

AMENABILITY OF WEIGHTED MEASURE ALGEBRAS*

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Abstract – Let G be a locally compact group, and let ω be a weight on G . We show that the weighted measure algebra $M(G, \omega)$ is amenable if and only if G is a discrete, amenable group and $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$, where $\omega(g) \geq 1$ ($g \in G$).

Keywords – Amenability, measure algebra, weight

1. INTRODUCTION

Let A be a Banach algebra, and let X be a Banach A -bimodule. A derivation from A into X is a continuous linear map $D : A \rightarrow X$ satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For example, let $x \in X$, and define $\delta_x(a) = a \cdot x - x \cdot a$, $a \in A$. Then it is a derivation; maps of this form are called inner derivations. The cohomology group $H^1(A, X)$ is the quotient of the space of continuous derivations from A into X by the space of inner derivations.

Let A be a Banach algebra and let X be a Banach A -bimodule. Then the dual space X' is also a Banach A -bimodule for the products $a \cdot \lambda$ and $\lambda \cdot a$ specified by

$$a \cdot \lambda(x) = \lambda(x \cdot a), \quad \lambda \cdot a(x) = \lambda(a \cdot x) \quad (a \in A, x \in X, \lambda \in X')$$

The Banach algebra A is amenable if $H^1(A, X') = 0$ for every Banach A -bimodule X ; this definition was introduced by Johnson in [1]. In his paper Johnson showed that $L^1(G)$ is amenable if and only if G is amenable.

Grønbæk in [2] showed that $L^1(G, \omega)$ is amenable if and only if G is amenable and

$$\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty.$$

Dales et al. in [3] showed that $M(G)$ is amenable if and only if G is amenable and discrete. In this paper we will generalize this result. We will show that the weighted measure algebra $M(G, \omega)$ is amenable if and only if G is amenable, discrete and $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$, where $\omega(g) \geq 1$ ($g \in G$).

Throughout G is a locally compact group. A weight on G is a continuous function $\omega : G \rightarrow \mathbb{R}^+$ satisfying

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$$\omega(e) = 1, \quad \omega(gh) \leq \omega(g)\omega(h) \quad (g, h \in G).$$

In the case where $\omega(g) \geq 1$ ($g \in G$), we denote by $M(G, \omega)$ the Banach algebra of all complex-valued regular Borel measures μ on G such that

$$\|\mu\|_\omega = \int_G \omega(t) d|\mu|(t) < \infty.$$

Note that $M(G, \omega)$ is identified with the dual of $C_0(G, \omega^{-1})$.

The subspace of the continuous measures is denoted by $M_c(G, \omega)$. The subspace of the discrete measures is denoted by $M_d(G, \omega)$, and is identified with $\ell^1(G, \omega)$. Note that $M_d(G, \omega)$ is a closed subalgebra of $M(G, \omega)$, and $M_c(G, \omega)$ is a closed ideal of $M(G, \omega)$. We have $M(G, \omega) = M_d(G, \omega) \oplus M_c(G, \omega)$ as a Banach space. In the case where G is discrete we have $M(G, \omega) = \ell^1(G, \omega)$ and $M_c(G, \omega) = \{0\}$; if G is not discrete, then $M_c(G, \omega) \neq \{0\}$.

2. THE MAIN RESULTS

In this section we investigate the amenability of the weighted measure algebra $M(G, \omega)$ for a locally compact group G , where $\omega(g) \geq 1$ for every $g \in G$. The method that we use is the same as [3], though in order to deal with the difficulty of the weighted case our presentation differs from [3] in some cases.

The subset $\{1, \dots, n\}$ of N is denoted by N_n , and N_4^m is the set of elements (i_1, \dots, i_m) such that $i_k \in N_4$ for $k \in N_m$. Let (K_n) be a sequence of compact subsets of group H as specified in [3]. The family of sets $K_n(i_1, \dots, i_n)$ for $n \in N$ and $(i_1, \dots, i_n) \in N_4^n$ is denoted by Ω . We set $K = \bigcap \{K_n : n \in N\}$.

Lemma 2.1. Let H be a non-discrete, metrizable, locally compact group, and let ω be a weight on H with $\omega(t) \geq 1$ for every $t \in H$. Then for every $L \in \Omega$, there exists $\mu_L \in M_c(H, \omega)^+$ such that

$$\frac{1}{M_L} \leq \int_{L \cap K} \omega(t) d\mu_L(t) \leq 1 \quad \text{and} \quad \int_{H \setminus (L \cap K)} \omega(t) d\mu_L(t) = 0,$$

where $M_L = \sup \{\omega(t) : t \in L\}$.

Proof: Let $L = K_m(i_1, \dots, i_m) \in \Omega$ be fixed, where $(i_1, \dots, i_m) \in N_4^m$. Let $n \in N$ and $i \in N_4^n$. Then $\int_{K_n(i)} \omega(t) d(t) > 0$, because $\text{int} K_n(i) \neq \emptyset$ and $\omega(t) \geq 1$.

For every $n \geq m$ and $i \in N_4^n$ such that $K_n(i) \subset L$, we define

$$\mu_{n,i}(E) = \frac{\int_{E \cap K_n(i)} \omega(t) d(t)}{4^{n-m} \int_{K_n(i)} \omega(t) dt},$$

clearly $\mu_{n,i}(H) = \mu_{n,i}(K_n(i)) = 4^{m-n}$ and $\mu_{n,i}(H \setminus K_n(i)) = 0$. Thus every $\mu_{n,i}$ is a positive measure with compact support. Now for every $n \in N$ we define

$$\mu_n = \frac{1}{M_L} \sum \{\mu_{n,i} : i \in N_4^n, K_n(i) \subset L\}.$$

Since for $n \geq m$ the set $\{i \in N_4^n, K_n(i) \subset L\}$ has 4^{n-m} elements, we have $\mu_n(H) = \frac{1}{M_L}$ and

$$\begin{aligned}\mu_n(H) &\leq \int_H \omega(t) d\mu_n(t) \leq \sum_{\substack{i \in N_4^n \\ K_n(i) \subset L}} \int_{K_n(i)} \frac{\omega(t)}{M_L} d\mu_{n,i}(t) \\ &\leq \sum_{\substack{i \in N_4^n \\ K_n(i) \subset L}} \mu_{n,i}(K_n(i)) = 1.\end{aligned}$$

Thus every μ_n is a positive measure with $\frac{1}{M_L} \leq \|\mu_n\|_\omega \leq 1$. We claim that the sequence $\{\mu_n\}$ has a weak accumulation point μ_L in $M_c(H, \omega)^+$ such that $\|\mu_L\|_\omega \leq 1$. The proof of the claim is similar to [3].

Now fix $n \in N$. Let $f \in C_0(H)$ be such that $f(K_n \cap L) = \{1\}$ and $f(H) \subset [0, 1]$. Since $f\omega \in C_0(H, \omega^{-1})$ and for every $r \geq n$, $\text{supp } \mu_r \subseteq K_n \cap L$, then we have

$$\langle f\omega, \mu_r \rangle = \int_H f(t)\omega(t)d\mu_r(t) = \int_{K_n \cap L} \omega(t)d\mu_r(t) = \|\mu_r\|_\omega \quad (r \geq n).$$

Hence $\frac{1}{M_L} \leq \langle f\omega, \mu_L \rangle \leq 1$. If we consider f to be the characteristic function of an arbitrary neighborhood U of $K_n \cap L$, then we have $\frac{1}{M_L} \leq \int_U \omega(t)d\mu_L(t) \leq 1$, and since μ_L is regular we have $\frac{1}{M_L} \leq \int_{K_n \cap L} \omega(t)d\mu_L(t) \leq 1$. But $(K_n \cap L : n \in N)$ is a decreasing sequence with $\bigcap_{n=1}^\infty K_n \cap L = K \cap L$. This implies

$$\frac{1}{M_L} \leq \int_{K \cap L} \omega(t)d\mu_L(t) \leq 1 \quad \text{and} \quad \int_{H \setminus (K \cap L)} \omega(t)d\mu_L(t) = 0.$$

With the same method as in [3] one can 'lift' the above result in the case where the underlying group is not metrizable.

Theorem 2.2. Let G be a non-discrete, locally compact group. Then

1. $M_c(G, \omega)^2$ has infinite co-dimension in $M_c(G, \omega)$.
2. There is a continuous, positive linear functional Ψ on $M(G, \omega)$ and $\mu_0 \in M_c(G, \omega)^+$ with $\frac{1}{M_L} \leq \|\mu_0\|_\omega \leq 1$ such that $\langle \mu_0, \Psi \rangle = \|\mu_0\|_\omega$, $\Psi|_{M_d(G, \omega)} = 0$ and $\Psi|_{M_c(G, \omega)^2} = 0$.

Proof: Let $\mu \in M_c(G, \omega)$, and let V be a set which is defined in [3]. Define

$$E_k(\mu) = \left\{ x \in G : \int_{xV} \omega(t)d|\mu|(t) > \frac{1}{k} \right\}$$

for every $k \in N$. With the same argument as in [1], one can show that $E_k(\mu)$ is a finite set. Now we define $E(\mu) = \bigcup_{k \in N} E_k(\mu) = \left\{ x : \int_{xV} \omega(t)d|\mu|(t) > 0 \right\}$. Then $E(\mu)$ is a countable set and $\int_{xV} \omega(t)d|\mu|(t) = 0$ whenever $x \in G \setminus E(\mu)$. For every $L \in \Omega$ and every μ in $M_c(G, \omega)$, we define

$$\langle \mu, \Psi_L \rangle = \int_G \omega(t)\chi_{V_L}(t)d\mu(t).$$

It is clear that $\Psi_L \in M_c(G, \omega)^{'+}$ and

$$\|\Psi_L\| = \sup \{ |\langle \mu, \Psi_L \rangle| : \mu \in M_c(G, \omega), \|\mu\|_\omega \leq 1 \} \leq 1.$$

Let μ_L be a measure on H as in the Lemma 2.1. Then with the same method as in [3] μ_L can be transferred to a measure $\nu_L \in M_c(G, \omega)^+$ and $\frac{1}{M_L} \leq \|\nu_L\|_\omega \leq 1$. Also

$$\langle \nu_L, \Psi_L \rangle = \int_G \chi_{V_L}(t)\omega(t)d\nu_L(t) = \|\nu_L\|_\omega.$$

Let $\mu, \nu \in M_c(G, \omega)$. Since for every $x \in G$, where $x^{-1} \notin E(\nu)$, we have $\int_{x^{-1}V_L} \omega(t)d\nu(t) = 0$. Thus

$$\begin{aligned}
\langle \mu * \nu, \Psi_L \rangle &= \int_G \chi_{V_L}(t) \omega(t) d\mu * \nu(t) \\
&\leq \int_G \int_G \chi_{V_L}(ty) \omega(t) \omega(y) d\mu(t) d\nu(y) \\
&= \int_G \left\{ \int_{t^{-1}V_L} \omega(y) d\nu(y) \right\} \omega(t) d\mu(t). \\
&\leq \|\nu\|_\omega \int_{E(\nu)^{-1}} \omega(t) d\mu(t).
\end{aligned}$$

However $E(\nu)$, hence $E(\nu)^{-1}$ is a countable set. Thus $\int_{E(\nu)^{-1}} \omega(t) d\mu(t) = 0$, so $\langle \mu * \nu, \Psi_L \rangle = 0$ and this implies that $\Psi_L|_{M_c(G, \omega)^2} = 0$, so $\Psi_L|_{\overline{M_c(G, \omega)^2}} = 0$.

Now let $\{L_n : n \in N\}$ be an infinite, pairwise disjoint subfamily of Ω , and let $a_n = \langle \nu_{L_n}, \Psi_{L_n} \rangle$. Then $\frac{1}{M_L} \leq a_n \leq 1$ and

$$\langle \mu_{L_n}, \Psi_{L_m} \rangle = \int_G \omega(t) \chi_{V_{L_m}}(t) d\mu_{L_n}(t) = \begin{cases} 0 & m \neq n, \\ a_n & m = n. \end{cases}$$

Therefore the set $\{\mu_{L_n} + \overline{M_c(G, \omega)^2} : n \in N\}$ is a linearly independent subset of $M_c(G, \omega)/\overline{M_c(G, \omega)^2}$. This shows that $\overline{M_c(G, \omega)^2}$ has infinite codimension in $M_c(G, \omega)$.

The proof of (ii) is similar to the proof of [3].

Theorem 2. 3. Let G be a non-discrete, locally compact group. Then $M(G, \omega)$ is not amenable, and for every weight ω on G such that $\omega(g) \geq 1$, $g \in G$.

Proof: We recall that $M_c(G, \omega)$ is a complemented closed ideal in $M(G, \omega)$. By Theorem 1 we have $M_c(G, \omega) \neq M_c(G, \omega)^{[2]}$ and also, by [4, Theorem 2.9.58], $M(G, \omega)$ is not amenable. Our final result is as follows.

Theorem 2. 4. Let G be a locally compact group and let ω be a weight of G such that $\omega(g) \geq 1$ for all $g \in G$. Then $M(G, \omega)$ is amenable if and only if G is discrete, amenable and $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$.

Proof: If G is discrete, then $M(G, \omega) = \ell^1(G, \omega)$. So by [2] $M(G, \omega)$ is amenable if and only if G is amenable and $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. If G is not discrete, then by Theorem 2.3 $M(G, \omega)$ is not amenable.

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