

## COMPACT HYPERSURFACES IN EUCLIDEAN SPACE AND SOME INEQUALITIES\*

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**Abstract** – Let  $(M, g)$  be a compact immersed hypersurface of  $(R^{n+1}, \langle, \rangle)$ ,  $\lambda_1$  the first nonzero eigenvalue,  $\alpha$  the mean curvature,  $\rho$  the support function,  $A$  the shape operator,  $vol(M)$  the volume of  $M$ , and  $S$  the scalar curvature of  $M$ . In this paper, we established some eigenvalue inequalities and proved the above.

- 1)  $\frac{1}{n} \int_M \|A\|^2 \rho^2 dv \geq \int_M \alpha^2 \rho^2 dv,$
- 2)  $\int_M \alpha^2 \rho^2 dv \geq \frac{1}{n(n-1)} \int_M S \rho^2 dv,$
- 3) If the scalar curvature  $S$  and the first nonzero eigenvalue  $\lambda_1$  satisfy  $S = \lambda_1 (n-1)$ , then

$$\int_M \left( \alpha^2 - \frac{\lambda_1}{n} \right) \rho^2 dv \geq 0,$$

- 4) Suppose that the Ricci curvature of  $M$  is bounded below by a positive constant  $k$ . Thus

$$\int_M \alpha^2 \rho^2 dv \geq \frac{k}{n(n-1)} \int_M \|\text{grad} f\|^2 dv + vol(M),$$

- 5) Suppose that the Ricci curvature is bounded and the scalar curvature satisfy  $S = \lambda_1 (n-1)$  and  $L=k-2S>0$  is a constant. Thus

$$vol(M) \geq -\frac{k\lambda_1}{L} \int_M \|\psi\|^2 \alpha \rho dv - \frac{2S}{L} \int_M \alpha^2 \rho^2 dv.$$

**Keywords** – First Eigenvalue, Support Function

### 1. PRELIMINARIES

We will use the same notations and terminologies as in [1] unless otherwise stated. Let  $M$  be a compact immersed hypersurface of  $R^{n+1}$ . We denote by  $\Psi : M \rightarrow R^{n+1}$  the smooth immersion by  $\langle, \rangle$  and  $g$ , the Euclidean metric on  $R^{n+1}$  and the induced metric on  $M$  respectively. Let  $N$  be the unit normal vector field and  $A$  the shape operator on  $M$ . We then have the Gauss and Weingarten formulas

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \bar{\nabla}_X N = -AX, X, Y \in \chi(M) \quad (1)$$

where  $\bar{\nabla}$  and  $\nabla$  are the Riemannian connections on  $R^{n+1}$  and  $M$  respectively,  $\chi(M)$  is the Lie-algebra of smooth vector fields on  $M$  and  $h$  is the second fundamental form which is related to  $A$  by  $g(AX, Y) = h(X, Y)$ . The shape operator  $A$  satisfies the Codazzi equation

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$$(\nabla A)(X, Y) = (\nabla A)(Y, X), X, Y \in \chi(M), \quad (2)$$

Since the shape operator  $A$  is symmetric and satisfies (2) it can be easily verified that the mean curvature  $\alpha = \frac{1}{n} \text{tr} A$  satisfies

$$\text{grad} \alpha = \frac{1}{n} \sum_{i=1}^n (\nabla A)(e_i, e_i), X(\alpha) = \frac{1}{n} \sum_{i=1}^n g(\nabla A(e_i, e_i), X), X \in \chi(M) \quad (3)$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame on  $M$ .

If we define  $f : M \rightarrow R$  by  $f = \frac{1}{2} \|\Psi\|^2$  and treat  $\Psi$  as a position vector field of  $M$  in  $R^{n+1}$ , we get

$$\Psi = \text{grad} f + \rho N \quad (4)$$

where  $\rho : M \rightarrow R$ , defined by  $\rho = \langle \Psi, N \rangle$ , is a support function of  $M$ . Then, using the equations in (1), we obtain

$$\nabla_X \text{grad} f = X + \rho A X$$

and

$$X(\rho) = -\rho(A X, \text{grad} f), X \in \chi(M) \quad (5)$$

From the first equation in (5) we get

$$\Delta f = n(1 + \alpha \rho) \quad (6)$$

which, on integration, yields the following formula Minkowski

$$\int_M (1 + \alpha \rho) dv = 0. \quad (7)$$

## 2. MAIN THEOREM

**Theorem 3. 1.** Let  $M$  be compact and the connected immersed hypersurface of  $R^{n+1}$ . The shape operator on  $M$  and the mean curvature  $\alpha$  of  $M$  satisfies the following inequality:

$$\frac{1}{n} \int_M \|A\|^2 \rho^2 \geq \int_M \alpha^2 \rho^2 dv \quad (8)$$

Proof: From the Gauss equation, we have the following expression for the Ricci curvature tensor of  $M$  [2].

$$\text{Ric}(X, Y) = n\alpha g(AX, Y) - g(AX, AY), X, Y \in \chi(M) \quad (9)$$

Thus, we have

$$\int_M \text{Ric}(\text{grad} f, \text{grad} f) dv = n \int_M \alpha g(A(\text{grad} f), \text{grad} f) dv - \int_M \|A(\text{grad} f)\|^2 dv \quad (10)$$

The second equation (5) gives  $\text{grad}(\rho) = -A(\text{grad} f)$  and we obtain

$$\begin{aligned} g(A(\text{grad} f), \text{grad} f) &= -g(\text{grad} \rho, \text{grad} f) = -\text{grad} f(\rho) \\ &= -\text{div}(\rho \text{grad} f) + \rho \Delta f \\ &= -\text{div}(\rho \text{grad} f) + n\rho(1 + \alpha \rho). \end{aligned}$$

Thus we have

$$\alpha g(A(\text{grad} f), \text{grad} f) = -\alpha \text{div}(\rho \text{grad} f) + n\alpha \rho(1 + \alpha \rho) \quad (11)$$

For a local orthonormal frame  $\{e_1, \dots, e_n\}$  on  $M$  we also have

$$\operatorname{div}(A(\operatorname{grad} f)) = \sum [g((\nabla A)(e_i, \operatorname{grad} f), e_i) + g(A(\nabla e_i \operatorname{grad} f), e_i)]$$

which, together with (3) and (5), gives

$$\operatorname{div}(A(\operatorname{grad} f)) = n(\operatorname{grad} f)\alpha + n\alpha + \rho\|A\|^2 \quad (12)$$

Using the identity  $\operatorname{div}(fX) = X(f) + f\operatorname{div}X$ ,  $X \in \chi(M)$  for any smooth function  $f : M \rightarrow R$ , we get

$$\begin{aligned} \rho \operatorname{div}(A(\operatorname{grad} f)) &= \operatorname{div}(\rho A(\operatorname{grad} f)) - A(\operatorname{grad} f)\rho \\ &= \operatorname{div}(\rho A(\operatorname{grad} f)) + \|A(\operatorname{grad} f)\|^2. \end{aligned}$$

Combining the above equation with (11) and (12), we arrive at

$$n\rho(\operatorname{grad} f)\alpha + n\alpha\rho + \rho^2\|A\|^2 = \operatorname{div}(\rho A(\operatorname{grad} f)) + \|A(\operatorname{grad} f)\|^2 \quad (13)$$

Since  $\operatorname{div}(\alpha\rho\operatorname{grad} f) = \rho(\operatorname{grad} f)\alpha + \alpha\operatorname{div}(\rho\operatorname{grad} f)$ , we can use this in (13) to get

$$-n\alpha\operatorname{div}(\rho\operatorname{grad} f) + \operatorname{div}(n\alpha\rho\operatorname{grad} f) + n\alpha\rho + \rho^2\|A\|^2 = \operatorname{div}(\rho A(\operatorname{grad} f)) + \|A(\operatorname{grad} f)\|^2 \quad (14)$$

Substituting the expression for  $-n\alpha\operatorname{div}(\rho\operatorname{grad} f)$  from (14) into (11), and using Stokes theorem, we arrive at

$$\int_M n\alpha g(A(\operatorname{grad} f), \operatorname{grad} f) dv = \int_M [\|A(\operatorname{grad} f)\|^2 - n\alpha\rho - \rho^2\|A\|^2 + n^2\rho\alpha(1 + \alpha\rho)] dv \quad (15)$$

Together with (15) and (10) gives

$$\int_M \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) dv = \int_M [-n\alpha\rho - \rho^2\|A\|^2 + n^2\rho\alpha(1 + \alpha\rho)] dv \quad (16)$$

From the Bochner-Lichnerowicz formula [3, 4]

$$\int_M [(\Delta f)^2 - \|\operatorname{Hess} f\|^2 - \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)] dv = 0 \quad (17)$$

and (16), we have

$$\int_M [(\Delta f)^2 - \|\operatorname{Hess} f\|^2 + n\alpha\rho + \rho^2\|A\|^2 - n^2\rho\alpha(1 + \alpha\rho)] dv = 0. \quad (18)$$

Newton's inequality  $(\Delta f)^2 \leq n\|\operatorname{Hess} f\|^2$  yields and using the Minkowski formula (7), we have

$$\frac{1}{n} \int_M \|A\|^2 \rho^2 dv \geq \int_M \alpha^2 \rho^2 dv.$$

**Corollary 3. 1.** Let  $M$  be a compact and connected immersed hypersurface of  $R^{n+1}$ . The mean curvature  $\alpha$  of  $M$  and the scalar curvature  $S$  of  $M$  satisfy the following inequality:

$$\int_M \alpha^2 \rho^2 dv \geq \frac{1}{n(n-1)} \int_M S \rho^2 dv \quad (19)$$

Proof: From the Gauss equation, we have the following expression for the scalar curvature of  $M$  [2].

$$S = n^2\alpha^2 - \|A\|^2 \quad (20)$$

From (20) and (8) we obtain (19).

Without loss of generality we can assume that the center of the mass of  $M$  is at the origin of  $R^{n+1}$  (for

otherwise an isometry  $\Phi : R^{n+1} \rightarrow R^{n+1}$  can be chosen which maps the center of mass of  $M$  to the origin of  $R^{n+1}$ , and then  $\Psi' = \Phi \circ \Psi$  will be the desired immersion). Thus the immersion  $\Psi : M \rightarrow R^{n+1}$  satisfies  $\int_M \Psi dv = 0$ . Hence we can apply the minimum principle to get

$$\lambda_1 \leq n \cdot \text{vol}(M) / \int_M \|\Psi\|^2 dv$$

Where,  $\lambda_1$  is the nonzero eigenvalue of the Laplacian operator on  $M$ . Consequently we have

$$\int_M \|\Psi\|^2 dv \leq \frac{n \cdot \text{vol}(M)}{\lambda_1}. \quad (21)$$

**Corollary 3. 2.** Let  $M$  be a compact and connected immersed hypersurface of  $R^{n+1}$ . If the scalar curvature  $S$  and the first nonzero eigenvalue  $\lambda_1$  of the Laplacian operator  $\Delta$  on  $M$ , with respect to the induced metric, satisfy  $S = \lambda_1 (n - 1)$ , then

$$\int_M \left( \alpha^2 - \frac{\lambda_1}{n} \right) \rho^2 dv \geq 0. \quad (22)$$

Thus  $M$  is isometric to a sphere  $S^n(c)$ .

**Proof:** By the hypothesis of the theorem and (19), hence

$$\int_M \left( \alpha^2 - \frac{\lambda_1}{n} \right) \rho^2 dv \geq 0.$$

### 3. THE RICCI CURVATURE IS BOUNDED

**Theorem 4. 1.** Let  $M$  be a compact and connected immersed hypersurface of  $R^{n+1}$  with positive Ricci curvature. Suppose that the Ricci curvature of  $M$  is bounded below by a positive constant  $k$ . Thus

$$\int_M \alpha^2 \rho^2 dv \geq \frac{k}{n(n-1)} \int_M \|\text{grad} f\|^2 dv + \text{vol}(M) \quad (23)$$

**Proof:** From (17), Newton's inequality, (6) and by the hypothesis of theorem

$$n(n-1) \int_M (1 + \alpha \rho)^2 dv \geq k \int_M \|\text{grad} f\|^2 dv.$$

Or

$$-n(n-1) \text{vol}(M) + n(n-1) \int_M \alpha^2 \rho^2 dv \geq k \int_M \|\text{grad} f\|^2 dv \quad (24)$$

where we have used the Minkowski formula (7). Thus, we get (23).

**Theorem 4. 2.** Let  $M$  be a compact and connected immersed hypersurface of  $R^{n+1}$  with positive Ricci curvature. Suppose that the Ricci curvature of  $M$  is bounded below by a positive constant  $k$ . If the scalar curvature  $S$  and the first nonzero eigenvalue  $\lambda_1$  of the Laplacian operator  $\Delta$  on  $M$ , with respect to the induced metric satisfy  $S = \lambda_1 (n - 1)$ , and  $L = k - 2S > 0$  is a constant, then

$$\text{vol}(M) \geq -\frac{k\lambda_1}{L} \int_M \|\Psi\|^2 \alpha \rho dv - \frac{2S}{L} \int_M \alpha^2 \rho^2 dv. \quad (25)$$

**Proof:** For the immersion  $\psi : M \rightarrow IR^{n+1}$  we know that the function  $f = \frac{1}{2} \|\Psi\|^2$  satisfies (7). We can compute  $\text{div}(\text{grad} f)$  to obtain

$$\operatorname{div}(f \operatorname{grad} f) = \|\operatorname{grad} f\|^2 + \frac{n}{2} \|\psi\|^2 (1 + \alpha \rho). \quad (26)$$

Integrating this equation, we obtain

$$\int_M \|\operatorname{grad} f\|^2 dv + \frac{n}{2} \int_M \|\psi\|^2 (1 + \alpha \rho) dv. \quad (27)$$

From (27), (24) and (21), we obtain (25).

**Example:** We can take ellipsoid

$$M = \{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1 \}$$

which is a compact hypersurface of  $\mathbb{R}^3$ , and locally express the immersion  $\psi$  as  $\psi(t, \theta) = (2 \cos t \cos \theta, 2 \cos t \sin \theta, \sin t)$

Further, we can show that, on this coordinate patch of ellipsoid the shape operator  $A$ , the mean curvature  $\alpha$  and the support function  $\rho$  are respectively given by

$$A = \begin{pmatrix} \frac{2}{\sqrt{\cos^2 t + 4 \sin^2 t}} & 0 \\ 0 & \frac{1}{2\sqrt{\cos^2 t + 4 \sin^2 t}} \end{pmatrix}$$

$$\alpha = \frac{5}{4\sqrt{\cos^2 t + 4 \sin^2 t}} \text{ and } \rho = -\frac{2}{\sqrt{\cos^2 t + 4 \sin^2 t}}$$

and consequently we arrive at

$$\frac{1}{2} \|A\|^2 \rho^2 = \frac{17}{2} \frac{1}{(\cos^2 t + 4 \sin^2 t)^2} > \frac{25}{4} \frac{1}{(\cos^2 t + 4 \sin^2 t)^2} = \alpha^2 \rho^2$$

that is

$$\frac{1}{n} \int_M \|A\|^2 \rho^2 dv \geq \int_M \alpha^2 \rho^2 dv.$$

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