced by Dörnte [9], which is a natural generalization of the notion of a group. *n*-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various *n*-ary algebra have appeared in the literature, (for example see [10-15]).

In this paper, *n*-ary hypergroups are defined and considered. Examples of *n*-ary hypergroups are given and some of their properties described. *n*-ary hypergroups are a generalization of hypergroups in the sense of Marty. Also, we can consider *n*-ary hypergroups as a good generalization of *n*-ary groups. We define the fundamental relation $\beta^*$ on an *n*-ary hypergroup *H* as the smallest equivalence relation such that $H/\beta^*$ is the *n*-ary group, and then some related properties are investigated.

## 2. BASIC DEFINITIONS AND RESULTS

Let *H* be a non-empty set and *f* be a mapping $f : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of *H*. Then *f* is called a *binary hyperoperation* on *H*. We denote by $H^n$ the cartesian product $H \times \cdots \times H$, where *H* appears *n* times. An element of $H^n$ will be denoted by $(x_1, \cdots, x_n)$, where $x_i \in H$ for any *i* with $1 \le i \le n$. In general, a mapping $f : H^n \longrightarrow P^*(H)$ is called an *n-ary*

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*hyperoperation* and $n$ is called the *arity of hyperoperation*.

Let $f$ be an $n$-ary hyperoperation on $H$ and $A_1,\cdots,A_n$ subsets of $H$. We define

$$f(A_1,\cdots,A_n)=\bigcup\Big\{f(x_1,\cdots,x_n)\mid x_i\in A_i\ ,\ i=1,\cdots,n\Big\}.$$

We shall use the following abbreviated notation: the sequence $x_i,x_{i+1},\cdots,x_j$ will be denoted by $x_i^j$. For $j<i$, $x_i^j$ is the empty set. In this convention

$$f\big(x_1,\cdots,x_i,y_{i+1},\cdots y_j,z_{j+1},\cdots,z_n\big)$$

will be written as $f\big(x_1^i,y_{i+1}^j,z_{j+1}^n\big)$.

**Definition 2. 1.** A non-empty set $H$ with an $n$-ary hyperoperation $f:H^n\longrightarrow \mathrm{P}^*(H)$ will be called an *n-ary hypergroupoid* and will be denoted by $(H,f)$. An $n$-ary hypergroupoid $(H,f)$ will be called an *n-ary semihypergroup* if and only if the following associative axiom holds:

$$f\Big(x_1^{i-1},f\big(x_i^{n+i-1}\big),x_{x+i}^{2n-1}\Big)=f\Big(x_1^{j-1},f\big(x_j^{n+j-1}\big),x_{n+j}^{2n-1}\Big)$$

for every $i,j\in\{1,2,\cdots,n\}$ and $x_1,x_2,\cdots,x_{2n-1}\in H$.

If for all $(a_1,a_2,\cdots,a_n)\in H^n$, the set $f(a_1,a_2,\cdots,a_n)$ is singleton, then $f$ is called an *n-ary operation* and $(H,f)$ is called an *n-ary groupoid* (resp. *n-ary semigroup*).

If $m=k(n-1)+1$, then the $m$-ary hyperoperation $g$ given by

$$g\big(x_1^{k(n-1)+1}\big)=f\Big(\underbrace{f\Big(\cdots,f\big(f(x_1^n}_{k}),x_{n+1}^{2n-1}\big),\cdots\Big),x_{(k-1)(n-1)+2}^{k(n-1)+1}\Big)$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of $g$ does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(\cdot)}$, to mean $f_{(k)}$ for some $k=1,2,\cdots$.

**Definition 2. 2.** An $n$-ary semihypergroup $(H,f)$, in which the equation

$$b\in f\big(a_1^{i-1},x_i,a_{i+1}^n\big) \tag{$*$}$$

has the solution $x_i\in H$ for every $a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n,b\in H$ and $1\le i\le n$, is called an *n-ary hypergroup*.

In Definition 2.2, if $f$ is $n$-ary operation then the equation ($*$) is as follows:

$$b=f\big(a_1^{i-1},x_i,a_{i+1}^n\big). \tag{$**$}$$

In this case $(H,f)$ is an *n-ary group*.

The important question is the solvability of the equation ($*$). The classical $n$-ary semigroup is an $n$-ary group if and only if the equation ($**$) is solvable at the place $i=1$ and $i=n$, or at least one place $1<i<n$, (see [12] or [13]). The following theorem shows that it is true for hypergroups.

**Theorem 2. 3.** *Let $(H,f)$ be an n-ary semihypergroup. Then $(H,f)$ is an n-ary hypergroup if and only if the equation ($*$) is solvable at the place $i=1$ and $i=n$ or at least one place $1<i<n$.*

**Proof:** If ($*$) is solvable at the place $i=1$ and $i=n$, then for every $a_1,\cdots,a_n,b\in H$ there exist $x_0,z_0\in H$ such that

$$b \in f(x_0, a_2^n) \text{ and } x_0 \in f(a_1^{n-1}, z_0).$$

Assume that $1 < j < n$ be arbitrary. Then

$$b \in f\left(f\left(a_1^{n-1}, z_0\right), a_2^n\right) = f\left(a_1^{j-1}, f\left(a_j^{n-1}, z_0, a_2^j\right), a_{j+1}^n\right).$$

Therefore there exists $x \in f\left(a_j^{n-1}, z_0, a_2^j\right)$ such that $b \in f\left(a_1^{j-1}, x, a_{j+1}^n\right)$.

Now, let (∗) be solvable at place $1 < i < n$. Assume that $j < i$, then for every $a_1, \cdots, a_n, b \in H$ there exists $y_1 \in H$ such that

$$b \in f(a_1^{i-1}, y_1, f(\underbrace{a_1, \cdots, a_1}_{n-(i-j+1)}, a_{j+1}^{i+1}), a_{i+2}^n)$$

and so

$$b \in f(a_1^{j-1}, f(a_j^{i-1}, y_1, \underbrace{a_1, \cdots, a_1}_{n-(i-j+1)}), a_{j+1}^n).$$

Therefore there exists $x \in f\left(a_j^{i-1}, y_1, a_1, \cdots, a_1\right)$ such that $b \in f\left(a_1^{j-1}, x, a_{j+1}^n\right)$. If we choose $i < j$, then similarly we can prove that (∗) is solvable.

Definition 2.2 is a generalization of Marty's formulation of axiom of a hypergroup. Let $\circ$ be a binary algebraic hyperoperation on *H*, then $(H, \circ)$ is called a *hypergroupoid*. A *hypergroup* is a hypergroupoid $(H, \circ)$ that satisfies:

1) $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$,
2) $x \circ H = H \circ x = H$ for all $x \in H$.

The second condition is frequently used in the form: Given $a, b \in H$, there exist $u, v \in H$ such that $b \in a \circ u$ and $b \in v \circ a$.

Condition 2 can be formulated for *n*-ary hypergroups as follows:

$$f\left(H^{i-1}, x, H^{n-i}\right) = H$$

for all $x \in H$ and $i = 1, \cdots, n$.

Let $(H, f)$ be an *n*-ary hypergroup, $a_2^{n-1} \in H$ be fixed and let $x \odot y = f\left(x, a_2^{n-1}, y\right)$. Then the hypergroupoid $(H, \odot)$ is a hypergroup and it is called a *retract of the n-ary hypergroup* $(H, f)$.

**Example 2. 4.** Let $H = \{x, y, z\}$ be a set with a 3-ary hyperoperation *f* as follows:

| | | |
|---|---|---|
| $f(x,x,x) = x$ | $f(y,y,x) = \{x,z\}$ | $f(z,x,x) = z$ |
| $f(x,x,y) = y$ | $f(y,y,y) = \{y,z\}$ | $f(z,x,y) = \{y,z\}$ |
| $f(x,x,z) = z$ | $f(y,y,z) = H$ | $f(z,x,z) = \{x,y\}$ |
| $f(x,y,x) = y$ | $f(y,x,x) = y$ | $f(z,y,x) = \{y,z\}$ |
| $f(x,y,y) = \{x,z\}$ | $f(y,x,y) = \{x,z\}$ | $f(z,y,y) = H$ |
| $f(x,y,z) = \{y,z\}$ | $f(y,x,z) = \{y,z\}$ | $f(z,y,x) = H$ |
| $f(x,z,x) = z$ | $f(y,z,x) = \{y,z\}$ | $f(z,z,x) = \{x,y\}$ |
| $f(x,z,y) = \{y,z\}$ | $f(y,z,y) = H$ | $f(z,z,y) = H$ |
| $f(x,z,z) = \{x,y\}$ | $f(y,z,z) = H$ | $f(z,z,z) = \{y,z\}$. |

For every $x_i \in H$ ($i = 1, \cdots, 5$), we have

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

i.e., *f* is associative, and it is easy to see that *f* is a 3-ary hypergroup.

Let $(H,f)$ be an *n*-ary hypergroup. If the value of $f(x_1,x_2,\cdots,x_n)$ is independent on the permutation of elements $x_1,x_2,\cdots,x_n$, then $(H,f)$ is called a *commutative n-ary hypergroup*.

The element $a\in H$ is called a *scalar* if

$$\left|f\left(x_1^i,a,x_{i+2}^n\right)\right|=1$$

for all $x_1,\cdots,x_i,x_{i+2},\cdots,x_n\in H$.

Element *e* of an *n*-ary hypergroup $(H,f)$ is called a *neutral* (*identity*) element if

$$f(\underbrace{e,\cdots,e}_{i-1},x,\underbrace{e,\cdots,e}_{n-i})$$

includes *x*, for all $x\in H$ and all $1\le i\le n$.

**Lemma 2. 5.** *Let $(H,f)$ be a commutative n-ary hypergroup and $a\in H$ a scalar element such that $f(a,e,\cdots,e)=a$ for some $e\in H$. Then e is a neutral element.*

**Proof:** We have

$$f(f(x,a,\underbrace{e,\cdots,e}_{n-2}),\underbrace{e,\cdots,e}_{n-1})=f(x,f(a,\underbrace{e,\cdots,e}_{n-1}),\underbrace{e,\cdots,e}_{n-2})=f(x,a,\underbrace{e,\cdots,e}_{n-2}).$$

Since every element of *H* is representable in the form $f(x,a,e,\cdots,e)$ and *f* is commutative, this means that *e* is a neutral element.

It is to be noted that in Lemma 2.4, the condition $f(a,x,\cdots,x)=a$ can be replaced by the condition $f(x,\cdots,x,a,x,\cdots,x)=a$, where *a* appears at one fixed place $i=1,\cdots,n$.

**Proposition 2. 6.** *If the set of all scalar neutral elements of a given commutative n-ary hypergroup is non-empty, then it is an n-ary group.*

**Proof:** To prove that the set $N_H$ of all scalar neutral elements is closed under the hyperoperation *f*, let $a=f(e_1^n)$, where $e_1,\cdots,e_n\in N_H$. Then

$$\begin{aligned}
f(\underbrace{a,\cdots,a}_{i-1},x,\underbrace{a,\cdots,a}_{n-i}) &= f(\underbrace{f(e_1^n),\cdots,f(e_1^n)}_{i-1},x,\underbrace{f(e_1^n),\cdots,f(e_1^n)}_{n-i})\\
&= f(\underbrace{e_1,\cdots,e_1}_{n-1},f(\underbrace{e_2,\cdots,e_2}_{n-1},f(\cdots,f(\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-1},f(\underbrace{e_n,\cdots,e_n}_{n-1},x))\cdots)))\\
&= f(\underbrace{e_1,\cdots,e_1}_{n-1},f(\underbrace{e_2,\cdots,e_2}_{n-1},f(\cdots,f(\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-1},x)\cdots)))\\
&= \cdots = f(\underbrace{e_1,\cdots,e_1}_{n-1},x)=x,
\end{aligned}$$

which proves that an element $a=f(e_1^n)$ is neutral. Therefore, $N_H$ is closed under *f*. Also, for all $e_2,\cdots,e_n,e\in N_H$, the equation $e=f(x,e_2^n)$ has the solution

$$x=f_{(\cdot)}(e,\underbrace{e_n,\cdots,e_n}_{n-2},\underbrace{e_{n-1},\cdots,e_{n-1}}_{n-2},\cdots,\underbrace{e_3,\cdots,e_3}_{n-2},\underbrace{e_2,\cdots,e_2}_{n-2})$$

which is contained in $N_H$.

**Definition 2. 7.** Let $(H,f)$ be an *n*-ary hypergroup and *B* be a non-empty subset of *H*. Then *B* is an *n-ary* *subhypergroup* of *H* if the following conditions hold:

1) *B* is closed under the *n*-ary hyperoperation *f*, i.e., for every $(x_1,\cdots,x_n)\in B^n$ implies that $f(x_1,\cdots,x_n)\subseteq B$.

2) Equation $b\in f\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ has the solution $x_i\in B$ for every $b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,b\in B$ and $1\le i\le n$.

**Definition 2. 8.** Let $(A,f)$ and $(B,g)$ be two *n*-ary hypergroups. A *homomorphism* from *A* to *B* is a mapping $\varphi : A \longrightarrow B$ such that

$$\varphi(f(a_1,\cdots,a_n)) = g(\varphi(a_1),\cdots,\varphi(a_n))$$

holds for all $a_1,\cdots,a_n\in A$.

If $\varphi$ is injective, then it is called an *embedding*. The map $\varphi$ is an *isomorphism* if $\varphi$ is injective and onto. We say that *A* is *isomorphic* to *B*, denoted by $A\cong B$, if there is an isomorphism from *A* to *B*.

**Theorem 2. 9.** *Let $(A,f)$ and $(B,g)$ be two n-ary hypergroups and $\varphi : A \longrightarrow B$ a homomorphism. Then*

1) *If S is an n-ary subhypergroup of A, then $\varphi(S)$ is an n-ary subhypergroup of B,*

2) *If K is an n-ary subhypergroup of B such that $\varphi^{-1}(K)\neq\phi$, then $\varphi^{-1}(K)$ is an n-ary subhypergroup of A.*

**Proof:** 1) Suppose that $y_1,\cdots,y_n\in\varphi(S)$. Then there exist $x_1,\cdots x_n\in S$ such that $\varphi(x_i)=y_i$ for all $1\le i\le n$. We have $\varphi(f(x_1,\cdots,x_n))\subseteq\varphi(S)$ and so $g(\varphi(x_1),\cdots,\varphi(x_n))\subseteq\varphi(S)$ or $g(y_1,\cdots,y_n)\subseteq\varphi(S)$. Therefore the first condition of Definition 2.7 is satisfied. For the second condition of Definition 2.7, we consider the equation $b\in g\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ for all $b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,b\in\varphi(S)$.

Then there exist $a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n,a\in S$ such that $\varphi(a)=b$ and $\varphi(a_i)=b_i$. Since *S* is an *n*-ary subhypergroup of *A*, the equation

$$a\in f\left(a_1^{i-1},y_i,a_{i+1}^n\right)$$

has a solution $y_i\in S$. From the equation $a\in f\left(a_1^{i-1},y_i,a_{i+1}^n\right)$ we obtain the equation $\varphi(a)\in\varphi\left(f\left(a_1^{i-1},y_i,a_{i+1}^n\right)\right)$ or $b\in g\left(b_1^{i-1},\varphi(y_i),b_{i+1}^n\right)$. Therefore the equation $b\in g\left(b_1^{i-1},x_i,b_{i+1}^n\right)$ has the solution $\varphi(y_i)$.

2) The proof of this part is similar to (1).

## 3. QUOTIENT *N*-ARY HYPERGROUPS

Let $(H,f)$ be an *n*-ary hypergroup. An equivalence relation $\theta$ on *H* is called *compatible* if $a_1\theta b_1,\cdots,a_n\theta b_n$, then for all $a\in f(a_1,\cdots,a_n)$ there exists $b\in f(b_1,\cdots,b_n)$ such that $a\theta b$. An equivalence relation $\theta$ is called *strongly compatible* if $a_1\theta b_1,\cdots,a_n\theta b_n$ implies that $a\theta b$ for all $a\in f(a_1,\cdots,a_n)$ and $b\in f(b_1,\cdots,b_n)$.

**Theorem 3. 1.** *Let $(H,f)$ be an n-ary hypergroup and $\theta$ a compatible relation on H. Then $(H/\theta,f|_\theta)$ is an n-ary hypergroup where*

$$f|_\theta\ (\theta(a_1),\cdots,\theta(a_n)) = \{\theta(a)\mid a\in f(a_1,\cdots,a_n)\}.$$

**Proof:** We shall use the following abbreviated notation: the sequence $\theta(a_i),\theta(a_{i+1}),\cdots,\theta(a_j)$ will be denoted by $\theta_{a_i}^{a_j}$. Since $\theta$ is a compatible relation, then we conclude that $f|_\theta$ is well-defined. We show that $f|_\theta$ is associative. We have

$$f|_\theta\left(\theta_{a_1}^{a_{i-1}}, f|_\theta\left(\theta_{a_i}^{a_{n+i-1}}\right), \theta_{a_{n+i}}^{a_{2n-1}}\right) = \bigcup\left\{ f|_\theta\left(\theta_{a_1}^{a_{i-1}}, \theta(y), \theta_{a_{n+i}}^{a_{2n-1}}\right) \mid y \in f\left(a_i^{n+i-1}\right)\right\}$$

$$= \left\{ \theta(a) \mid a \in f\left(a_1^{i-1}, y, a_{n+i}^{2n-1}\right), y \in f\left(a_i^{n+i-1}\right)\right\}$$
$$= \left\{ \theta(a) \mid a \in f\left(a_1^{i-1}, f\left(a_i^{n+i-1}\right), a_{n+i}^{2n-1}\right)\right\}$$
$$= \left\{ \theta(a) \mid a \in f\left(a_1^{j-1}, f\left(a_j^{n+j-1}\right), a_{n+j}^{2n-1}\right)\right\}$$
$$= f|_\theta\left(\theta_{a_1}^{a_{j-1}}, f|_\theta\left(a_j^{n+j-1}\right), \theta_{a_{n+j}}^{a_{2n-1}}\right).$$

Therefore $f|_\theta$ is associative. Now, we consider the equation

$$\theta(b) \in f|_\theta\left(\theta_{a_1}^{a_{i-1}}, \theta(x_i), \theta_{a_{i+1}}^{a_n}\right) \qquad (*)$$

for every $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b \in H$. Since $H$ is an *n*-ary hypergroup, the equation $b \in f\left(a_1^{i-1}, x_i, a_{i+1}^n\right)$ has the solution $x_i \in H$, and so $\theta(x_i)$ is a solution for (∗).

The natural map $\pi : H \longrightarrow H/\theta$, where $\pi(x) = \theta(x)$ is an onto homomorphism.

**Definition 3. 2.** Let $(A, f)$ and $(B, g)$ be two *n*-ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then the *kernel* φ, written $ker\varphi$, is defined by

$$ker\varphi = \left\{ (a,b) \in A^2 \mid \varphi(a) = \varphi(b) \right\}.$$

It is easy to see that $ker\varphi$ is a compatible relation.

**Theorem 3. 3.** *Let $(A, f)$ and $(B, g)$ be two n-ary hypergroups and let $\varphi : A \longrightarrow B$ be a homomorphism. Then there exists a compatible relation θ on A and a monomorphism $\psi : A/\theta \longrightarrow B$ such that $\psi \circ \pi = \varphi$.*

**Proof:** We consider $\theta = ker\varphi$. Now, let $\theta(a) \in A/\theta$ and define $\psi(\theta(a)) = \varphi(a)$.

**Theorem 3. 4.** *Let ρ and θ be compatible relations on an n-ary hypergroup $(H, f)$ such that $\rho \subseteq \theta$. Then there exists a compatible relation μ on $\left(H/\rho, f|_\rho\right)$ such that $(H/\rho)/\mu$ is isomorphic to $H/\theta$.*

**Proof:** We consider the map $\varphi : H/\rho \longrightarrow H/\theta$ by $\varphi(\rho(x)) = \theta(x)$. Since $\rho \subseteq \theta$, $\varphi$ is well-defined. Clearly $\varphi$ is a homomorphism. Now, by Theorem 3.3, there exists a compatible relation $\mu$ and a monomorphism $\psi : (H/\rho)/\mu \longrightarrow H/\theta$ such that $\psi \circ \pi = \varphi$, and so $\psi$ is an isomorphism.

The diagonal relation Δ on $H$ is the set $\left\{ (a,a) \mid a \in H \right\}$ and the full relation $H^2$ is denoted by ∇. The set of all equivalence relations on a set $H$, with ⊆ as the partial ordering, is a complete lattice. Let $\theta_1$ and $\theta_2$ be two equivalence relations on $H$. It is clear that $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$. Also, we have

$$\theta_1 \vee \theta_2 = \theta_1 \bigcup (\theta_1 \circ \theta_2) \bigcup (\theta_1 \circ \theta_2 \circ \theta_1) \bigcup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \bigcup \cdots.$$

We suppose that analogous results on other products of hyperstructures can be obtained [7], [16].

**Definition 3. 5.** Let $(A_1, f_1)$ and $(A_2, f_2)$ be two *n*-ary hypergroups. Define the direct hyperproduct $(A_1 \times A_2, f_1 \times f_2)$ to be the *n*-ary hypergroup whose universe is the set $A_1 \times A_2$ and such that for $a_i \in A_1$, $a_i' \in A_2$, $1 \le i \le n$,

$$(f_1 \times f_2)((a_1,a_1'),\cdots,(a_n,a_n')) = \Big\{ (a,a') \mid a \in f_1(a_1,\cdots,a_n), a' \in f_2(a_1',\cdots,a_n') \Big\}.$$

The mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$, $i = 1,2$, defined by $\pi_i((a_1,a_2)) = a_i$, is called the *projection map* on the *i*th coordinate of $A_1 \times A_2$. For $i = 1,2$, the mapping $\pi_i : A_1 \times A_2 \longrightarrow A_i$ is an onto homomorphism. Furthermore, we have

i) $ker\,\pi_1 \bigcap ker\,\pi_2 = \Delta$,
ii) $ker\,\pi_1$ and $ker\,\pi_2$ permute,
iii) $ker\,\pi_1 \wedge ker\,\pi_2 = \nabla$,

where

$$ker\,\pi_i = \Big\{((a_1,a_2),(b_1,b_2)) \mid \pi_i(a_1,a_2) = \pi_i(b_1,b_2)\Big\}, \qquad i = 1,2.$$

Note that

$$((a_1,a_2),(b_1,b_2)) \in ker\,\pi_i \quad \Leftrightarrow \quad \pi_i((a_1,a_2)) = \pi_i((b_1,b_2)) \quad \Leftrightarrow \quad a_i = b_i.$$

Thus $ker\,\pi_1 \bigcap ker\,\pi_2 = \Delta$. Also, if $(a_1,a_2)$, $(b_1,b_2)$ are any two elements of $A_1 \times A_2$, then

$$(a_1,a_2)\ ker\pi_1\ (a_1,b_2),$$

$$(a_1,b_2)\ ker\pi_2\ (b_1,b_2),$$

so $\nabla = ker\,\pi_1 \circ ker\,\pi_2$. But, then $ker\,\pi_1$ and $ker\,\pi_2$ permute, and their joining is $\nabla$.

**Definition 3. 6.** Let $(H,f)$ be an *n*-ary hypergroup. A compatible relation $\theta$ on *H* is a *factor compatible relation* if there is a compatible relation $\theta^*$ on *H* such that $\theta \bigcap \theta^* = \Delta$, $\theta \wedge \theta^* = \nabla$ and $\theta$ permutes with $\theta^*$.

The pair $\theta$, $\theta^*$ is called a *pair of factor compatible relations* on *H.*

**Theorem 3. 7.** *If* $\theta$, $\theta^*$ *is a pair of factor compatible relations on H, then*

$$H \cong H/\theta \times H/\theta^*$$

*under the map* $\psi(a) = \big(\theta(a),\theta^*(a)\big)$.

**Proof:** If $a,b \in H$ and $\psi(a) = \psi(b)$, then $\theta(a) = \theta(b)$ and $\theta^*(a) = \theta^*(b)$, so $(a,b) \in \theta \bigcap \theta^*$; hence $a = b$. This means that $\psi$ is injective. Next, given $a,b \in H$, there is $c \in H$ such that $a\theta c$ and $c\theta^* b$, hence $\psi(c) = \big(\theta(c),\theta^*(c)\big) = \big(\theta(a),\theta^*(b)\big)$, so $\psi$ is onto. Finally, for $a_1,\cdots,a_n \in H$, we show that

$$\psi(f(a_1,\cdots,a_n)) = (f\,|_\theta \times f\,|_{\theta^*})(\psi(a_1),\cdots,\psi(a_n)).$$

We have

$$\begin{aligned}\psi(f(a_1,\cdots,a_n)) &= \left\{ \psi(a) \mid a \in f(a_1,\cdots,a_n) \right\} \\ &= \left\{ \left(\theta(a),\theta^*(a)\right) \mid a \in f(a_1,\cdots,a_n) \right\} \\ &\subseteq \left\{ \left(\theta(a),\theta^*(b)\right) \mid a \in f(a_1,\cdots,a_n), b \in f(a_1,\cdots,a_n) \right\} \\ &\subseteq f\mid_\theta (\theta(a_1),\cdots,\theta(a_n)) \times f\mid_{\theta^*} (\theta^*(a_1),\cdots,\theta^*(a_n)) \\ &= (f\mid_\theta \times f\mid_{\theta^*})\left((\theta(a_1),\theta^*(a_1)),\cdots,(\theta(a_n),\theta^*(a_n))\right) \\ &= (f\mid_\theta \times f\mid_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))\end{aligned}$$

and so $\psi(f(a_1,\cdots,a_n)) \subseteq (f\mid_\theta \times f\mid_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))$.

Conversely, suppose that $\left(\theta(x),\theta^*(y)\right) \in (f\mid_\theta \times f\mid_{\theta^*})(\psi(a_1),\cdots,\psi(a_n))$, then $\left(\theta(x),\theta^*(y)\right) \in \left\{ \left(\theta(a),\theta^*(b)\right) \mid \theta(a) \in f\mid_\theta (\theta(a_1),\cdots,\theta(a_n)),\ \theta^*(b) \in f\mid_{\theta^*} \left(\theta^*(a_1),\cdots,\theta^*(a_n)\right) \right\}$. Now there exists $c \in H$ such that $x\theta c$ and $c\theta^* y$, and so $(\theta(x),\theta(y)) = \left(\theta(c),\theta^*(c)\right)$ where $c \in f(a_1,\cdots,a_n)$. Therefore $(\theta(x),\theta(y)) \in \psi(f(a_1,\cdots,a_n))$.

## 4. FUNDAMENTAL N-ARY GROUPS

If $(H,f)$ is an *n*-ary hypergroup, then $\hat{\beta}$ denotes the transitive closure of the relation $\beta = \bigcup_{k\geq 1} \beta_k$, where $\beta_1$ is the diagonal relation, i.e., $\beta_1 = \left\{ (x,x) \mid x \in H \right\}$ and for every integer $k > 1$, $\beta_k$ is the relation defined as follows:

$$x\beta_k y \text{ if and only if } \{x,y\} \subseteq f_{(\cdot)},$$

where $f_{(\cdot)}$ means that $f_{(k)}$ for some $k = 1,2,\cdots$. When $x\beta_1 y$ (i.e., $x = y$) then we write $\{x,y\} \subseteq f_{(0)}$, we define $\beta^*$ as the smallest equivalence relation such that the quotient $\left(H/\beta^*, f/\beta^*\right)$ is an *n*-ary group, where $H/\beta^*$ is the set of all equivalence classes. The $\beta^*$ is called *fundamental equivalence relation*. The equivalence relation $\beta^*$ was first introduced on hypergroups by Koskas [17] and studied mainly by Corsini [6] concerning hypergroups, Vougiouklis [16] and Davvaz [7] concerning $H_v$-structures.

**Theorem 4. 1.** *The fundamental relation* $\beta^*$ *is the transitive closure of the relation β, i.e.,* ($\beta^* = \hat{\beta}$).

**Proof:** First we show that the quotient set $H/\hat{\beta}$ is an *n*-ary semigroup. The *n*-ary operation $f/\hat{\beta}$ in $H/\hat{\beta}$ is defined in the usual manner:

$$f/\hat{\beta}\left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right) = \left\{\hat{\beta}(y) \mid y \in \left(\hat{\beta}(x_1),\cdots,\hat{\beta}(x_n)\right)\right\}$$

for all $x_1,\cdots,x_n \in H$. Suppose $a_1 \in \hat{\beta}(x_1),\cdots,a_n \in \hat{\beta}(x_n)$. Then we have
$a_1\hat{\beta}x_1$ if there exist $x_{11},\cdots,x_{1m_1+1}$ with $x_{11} = a_1$, $x_{1m_1+1} = x_1$ such that

$$\left\{ x_{1i_1}, x_{1i_1+1} \right\} \subseteq f_{(k_1)} \qquad (0 < i_1 \leq m_1)$$

$a_n\hat{\beta}x_n$ if there exist $x_{n1},\cdots,x_{nm_n+1}$ with $x_{n1} = a_n$, $x_{nm_n+1} = x_n$ such that

$$\left\{ x_{ni_n}, x_{ni_n+1} \right\} \subseteq f_{(k_n)} \qquad (0 < i_n \leq m_n).$$

Therefore, we obtain

$$f\left(\{x_{1i_1}, x_{1i_1+1}\}, x_{21}, \cdots, x_{n1}\right) \subseteq f_{(k_1)} \qquad 1 \le i_1 \le m_1,$$
$$f\left(x_{1m_1+1}, \{x_{2i_2}, x_{2i_2+1}\}, \cdots, x_{n1}\right) \subseteq f_{(k_2)} \qquad 1 \le i_2 \le m_2,$$
$$\vdots \qquad\qquad \vdots$$
$$f\left(x_{1m_1}, , x_{21m_2+1}, \cdots, \{x_{ni_n}, x_{ni_n+1}\}\right) \subseteq f_{(k_n)} \qquad 1 \le i_n \le m_n.$$

So, every element $z \in f(x_{11}, x_{21}, \cdots, x_{n1}) = f(a_1, a_2, \cdots, a_n)$ is equivalent to every element $t \in f\left(x_{1m_1+1}, x_{2m_2+1}, \cdots, x_{nm_n+1}\right) = f(x_1, x_2, \cdots, x_n)$. Therefore

$$f / \hat{\beta}\left(\hat{\beta}(x_1), \cdots, \hat{\beta}(x_n)\right)$$

is singleton. So we can write $f / \hat{\beta}\left(\hat{\beta}(x_1), \cdots, \hat{\beta}(x_n)\right) = \hat{\beta}(y)$ for all $y \in f\left(\hat{\beta}(x_1), \cdots, \hat{\beta}(x_n)\right)$.

Moreover, since *f* is associative, it is obvious that $f / \hat{\beta}$ is associative, and consequently, $H / \hat{\beta}$ is an *n*-ary semigroup.

Now, let $\theta$ be an equivalence relation on *H* such that $H / \theta$ is an *n*-ary semigroup. Denote $\theta\langle a \rangle$ the class of *a*. Then for all $x_1, \cdots x_n \in H$, $f|_\theta\ (\theta(x_1), \cdots, \theta(x_n)) = \theta(y)$ for all $y \in f(\theta(x_1), \cdots, \theta(x_n))$. But also, for every $x_1, \cdots x_n \in H$ and $A_i \subseteq \theta(x_i)$ ($i = 1, \cdots, n$) we have

$$f|_\theta\ (\theta(x_1), \cdots, \theta(x_n)) = \theta(f(x_1, \cdots, x_n)) = \theta(f(A_1, \cdots, A_n)).$$

Therefore $\theta\langle x \rangle = \theta\left(f_{(k)}\right)$ for all $k \ge 0$ and for all $x \in f_{(k)}$. So for every $a \in H$, $x \in \beta\langle a \rangle$ implies $x \in \theta\langle a \rangle$. But $\theta$ is transitively closed, so we obtain $x \in \hat{\beta}\langle a \rangle$ implies $x \in \theta\langle a \rangle$. Hence, the relation $\hat{\beta}$ is the smallest equivalence relation on *H* such that $H / \hat{\beta}$ is an *n*-ary semigroup, i.e., $\hat{\beta} = \beta^*$.

**Theorem 4. 2.** *$\beta^*$ is a strongly compatible relation.*

**Proof:** If $a_1\beta^* b_1, \cdots, a_n\beta^* b_n$, then $\beta^*(a_1) = \beta^*(b_1), \cdots, \beta^*(a_n) = \beta^*(b_n)$. For every $a \in f(a_1, \cdots, a_n)$ and $b \in f(b_1, \cdots, b_n)$ we have

$$\begin{aligned} \beta^*\langle a \rangle &= \beta^*(f(a_1, \cdots, a_n)) \\ &= f / \beta^*\left(\beta^*(a_1), \cdots, \beta^*(a_n)\right) \\ &= f / \beta^*\left(\beta^*(b_1), \cdots, \beta^*(b_n)\right) \\ &= \beta^*(f(b_1, \cdots, b_n)) \\ &= \beta^*(b). \end{aligned}$$

**Theorem 4. 3.** *Let $(A, f)$ and $(B, g)$ be two n-ary hypergroups and let $\beta_A^*$, $\beta_B^*$ and $\beta_{A\times B}^*$ be fundamental equivalence relations on A, B and $A \times B$ respectively. Then*

$$A \times B / \beta_{A\times B}^* \cong A / \beta_A^* \times B / \beta_B^*.$$

**Proof:** First we define the relation $\tilde{\beta}$ on $A \times B$ as follows:

$$(a_1, b_1)\tilde{\beta}(a_2, b_2) \quad \Leftrightarrow \quad a_1\beta_A^* a_2 \text{ and } b_1\beta_B^* b_2.$$

$\tilde{\beta}$ is an equivalence relation. We define *h* on $A \times B / \tilde{\beta}$ as follows:

$$h\left(\tilde{\beta}(a_1, b_1), \cdots, \tilde{\beta}(a_n, b_n)\right) = \tilde{\beta}(a, b)$$

for all $a \in f\left(\beta_A^*(a_1), \cdots, \beta_A^*(a_n)\right)$, $b \in g\left(\beta_B^*(b_1), \cdots, \beta_B^*(b_n)\right)$. Since *f g* are associative, we see that *h* is associative, and consequently, $A \times B / \tilde{\beta}$ is an *n*-ary semigroup. Now let $\theta$ be an equivalence relation on $A \times B$ such that $A \times B / \theta$ is an *n*-ary group. Similar to the proof of Theorem 4.1, we get

$$(a_1, b_1)\tilde{\beta}(a_2, b_2) \quad \Rightarrow \quad (a_1, b_1)\theta(a_2, b_2).$$

Therefore the relation $\tilde{\beta}$ is the smallest equivalence relation on $A \times B$ such that $A \times B / \tilde{\beta}$ is an *n*-ary group, i.e., $\tilde{\beta} = \beta^*_{A\times B}$. Now we consider the map $\varphi : A / \beta^*_A \times B / \beta^*_B \longrightarrow A \times B / \beta^*_{A\times B}$ by

$$\varphi\left(\beta^*_A(a), \beta^*_B(b)\right) = \beta^*_{A\times B}(a, b).$$

It is easy to see that $\varphi$ is an isomorphism.

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