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COMPACT HYPERSURFACES IN EUCLIDEAN SPACE AND SOME INEQUALITIES*

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Abstract – Let (M,g) be a compact immersed hypersurface of $\left(R^{n+1},<,>\right)$, λ_1 the first nonzero eigenvalue, α the mean curvature, ρ the support function, A the shape operator, $vol(M)$ the volume of M, and S the scalar curvature of M. In this paper, we established some eigenvalue inequalities and proved the above.

1) $\frac{1}{n}\int_M \|A\|^2 \rho^2 dv \geq \int_M \alpha^2\rho^2 dv$,

2) $\int_M \alpha^2\rho^2 dv \geq \frac{1}{n(n-1)}\int_M S\rho^2 dv$,

3) If the scalar curvature S and the first nonzero eigenvalue λ_1 satisfy $S=\lambda_1(n-1)$, then

$$\int_M \left[\alpha^2 - \frac{\lambda_1}{n}\right]\rho^2 dv \geq 0,$$

4) Suppose that the Ricci curvature of M is bounded below by a positive constant k. Thus

$$\int_M \alpha^2\rho^2 dv \geq \frac{k}{n(n-1)}\int_M \|gradf\|^2 dv + vol(M),$$

5) Suppose that the Ricci curvature is bounded and the scalar curvature satisfy $S=\lambda_1(n-1)$ and L=k-2S>0 is a constant. Thus

$$vol(M) \geq -\frac{k\lambda_1}{L}\int_M \|\psi\|^2 \alpha\rho dv - \frac{2S}{L}\int_M \alpha^2\rho^2 dv.$$

Keywords – First Eigenvalue, Support Function

1. PRELIMINARIES

We will use the same notations and terminologies as in [1] unless otherwise stated. Let M be a compact immersed hypersurface of R^{n+1}. We denote by $\Psi: M \to R^{n+1}$ the smooth immersion by $<,>$ and g, the Euclidean metric on R^{n+1} and the induced metric on M respectively. Let N be the unit normal vector field and A the shape operator on M. We then have the Gauss and Weingarten formulas

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)N \,, \overline{\nabla}_X N = -AX \,, X,Y \in \chi(M) \tag{1}$$

where $\overline{\nabla}$ and ∇ are the Riemannian connections on R^{n+1} and M respectively, $\chi(M)$ is the Lie-algebra of smooth vector fields on M and h is the second fundamental form which is related to A by $g(AX,Y)=h(X,Y)$. The shape operator A satisfies the Codazzi equation

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$$(\nabla A)(X,Y) = (\nabla A)(Y,X)\,, X,Y \in \chi(M)\,, \tag{2}$$

Since the shape operator A is symetric and satisfies (2) it can be easily verified that the mean curvature $\alpha = \frac{1}{n} trA$ satisfies

$$grad\alpha = \frac{1}{n}\sum_{i=1}^{n}(\nabla A)(e_i,e_i)\,, X(\alpha) = \frac{1}{n}\sum_{i=1}^{n} g(\nabla A(e_i,e_i),X)\,,\; X \in \chi(M) \tag{3}$$

where $\{e_1,...,e_n\}$ is a local orthonormal frame on M .

If we define $f : M \to R$ by $f = \frac{1}{2}\|\Psi\|^2$ and treat Ψ as a position vector field of M in R^{n+1}, we get

$$\Psi = gradf + \rho N \tag{4}$$

where $\rho : M \to R$, defined by $\rho =< \Psi, N >$, is a support function of M. Then, using the equations in (1), we obtain

$$\nabla_X gradf = X + \rho AX$$

and

$$X(\rho) = -\rho(AX, gradf)\,, X \in \chi(M) \tag{5}$$

From the first equation in (5) we get

$$\Delta f = n(1 + \alpha\rho) \tag{6}$$

which, on integration, yields the following formula Minkowski

$$\int_M (1 + \alpha\rho)\, dv = 0\,. \tag{7}$$

2. MAIN THEOREM

Theorem 3. 1. Let M be compact and the connected immersed hypersurface of R^{n+1}. The shape operator on M and the mean curvature α of M satisfies the following inequality:

$$\frac{1}{n}\int_M \|A\|^2 \rho^2 \geq \int_M \alpha^2\rho^2 dv \tag{8}$$

Proof: From the Gauss equation, we have the following expression for the Ricci curvature tensor of M [2].

$$Ric(X,Y) = n\alpha g(AX,Y) - g(AX,AY)\,, X,Y \in \chi(M) \tag{9}$$

Thus, we have

$$\int_M Ric(gradf, gradf)\, dv = n\int_M \alpha g(A(gradf), gradf)\, dv - \int_M \|A(gradf)\|^2\, dv \tag{10}$$

The second equation (5) gives $grad(\rho) = -A(gradf)$ and we obtain

$$\begin{aligned} g(A(gradf), gradf) &= -g(grad\rho, gradf) = -gradf(\rho) \\ &= -div(\rho gradf) + \rho\Delta f \\ &= -div(\rho gradf) + n\rho(1 + \rho\alpha)\,. \end{aligned}$$

Thus we have

$$\alpha g(A(gradf), gradf) = -\alpha div(\rho gradf) + n\alpha\rho(1 + \rho\alpha) \tag{11}$$

For a local orthonormal frame $\{e_1,...,e_n\}$ on M we also have

$$div(A(gradf)) = \sum[g((\nabla A)(e_i, gradf), e_i) + g(A(\nabla e_i gradf), e_i)]$$

which, together with (3) and (5), gives

$$div(A(gradf)) = n(gradf)\alpha + n\alpha + \rho\|A\|^2 \tag{12}$$

Using the identity $div(fX) = X(f) + fdivX$, $X \in \chi(M)$ for any smooth function $f : M \to R$, we get

$$\begin{aligned}\rho div(A(gradf)) &= div(\rho A(gradf)) - A(gradf)\rho \\ &= div(\rho A(gradf)) + \|A(gradf)\|^2.\end{aligned}$$

Combining the above equation with (11) and (12), we arrive at

$$n\rho(gradf)\alpha + n\alpha\rho + \rho^2\|A\|^2 = div(\rho A(gradf)) + \|A(gradf)\|^2 \tag{13}$$

Since $div(\alpha\rho gradf) = \rho(gradf)\alpha + \alpha div(\rho gradf)$, we can use this in (13) to get

$$-n\alpha div(\rho gradf) + div(n\alpha\rho gradf) + n\alpha\rho + \rho^2\|A\|^2 = div(\rho A(gradf)) + \|A(gradf)\|^2 \tag{14}$$

Substituting the expression for $-n\alpha div(\rho gradf)$ from (14) into (11), and using Stokes theorem, we arrive at

$$\int_M n\alpha g(A(gradf), gradf)dv = \int_M \left[\|A(gradf)\|^2 - n\alpha\rho - \rho^2\|A\|^2 + n^2\rho\alpha(1+\alpha\rho)\right]dv \tag{15}$$

Together with (15) and (10) gives

$$\int_M Ric(gradf, gradf)dv = \int_M \left[-n\alpha\rho - \rho^2\|A\|^2 + n^2\alpha\rho(1+\alpha\rho)\right]dv \tag{16}$$

From the Bochner-Lichnerowicz formula [3, 4]

$$\int_M \left[(\Delta f)^2 - \|Hessf\|^2 - Ric(gradf, gradf)\right]dv = 0 \tag{17}$$

and (16), we have

$$\int_M \left[(\Delta f)^2 - \|Hessf\|^2 + n\alpha\rho + \rho^2\|A\|^2 - n^2\alpha\rho(1+\alpha\rho)\right]dv = 0. \tag{18}$$

Newton's inequality $(\Delta f)^2 \le n\|Hessf\|^2$ yields and using the Minkowski formula (7), we have

$$\frac{1}{n}\int_M \|A\|^2\rho^2 dv \ge \int_M \alpha^2\rho^2 dv.$$

Corollary 3. 1. Let M be a compact and connected immersed hypersurface of R^{n+1}. The mean curvature α of M and the scalar curvature S of M satisfy the following inequality:

$$\int_M \alpha^2\rho^2 dv \ge \frac{1}{n(n-1)}\int_M S\rho^2 dv \tag{19}$$

Proof: From the Gauss equation, we have the following expression for the scalar curvature of M [2].

$$S = n^2\alpha^2 - \|A\|^2 \tag{20}$$

From (20) and (8) we obtain (19).

Without loss of generality we can assume that the center of the mass of M is at the origin of R^{n+1} (for otherwise an isometry $\Phi : R^{n+1} \to R^{n+1}$ can be chosen which maps the center of mass of M to the origin of R^{n+1}, and then $\Psi' = \Phi \circ \Psi$ will be the desired immersion). Thus the immersion $\Psi : M \to R^{n+1}$ satisfies $\int_M \Psi dv = 0.$ Hence we can apply the minimum principle to get

$$\lambda_1 \leq n.vol(M) \Big/ \int_M \|\Psi\|^2 dv$$

Where, λ_1 is the nonzero eigenvalue of the Laplacian operator on M. Consequently we have

$$\int_M \|\Psi\|^2 dv \leq \frac{n.vol(M)}{\lambda_1}. \tag{21}$$

Corollary 3. 2. Let M be a compact and connected immersed hypersurface of R^{n+1}. If the scalar curvature S and the first nonzero eigenvalue λ_1 of the Laplacian operator Δ on M, with respect to the induced metric, satisfy $S = \lambda_1(n-1)$, then

$$\int_M \left[\alpha^2 - \frac{\lambda_1}{n} \right] \rho^2 dv \geq 0. \tag{22}$$

Thus M is isometric to a sphere $S^n(c)$.
Proof: By the hypothesis of the theorem and (19), hence

$$\int_M \left[\alpha^2 - \frac{\lambda_1}{n} \right] \rho^2 dv \geq 0.$$

3. THE RICCI CURVATURE IS BOUNDED

Theorem 4. 1. Let M be a compact and connected immersed hypersurface of R^{n+1} with positive Ricci curvature. Suppose that the Ricci curvature of M is bounded below by a positive constant k. Thus

$$\int_M \alpha^2 \rho^2 dv \geq \frac{k}{n(n-1)} \int_M \|gradf\|^2 dv + vol(M) \tag{23}$$

Proof: From (17), Newton's inequality, (6) and by the hypothesis of theorem

$$n(n-1) \int_M (1 + \alpha\rho)^2 dv \geq k \int_M \|gradf\|^2 dv .$$

Or

$$-n(n-1)vol(M) + n(n-1) \int_M \alpha^2 \rho^2 dv \geq k \int_M \|gradf\|^2 dv \tag{24}$$

where we have used the Minkowski formula (7). Thus, we get (23).

Theorem 4. 2. Let M be a compact and connected immersed hypersurface of R^{n+1} with positive Ricci curvature. Suppose that the Ricci curvature of M is bounded below by a positive constant k. If the scalar curvature S and the first nonzero eigenvalue λ_1 of the Laplacian operator Δ on M, with respect to the induced metric satisfy $S = \lambda_1(n-1)$, and L=k-2S>0 is a constant, then

$$vol(M) \geq -\frac{k\lambda_1}{L} \int_M \|\psi\|^2 \alpha\rho dv - \frac{2S}{L} \int_M \alpha^2 \rho^2 dv. \tag{25}$$

Proof: For the immersion $\psi : M \to IR^{n+1}$ we know that the function $f = \frac{1}{2}\|\Psi\|^2$ satisfies (7). We can compute div (fgradf) to obtain

$$\text{div(fgradf)} = \|gradf\|^2 + \frac{n}{2}\|\psi\|^2(1+\alpha\rho). \quad (26)$$

Integrating this equation, we obtain

$$\int_M \|gradf\|^2 \text{dv} + \frac{n}{2}\int_M \|\psi\|^2(1+\alpha\rho)dv. \quad (27)$$

From (27), (24) and (21), we obtain (25).

Example: We can take ellipsoid

$$M = \{ (x,y,z) \in IR^3 : \frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1 \}$$

which is a compact hypersurface of IR^3, and locally express the immersion ψ as
ψ (t, θ) = (2costcosθ, 2costsinθ, sint)
Further, we can show that, on this coordinate patch of ellipsoid the shape operator A, the mean curvature α and the support function ρ are respectively given by

$$A = \begin{pmatrix} \frac{2}{\sqrt{\cos^2 t + 4\sin^2 t}} & 0 \\ 0 & \frac{1}{2\sqrt{\cos^2 t + 4\sin^2 t}} \end{pmatrix}$$

$$\alpha = \frac{5}{4\sqrt{\cos^2 t + 4\sin^2 t}} \text{ and } \rho = -\frac{2}{\sqrt{\cos^2 t + 4\sin^2 t}}$$

and consequently we arrive at

$$\frac{1}{2}\|A\|^2\rho^2 = \frac{17}{2}\frac{1}{(\cos^2 t + 4\sin^2 t)^2} > \frac{25}{4}\frac{1}{(\cos^2 t + 4\sin^2 t)^2} = \alpha^2\rho^2$$

that is

$$\frac{1}{n}\int_M \|A\|^2\rho^2 dv \geq \int_M \alpha^2\rho^2 dv\,.$$

REFERENCES

1. Deshmukh, S. (1988). Compact Hypersurfaces in a Euclidean Space. *Quart J. Math. Oxford 4*(2), 35-41.
2. Kobayashi, S. & Nomizu, K. (1969). *Foundations of Differential Geometry, vol II.* John. Wiley, New York.
3. Choı, H. I. & Waıg, A. N. (1983). A First Eigenvalue Estimate for Minimal Hypersurfaces. *J. Dif. Geo.18*, 559-562.
4. Bektas, M. & Ergüt, M. (1999)."Compact Space-Like Hypersurfaces in The De Sitter Space". *Proc. Of IMM of Azerbaijan AS, Vol X(XVIII*), 20-24.