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FUZZY β – IRRESOLUTE FUNCTIONS AND FUZZY β – COMPACT SPACES IN FUZZIFYING TOPOLOGY*

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Abstract – In this paper the concepts of fuzzifying β – irresolute functions and fuzzifying β – compact spaces are characterized in terms of fuzzifying β – open sets and some of their properties are discussed.

Keywords – Lukasiewicz logic, fuzzifying topology, β – irresoluteness, fuzzifying compactness, β – compactness

1. INTRODUCTION

Fuzzy topology, as an important research field in fuzzy set theory, has been developed into quite a mature discipline [1-6]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested in different ways to generalize certain classical concepts. So far, according to [2], the kind of topologies defined by Chang [7] and Goguen [8] are called the topologies of fuzzy subsets, and further, are naturally called *L*-topological spaces if a lattice *L* of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an *L*-topological space) is a family τ of fuzzy subsets (resp. *L*-fuzzy subsets) of nonempty set *X*, and τ satisfies the basic conditions of classical topologies [9].

On the other hand, the authors of [10, 11] proposed the terminologies *I*-fuzzy topologies (if the set of membership values is chosen to be the unit interval [0, 1]) and *L*-fuzzy topologies (if the corresponding set of membership values is chosen to be lattice *L*). In [12], an *L*-fuzzy topology is an *L*-valued mapping on the traditional power set *P*(*X*) of *X*. In [4-5, 10, 11] an *L*-fuzzy topology is an *L*-valued mapping on the *L*-valued mapping on the *L*-power set L^X of *X*.

In 1991, Ying [13-16] used the semantic method of continuous valued logic to propose so-called fuzzifying topology as preliminary research on bifuzzy topology and to give an elementary development of topology in the theory of fuzzy sets from a completely different direction. Briefly speaking, a fuzzifying topology on a set *X* assigns each crisp subset of *X* to a certain degree of being open, other than being definitely open or not. In fact fuzzifying topologies are a special case of the *L*-fuzzy topologies in [10, 11] since all the t-norms on *I* are included as a special class of tensor products in these paper. Ying uses *one* particular tensor product, namely the Lukasiewicz conjunction. Thus his fuzzifying topologies are a special class of all *I*-fuzzy topologies considered in the categorical framework of [10, 11].

Particularly, as the author [13-16] indicated, by investigating fuzzifying topology we may partially answer an important question proposed by Rosser and Turquette [17] in 1952, which asked whether there

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are many valued theories beyond the level of predicates calculus.

Roughly speaking, the semantically analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies. For example, Ying [16] introduced the concept of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [18] the concepts of fuzzifying $\beta-$open sets and $\beta-$continuity were introduced and studied. Also, Sayed [19] introduced and studied the concept of fuzzifying $\beta-$ Hausdorff separation axiom.

In classical mathematics, the $\beta-$irresolute function has been given in [20] and the concept of $\beta-$compact spaces has been defined and some of its properties have been obtained in [21].

In [22] the concept of fuzzy $\beta-$irresolute function was characterized and investigated. Also, in [23] the concept of $\beta-$compactness for fuzzy topological spaces was introduced and discussed.

In this paper we introduce and study the concept of the $\beta-$irresolute function between fuzzifying topological spaces. Furthermore, we introduce and study the concept of $\beta-$compactness in the framework of fuzzifying topology. We use the finite intersection property to give a characterization of the fuzzifying $\beta-$compact spaces.

2. PRELIMINARIES

In this section, we offer some concepts and results in fuzzifying topology which will be used in the sequel. For the details, we refer to [8, 13-16]. First, we display the Lukasiewicz logic and corresponding set theoretical notations used in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models\varphi$ if $[\varphi]=1$ for any interpretation. By $\models^{w}\varphi$ (φ is feebly valid) we mean that for any valuation it always holds that $[\varphi]>0$, and $\varphi\models^{ws}\psi$ we mean that $[\varphi]>0$ implies $[\psi]=1$. The truth valuation rules for primary fuzzy logical formulae and corresponding set theoretical notations are:

(1) (a) $[\alpha]=\alpha\,(\alpha\in[0,1])$;

(b) $[\varphi\wedge\psi]=\min([\varphi],[\psi])$;

(c) $[\varphi\to\psi]=\min(1,1-[\varphi]+[\psi])$.

(2) If $\tilde{A}\in\Im(X)$, $[x\in\tilde{A}]:=\tilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x\varphi(x)]:=\inf_{x\in X}[\varphi(x)]$.

In addition, the truth valuation rules for some derived formulae are

(1) $[\neg\varphi]:=[\varphi\to 0]=1-[\varphi]$;

(2) $[\varphi\vee\psi]:=[\neg(\neg\varphi\wedge\neg\psi)]=\max([\varphi],[\psi])$;

(3) $[\varphi\leftrightarrow\psi]:=[(\varphi\to\psi)\wedge(\psi\to\varphi)]$;

(4) $[\varphi\mathbin{\dot\wedge}\psi]:=[\neg(\varphi\to\neg\psi)]=\max(0,[\varphi]+[\psi]-1)$;

(5) $[\varphi\mathbin{\dot\vee}\psi]:=[\neg\varphi\to\psi]=\min(1,[\varphi]+[\psi])$;

(6) $[\exists x\varphi(x)]:=[\neg\forall x\neg\varphi(x)]:=\sup_{x\in X}[\varphi(x)]$;

(7) If $\tilde{A},\tilde{B}\in\Im(X)$, then

(a) $[\tilde{A}\subseteq\tilde{B}]:=[\forall x(x\in\tilde{A}\to x\in\tilde{B})]=\inf_{x\in X}\min(1,1-\tilde{A}(x)+\tilde{B}(x))$;

(b) $\left[\ \tilde{A}\equiv\tilde{B}\ \right]:=\left[\ \tilde{A}\subseteq\tilde{B}\ \right]\wedge\left[\ \tilde{B}\subseteq\tilde{A}\ \right]$;

(c) $\left[\ \tilde{A}\ \dot{\equiv}\ \tilde{B}\ \right]$.

where $\Im(X)$ is the family of all fuzzy sets in *X*.

Often we do not distinguish the connectives and their truth value functions, but strictly state our results on formalization as Ying [13-16] did.

Second, we give some definitions and results in fuzzifying topology.

Definition 2. 1. [13]. Let *X* be a universe of discourse, $\tau\in\Im(P(X))$, satisfying the following conditions:
(1) $\tau(X)=1, \tau(\phi)=1$;

(2) for any A, B, $\tau\left(\ A\cap B\right)\geq\tau(A)\wedge\tau(B)$;

(3) for any $\left\{\ A_\lambda:\lambda\in\Lambda\ \right\}, \tau\left(\ \bigcup_{\lambda\in\Lambda}A_\lambda\ \right)\geq\bigwedge_{\lambda\in\Lambda}\tau\left(A_\lambda\right)$.

Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

Definition 2. 2. [13]. The family of all fuzzifying closed sets, denoted by $F\in\Im\left(P(X)\right)$, is defined as $A\in F:=X-A\in\tau$, where $X-A$ is the complement of A.

Definition 2. 3. [13]. The fuzzifying neighborhood system of a point $x\in X$ is denoted by $N_x\in\Im\left(P(X)\right)$ and defined as follows: $N_x(A)=\sup_{x\in B\subseteq A}\tau(B)$.

Definition 2. 4. [13, Lemma 5. 2]. The closure $\overline{A}$ of *A* is defined as
$\overline{A}(x)=\ \ 1-N_x(X-A)$.

In Theorem 5.3 [20], Ying proved that the closure $^{-}:P(X)\rightarrow\Im(X)$ is a fuzzifying closure operator (see Definition 5.3 [13]) because its extension $^{-}$: $\Im(X)\rightarrow\Im(X), \overline{\tilde{A}}=\bigcup_{\alpha\in[0,1]}\alpha\overline{\tilde{A}_\alpha}$, $\tilde{A}\in\Im(X)$, where $\tilde{A}_\alpha=\left\{\ x:\tilde{A}(x)\geq\alpha\ \right\}$ is the $\alpha-$cut of A and $\alpha\tilde{A}(x)=\alpha\wedge\tilde{A}(x)$ satisfies the following Kuratowski closure axioms:
(1) $\models\overline{\phi}=\phi$;

(2) for any $\tilde{A}\in\Im(X), \models\tilde{A}\subseteq\overline{\tilde{A}}$;

(3) for any $\tilde{A},\tilde{B}\in\Im(X), \models\overline{\tilde{A}\cup\tilde{B}}\equiv\overline{\tilde{A}}\cup\overline{\tilde{B}}$;

(4) for any $\tilde{A}, \tilde{B}\in\Im(X), \models\overline{\left(\overline{\tilde{A}}\right)}\subseteq\overline{\tilde{A}}$, where $\left(\tilde{A}\cup\tilde{B}\ \right)(x)=\max\left(\ \tilde{A}(x),\tilde{B}(x)\ \right)$.

Definition 2. 5. [14]. For any $A\subseteq X$, the fuzzy set of the interior points of *A* is called the interior of *A*, and given as follows: $A^\circ(x):=N_x(A)$. From Lemma 3.1 [13] and the definitions of $N_x(A)$ and A° we have $\tau(A)=\inf_{x\in A}A^\circ(x)$.

Definition 2. 6. [18]. For any $\tilde{A}\in\Im(X), \models\left(\tilde{A}\right)^\circ\equiv X-\overline{\left(X-\tilde{A}\ \right)}$, where $X-\tilde{A}$ is the complement of $\tilde{A}$ and $\left(X-\tilde{A}\ \right)(x)=1-\tilde{A}(x)$.

Definition 2. 7. [14, Lemma 5.1]. If (X,τ) is a fuzzifying topological space, $Y\subseteq X$, then $\tau|_Y\in\Im\left(P(X)\right)$, which is given as

$V \in \tau|_Y := (\exists U)\big((U \in \tau) \wedge (V = U \cap Y) \big)$, i.e., $\tau|_Y(V) = \sup_{V = U \cap Y} \tau(U)$ is a fuzzifying topology on *Y* and is called the relative fuzzifying topology of τ with respect to *Y*. If $Y \subseteq X, \sigma = \tau|_Y$, then (Y, σ) is called a subspace of (X, τ).

Definition 2. 8. [14, Theorem 5.1]. Let (Y, σ) be a subspace of (X, τ). For any $A \subseteq Y$, $A \in F_Y := (\exists F)\big((F \in F_X) \wedge (A = F \cap Y) \big)$, where F_X and F_Y are fuzzy families of $\tau, \sigma-$ closed sets in *X* and *Y*, respectively.

Lemma 2. 1. [18]. If $\big[\tilde{A} \subseteq \tilde{B} \big] = 1$, then (1) $\models \overline{\tilde{A}} \subseteq \overline{\tilde{B}}$ (2) $\models \big(\tilde{A}\big)^{\circ} \subseteq \big(\tilde{B}\big)^{\circ}$.

Definition 2. 9. [18]. Let (X, τ) be a fuzzifying topological space.

(1) The family of all fuzzifying $\beta-$open sets, denoted by $\tau_\beta \in \Im(P(X))$, is defined as follows: $A \in \tau_\beta := \forall x\big(x \in A \rightarrow x \in A^{-\circ-} \big)$, i.e., $\tau_\beta(A) = \inf_{x \in A} A^{-\circ-}(x)$

(2) The family of all fuzzifying $\beta-$closed sets, denoted by $F_\beta \in \Im(P(X))$, is defined as follows: $A \in F_\beta := X - A \in \tau_\beta$.

(3) The fuzzifying $\beta-$neighborhood system of a point $x \in X$ is denoted by $N_x^\beta \in \Im(P(X))$ and defined as follows: $N_x^\beta(A) = \sup_{x \in B \subseteq A} \tau_\beta(B)$.

(4) The fuzzifying $\beta-$closure of a set $A \in P(X)$, denoted by $cl_\beta \in \Im(X)$, is defined as follows: $cl_\beta(A)(x) = 1 - N_x^\beta(X - A)$.

(5) Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $C_\beta \in \Im\big(Y^X\big)$, called fuzzifying $\beta-$continuity, is given as follows:

$$C_\beta(f) := \forall B\big(B \in \sigma \rightarrow f^{-1}(B) \in \tau_\beta \big).$$

Definition 2.10. [19]. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicate T_2^β (fuzzifying $\beta-$Hausdorff) $\in \Im(\Omega)$ is defined as follows:

$$T_2^\beta(X, \tau) := \forall x \forall y \Big(\big(x \in X \wedge y \in X \wedge x \neq y \big) \rightarrow \exists B \exists C \big(B \in N_x^\beta \wedge C \in N_y^\beta \wedge B \cap C = \phi \big) \Big).$$

Definition 2. 11. [16]. Let *X* be a set. If $\tilde{A} \in \Im(X)$ such that the support $\operatorname{supp} \tilde{A} = \big\{ x \in X : \tilde{A}(x) > 0 \big\}$ of *A* is finite, then $\tilde{A}$ is said to be finite and we write $F\big(\tilde{A}\big)$. A unary fuzzy predicate $FF \in \Im\big(\Im(X) \big)$, called fuzzy finiteness, is given as $FF\big(\tilde{A}\big); = \big(\exists \tilde{B} \big)\big(F\big(\tilde{B}\big) \wedge \big(\tilde{A} \equiv \tilde{B}\big) \big) = 1 - \inf\big\{ \alpha \in [0,1] : F\big(\tilde{A}_\alpha\big) \big\} = 1 - \inf\big\{ \alpha \in [0,1] : F\big(\tilde{A}_{[\alpha]}\big) \big\}$, where $\tilde{A}_\alpha = \big\{ x \in X : \tilde{A}(x) \geq \alpha \big\}$ and $\tilde{A}_{[\alpha]} = \big\{ x \in X : \tilde{A}(x) > \alpha \big\}$.

Definition 2. 12. [16]. Let *X* be a set.

(1) A binary fuzzy predicate $K \in \Im\big(\Im(P(X)) \times P(X) \big)$, called fuzzifying covering, is given as follows: $K(\Re, A) := \forall x\big(x \in A \rightarrow \exists B (B \in \Re \wedge x \in B) \big)$.

(2) Let (X, τ) be a fuzzifying topological space. A binary fuzzy predicate $K_\circ \in \Im\big(\Im\big(P(X)\big) \times P(X) \big)$, called fuzzifying open covering, is given as follows:

$$K_\circ(\Re, A) := K(\Re, A) \mathbin{\dot{\wedge}} (\Re \subseteq \tau).$$

Definition 2. 13. [16]. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma \in \Im\big(\Omega \big)$, called fuzzifying compactness, is given as follows:

$(X, \tau) \in \Gamma := \big(\forall \Re \big)\big(K_\circ(\Re, X) \rightarrow \big(\exists \wp \big)((\wp \leq \Re) \wedge K(\wp, A) \mathbin{\dot{\wedge}} FF(\wp) \big)$, where $\wp \leq \Re$ means that for any

$M \in P(X), \wp(M) \leq \Re(M)$.

Definition 2. 14. [16]. Let *X* be a set. A unary fuzzy predicate $fI \in \Im\left(\Im(P(X))\right)$, called fuzzifying finite intersection property, is given as follows:

$$fI(\Re) := \left(\forall \beta\right)\left((\beta \leq \Re) \wedge FF(\beta) \rightarrow (\exists x)(\forall B)((B \in \beta) \rightarrow (x \in B))\right).$$

Lemma 2. 2. [19]. Let (X, τ) be a fuzzifying topological space. Then

(1) $\models \tau \subseteq \tau_\beta$; (2) $\models F \subseteq F_\beta$, (3) $F_\beta\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \geq \bigwedge_{\lambda \in \Lambda} F_\beta(A_\lambda)$.

Corollary 2. 1. [18]. $\tau_\beta(A) = \inf_{x \in A} N_x^\beta(A)$.

Theorem 2. 1. [18]. For any $x, A, B, \models A \subseteq B \rightarrow \left(A \in N_x^\beta \rightarrow B \in N_x^\beta\right)$.

3. β – IRRESOLUTE FUNCTIONS

Definition 3. 1. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I_\beta \in \Im\left(Y^X\right)$, called fuzzifying $\beta-$ irresolute, is given as follows: $I_\beta(f) := \forall B\left(B \in \sigma_\beta \rightarrow f^{-1}(B) \in \tau_\beta\right)$.

Theorem 3.1. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. Then $\models f \in I_\beta \rightarrow f \in C_\beta$

Proof: From Lemma 2.2 we have $\sigma(B) \leq \sigma_\beta(B)$ and the result holds.

Definition 3. 2. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. We define the unary fuzzy predicates $w_k \in \Im\left(Y^X\right)$, where *k* = 1,…5, as follows:

(1) $f \in w_1 = \forall B\left(B \in F_\beta^Y \rightarrow f^{-1}(B) \in F_\beta^X\right)$, where F_β^X and F_β^Y are the fuzzifying $\beta-$ closed subsets of X and *Y*, respectively.

(2) $f \in w_2 = \forall x \forall u\left(u \in N_{f(x)}^{\beta^Y} \rightarrow f^{-1}(u) \in N_x^{\beta^X}\right)$, where N^{β^X} and N^{β^Y} are the family of fuzzifying $\beta-$neighborhood systems of *X* and *Y*, respectively;

(3) $f \in w_3 = \forall x \forall u\left(u \in N_{f(x)}^{\beta^Y} \rightarrow \exists v\left(f(v) \subseteq u \rightarrow v \in N_x^{\beta^X}\right)\right)$;

(4) $f \in w_4 = \forall A\left(f\left(cl_\beta^X(A)\right) \subseteq cl_\beta^Y(f(A))\right)$;

(5) $f \in w_5 = \forall B\left(cl_\beta^X\left(f^{-1}(B)\right) \subseteq f^{-1}\left(cl_\beta^Y(B)\right)\right)$.

Theorem 3. 2. $\models f \in I_\beta \leftrightarrow f \in w_k, k = 1, \ldots, 5$

Proof: (a) We will prove that $\models f \in I_\beta \leftrightarrow f \in w_1$.

$$\left[f \in w_1\right] = \left[\forall B\left(F_\beta^Y(B) \rightarrow F_\beta^X\left(f^{-1}(B)\right)\right)\right]$$

$$= \left[\forall B\left(\sigma_\beta(Y - B) \rightarrow \tau_\beta\left(X - f^{-1}(B)\right)\right)\right]$$

$$= \left[\forall B\left(\sigma_\beta(Y - B) \rightarrow \tau_\beta\left(f^{-1}(Y - B)\right)\right)\right]$$

$$=\left[\ \forall u\left(\ \sigma_{\beta}(u)\rightarrow\tau_{\beta}\ \left(f^{-1}(u)\right)\right)\ \right]$$

$$=\left[\ f\in I_{\beta}\ \right].$$

(b) First, we prove that $\left[\ f\in I_{\beta}\right]\leq\left[\ f\in w_2\right]$ employing the rules of *L*ukasiewicz logic and the clear fact that $f(x)\in A\subseteq u$ implies $x\in f^{-1}(A)\subseteq f^{-1}(u)$:

$$\left[f\in I_{\beta}\ \right]:=\left[\ \forall A\left(\ \sigma_{\beta}(A)\rightarrow\tau_{\beta}\left(f^{-1}(A)\right)\right)\ \right]$$

$$\leq\left[\ \forall B\left(f(x)\in A\subseteq u\rightarrow\left(\ \sigma_{\beta}(A)\rightarrow\tau_{\beta}\left(f^{-1}(A)\right)\right)\right)\ \right]$$

$$\leq\left[\ \exists A\left(f(x)\in A\subseteq u\ \dot\wedge\ \sigma_{\beta}(A)\right)\rightarrow\ \exists A\left(f(x)\in A\subseteq u\ \dot\wedge\ \tau_{\beta}\left(f^{-1}(A)\right)\right)\right]$$

$$\leq\left[\ \exists A\left(f(x)\in A\subseteq u\ \dot\wedge\ \sigma_{\beta}(A)\right)\rightarrow\ \exists A\left(x\in B\subseteq f^{-1}(u)\ \dot\wedge\ \tau_{\beta}(B)\right)\ \right].$$

From this, the required $\left[\ f\in I_{\beta}\right]\leq\left[\ f\in w_2\ \right]$ is followed by the rule of generalization (on *x* and *u*) in *L*ukasiewicz logic.

Second, we prove that $\left[\ f\in w_2\right]\leq\left[\ f\in I_{\beta}\ \right]$, by the rules of Lukasiewicz logic and employing Corollary 2.1:

$$\left[f\in w_2\ \right]=\left[\forall u\,\forall x\left(\ N_{f(x)}^{\beta^Y}(u)\rightarrow N_x^{\beta^X}\left(f^{-1}(u)\right)\ \right)\right]$$

$$\leq\left[\forall u\ \left(\forall x\in f^{-1}(u)\wedge N_{f(x)}^{\beta^Y}(u)\rightarrow\ \forall x\in f^{-1}(u)\wedge N_x^{\beta^X}\left(f^{-1}(u)\right)\ \right)\right]$$

$$=\left[\forall u\left(\ \sigma_{\beta}(u)\rightarrow\tau_{\beta}\left(f^{-1}(u)\right)\right)\right]=\left[\ f\in I_{\beta}\right].$$

(c) We prove that $\left[\ f\in w_2\right]=\left[\ f\in w_3\ \right]$. From Theorem 2.1 we have

$$\left[\ f\in w_3\right]=\inf_{x\in X}\ \inf_{u\in P(Y)}\min\left(\ 1,1-N_{f(x)}^{\beta^Y}(u)+\sup_{v\in P(X),f(v)\subseteq u}N_x^{\beta^X}(v)\right)$$

$$\geq\inf_{x\in X}\ \inf_{u\in P(Y)}\min\left(1,1-N_{f(x)}^{\beta^Y}(u)+\sup_{v\in P(X),f(v)\subseteq u}N_x^{\beta^X}\left(\ f^{-1}(u)\right)\right)=\left[\ f\in w_2\right].$$

(d) We prove that $\left[\ f\in w_4\right]=\left[\ f\in w_5\ \right]$. First, since for any fuzzy set $\tilde{A}$ we have $\left[\ f^{-1}\left(f\left(\tilde{A}\right)\right)\supseteq\tilde{A}\ \right]=1$, then $\left[\ f^{-1}\left(f\left(\ cl_{\beta}^X\left(f^{-1}(B)\right)\right)\right)\supseteq cl_{\beta}^X\left(f^{-1}(B)\right)\ \right]=1$ for any $B\in P(Y)$. Also, since $\left[\ f\left(f^{-1}((B))\right)\subseteq B\ \right]=1$, then $\left[\ cl_{\beta}^Y\left(f\left(f^{-1}(B)\right)\right)\subseteq cl_{\beta}^Y(B)\ \right]=1$. Therefore, from Lemma 1.2 (2) [22] we have

$$\left[\ cl_{\beta}^X\left(f^{-1}(B)\right)\subseteq f^{-1}\left(\ cl_{\beta}^Y(B)\right)\ \right]\geq\left[\ f^{-1}\left(f\left(\ cl_{\beta}^X\left(f^{-1}(B)\right)\right)\right)\subseteq f^{-1}\left(\ cl_{\beta}^Y(B)\ \right)\right]$$

$$\geq\left[\ f^{-1}\left(f\left(cl_{\beta}^X\left(f^{-1}(B)\right)\right)\right)\subseteq f^{-1}\left(cl_{\beta}^Y\left(f\left(f^{-1}(B)\right)\ \right)\right)\right]$$

$$\geq\left[\ f\left(cl_{\beta}^X\left(f^{-1}(B)\right)\right)\subseteq cl_{\beta}^Y\left(f\left(f^{-1}(B)\right)\ \right)\right].$$

Therefore

$$\left[\ f\in w_5\right]=\left[\forall B\left(\left(cl_{\beta}^X\left(f^{-1}(B)\right)\right)\subseteq f^{-1}\left(cl_{\beta}^Y(B)\right)\right)\right]$$

$$\geq\left[\forall B\left(f\left(cl_{\beta}^{X}\left(f^{-1}(B)\right)\right)\subseteq cl_{\beta}^{Y}\ \left(f\left(f^{-1}(B)\right)\right)\right)\right]$$

$$\geq\left[\forall A\left(f\left(cl_{\beta}^{X}(A)\right)\subseteq cl_{\beta}^{Y}\ \left(f(A)\right)\right)\right]=\left[\ f\in w_4\right].$$

Second, for each $A\subseteq X$, there exists $B\subseteq Y$ such that $f(A)=B$ and $f^{-1}(B)\supseteq A$. From Lemma 1.2 (1) [15] we have

$$\left[\ f\in w_4\right]=\left[\forall A\left(f\left(cl_{\beta}^{X}(A)\right)\subseteq cl_{\beta}^{Y}\ \left(f(A)\right)\right)\right]$$

$$\geq\left[\forall A\left(f\left(cl_{\beta}^{X}(A)\right)\subseteq f\left(f^{-1}\left(cl_{\beta}^{Y}\ \left(f(A)\right)\right)\right)\right)\right]$$

$$\geq\left[\forall A\left(cl_{\beta}^{X}(A)\subseteq f^{-1}\left(cl_{\beta}^{Y}\left(f(A)\right)\right)\right)\right]$$

$$\geq\left[\forall B=f(A)\left(cl_{\beta}^{X}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(cl_{\beta}^{Y}\left(B\right)\right)\right)\right]$$

$$\geq\left[\forall B\left(cl_{\beta}^{X}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(cl_{\beta}^{Y}(B)\right)\right)\right]=\left[\ f\in w_5\right].$$

(e) We want to prove that $\models f\in w_2\leftrightarrow f\in w_5$.

$$\left[\ f\in w_5\right]=\inf_{B\in P(Y)}\left[cl_{\beta}^{X}\left(\ f^{-1}(B)\right)\subseteq f^{-1}\left(cl_{\beta}^{Y}(B)\right)\right]$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\left(\ 1,1-\left(\ 1-N_{x}^{\beta^{X}}\left(\ X-f^{-1}(B)\right)\right)+1-N_{f(x)}^{\beta^{Y}}(Y-B)\ \right)$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\min\left(\ 1,1-N_{f(x)}^{\beta^{Y}}\left(Y-B\ \right)+N_{x}^{\beta^{X}}\left(\ f^{-1}(Y-B)\right)\right)$$

$$=\inf_{B\in P(Y)}\inf_{x\in X}\min\left(\ 1,1-N_{f(x)}^{\beta^{Y}}(u)+N_{x}^{\beta^{X}}\left(f^{-1}(u)\right)\right)=\left[\ f\in w_2\ \right].$$

4. $\beta-$COMPACTNESS IN FUZZIFYING TOPOLOGY

Definition 4.1. A fuzzifying topological space (X,τ) is said to be $\beta-$ fuzzifying topological space if $\tau_{\beta}(A\cap B)\geq\tau_{\beta}(A)\wedge\tau_{\beta}(B)$.

Definition 4.2. A binary fuzzy predicate $K_{\beta}\in\Im\left(\Im(P(X))\times P(X)\right)$, called fuzzifying $\beta-$open covering, is given as $K_{\beta}(\Re,A):=K(\Re,A)\dot{\wedge}\left(\Re\subseteq\tau_{\beta}\right)$.

Definition 4.3. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma_{\beta}\in\Im(\Omega)$, called fuzzifying $\beta-$compactness, is given as follows:

(1) $(X,\tau)\in\Gamma_{\beta}:=(\forall\Re)\left(K_{\beta}(\Re,X)\rightarrow(\exists\wp)\left((\wp\leq\Re)\wedge K(\wp,X)\dot{\wedge}FF(\wp)\right)\right)$.

(2) If $A\subseteq X$, then $\Gamma_{\beta}(A):=\Gamma_{\beta}\left(A,\tau|_A\right)$.

Lemma 4. 1. $\models K_{\circ}(\Re,A)\rightarrow K_{\beta}(\Re,A)$.

Proof: Since from Lemma 2.2 $\models\tau\subseteq\tau_{\beta}$, then we have $\left[\Re\subseteq\tau\right]\leq\left[R\subseteq\tau_{\beta}\right]$. So,

$$\left[\ K_{\circ}(\Re,A)\right]\leq\left[K_{\beta}(\Re,A)\right].$$

Theorem 4. 1. $\models (X,\tau)\in \Gamma_\beta \to (X,\tau)\in\Gamma$.

Proof: From Lemma 4.1 the proof is immediate.

Theorem 4. 2. For any fuzzifying topological space (X,τ) and $A\subseteq X$,
$\Gamma_\beta(A)\leftrightarrow(\forall\Re)\Big(K_\beta(\Re,A)\to(\exists\wp)\big((\wp\le\Re)\wedge K(\wp,A)\dot{\wedge} FF(\wp)\big)\Big)$, K_β is related to τ.

Proof: For any $\Re\in\Im\big(\Im(X)\big)$, we set $\bar{\Re}\in\Im\big(\Im(A)\big)$ defined as $\bar{\Re}(C)=\Re(B)$ with $C=A\cap B$, $B\subseteq X$. Then $K\big(\bar{\Re},A\big)=\inf_{x\in A}\sup_{x\in C}\bar{\Re}(C)=\inf_{x\in A}\sup_{x\in C=A\cap B}\Re(B)=\inf_{x\in A}\sup_{x\in B}\Re(B)=K\big(\Re,A\big)$, because $x\in A$ and $x\in B$ if and only if $x\in A\cap B$. Therefore

$$\big[\bar{\Re}\subseteq\tau_\beta\,|A\big]=\inf_{C\subseteq A}\min\big(1,1-\bar{\Re}(C)+\tau_\beta\,|A(C)\big)$$

$$=\inf_{C\subseteq A}\min\Bigg(1,1-\sup_{C=A\cap B,B\in X}\Re(B)+\sup_{C=A\cap B,B\subseteq X}\tau_\beta(B)\Bigg)$$

$$\ge\sup_{C\subseteq A,C=A\cap B,B\in X}\min\big(1,1-\Re(B)+\tau_\beta(B)\big)$$

$$\ge\inf_{B\in X}\min\big(1,1-\Re(B)+\tau_\beta(B)\big)=\big[\Re\subseteq\tau_\beta\big].$$

For any $\wp\le\bar{\Re}$, we define $\wp'\in\Im(P(X))$ as follows:

$$\wp'(B)=\begin{cases}\wp(B) & \text{if } B\subseteq A,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\wp'\le\Re, FF(\wp')=FF(\wp)$ and $K\big(\wp',A\big)=K(\wp,A)$.
Furthermore, we have

$$\Big[\Gamma_\beta(A)\dot{\wedge}K_\beta(\Re,A)\Big]\le\Big[\Gamma_\beta(A)\dot{\wedge}K'_\beta\big(\bar{\Re},A\big)\Big]$$

$$\le\Big[(\exists\wp)\Big(\big(\wp\le\bar{\Re}\big)\wedge K\big(\wp,A\big)\dot{\wedge}FF(\wp)\Big)\Big]$$

$$\le\Big[(\exists\wp')\Big(\big(\wp'\le\Re\big)\wedge K\big(\wp',A\big)\dot{\wedge}FF(\wp')\Big)\Big]$$

$$\le\Big[(\exists \mathrm{B})\Big(\big(\mathrm{B}\le\Re\big)\wedge K\big(\mathrm{B},A\big)\dot{\wedge}FF(\mathrm{B})\Big)\Big]$$

Then $\Gamma_\beta(A)\le\big[K_\beta(\Re,A)\big]\to\Big[(\exists\mathrm{B})\Big(\big(\mathrm{B}\le\Re\big)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\Big)\Big]$,
where $K'_\beta\big(\bar{\Re},A\big)=\Big[K_\beta\big(\bar{\Re},A\big)\dot{\wedge}\big(\bar{\Re}\subseteq\tau_\beta\,|_A\big)\Big]$. Therefore

$$\Gamma_\beta(A)\le\inf_{\Re\in\Im(P(X))}\Big[K_\beta(\Re,A)\to(\exists\mathrm{B})\Big(\big(\mathrm{B}\le\Re\big)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\Big)\Big]$$

$$=\Big[(\exists\Re)\Big(K_\beta(\Re,A)\to(\exists\mathrm{B})\Big(\big(\mathrm{B}\le\Re\big)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\Big)\Big)\Big]$$

Conversely, for any $\Re\in\Im(P(A))$, if $\big[\Re\subseteq\tau_\beta\,|_A\big]=\inf_{B\subseteq A}\min\big(1,1-\Re(B)+\tau_\beta\,|_A(B)\big)$

$=\lambda$, then for any $n\in N$ and $B\subseteq A$, $\sup_{B=A\cap C, C\subseteq X}\tau_\beta(C)=\tau_\beta|_A(B)>\lambda+\Re(B)-1-\frac{1}{n}$, and there exists $C_B\subseteq X$ such that $C_B\cap A=B$ and $\tau_\beta(C_B)>\lambda+\Re(B)-1-\frac{1}{n}$. Now, we define $\bar{\Re}\in\Im(P(X))$ as $\bar{\Re}(C)=\max_{B\subseteq A}\left(0,\lambda+\Re(B)-1-\frac{1}{n}\right)$. Then $\left[\bar{\Re}\subseteq\tau_\beta\right]=1$ and

$$K\left(\bar{\Re},A\right)=\inf_{x\in A}\sup_{x\in C\subseteq X}\bar{\Re}(C)=\inf_{x\in A}\sup_{x\in B}\bar{\Re}(C_B)\ \ \geq\inf_{x\in A}\sup_{x\in B}\left(\lambda+\Re(B)-1-\frac{1}{n}\right)$$

$$=\inf_{x\in A}\sup_{x\in B}\Re(B)+\lambda-1-\frac{1}{n}=K\left(\Re,A\right)+\lambda-1-\frac{1}{n},$$

$$K_\beta\left(\bar{\Re},A\right)=\left[K\left(\bar{\Re},A\right)\dot{\wedge}\left(\bar{\Re}\subseteq\tau_\beta\right)\right]=\left[K\left(\bar{\Re},A\right)\right]\geq\max\left(0,K\left(\Re,A\right)+\lambda-1-\frac{1}{n}\right)$$

$$\geq\max\left(0,K\left(\Re,A\right)+\lambda-1\right)-\frac{1}{n}=K'_\beta\left(\Re,A\right)-\frac{1}{n}.$$

For any $\wp\leq\bar{\Re}$, we set $\wp'\in\Im(P(A))$ as $\wp'(B)=\wp(C_B), B\subseteq A$. Then $\wp'\leq\Re, FF(\wp')=FF(\wp)$ and $K\left(\wp',A\right)=K\left(\wp,A\right)$. Therefore

$$\left[(\forall\Re)\left(K_\beta(\Re,A)\to(\exists\wp)\left(\left(\wp\leq\Re\right)\wedge K\left(\wp,A\right)\dot{\wedge}FF(\wp)\right)\right)\right]\dot{\wedge}\left[K'_\beta(\Re,A)\right]-\frac{1}{n}$$

$$\leq\left[(\forall\Re)\left(K_\beta(\Re,A)\to(\exists\wp)\left(\left(\wp\leq\Re\right)\wedge K\left(\wp,A\right)\dot{\wedge}FF(\wp)\right)\right)\right]\dot{\wedge}\left(\left[K'_\beta(\Re,A)\right]-\frac{1}{n}\right)$$

$$\leq\left[K_\beta\left(\bar{\Re},A\right)\to(\exists\wp)\left(\left(\wp\leq\bar{\Re}\right)\wedge K\left(\wp,A\right)\dot{\wedge}FF(\wp)\right)\right]\dot{\wedge}\left[K_\beta\left(\bar{\Re},A\right)\right]$$

$$\leq\left[(\exists\wp)\left(\left(\wp\leq\bar{\Re}\right)\wedge K\left(\wp,A\right)\dot{\wedge}FF(\wp)\right)\right]\ \ \leq\left[\left(\exists\wp'\right)\left(\left(\wp'\leq\Re\right)\wedge K\left(\wp',A\right)\dot{\wedge}FF(\wp')\right)\right]$$

$$\leq\left[\left(\exists\mathrm{B}\right)\left(\left(\mathrm{B}\leq\Re\right)\wedge K\left(\mathrm{B},A\right)\dot{\wedge}FF(\mathrm{B})\right)\right].$$ Let $n\to\infty$. We obtain

$$\left[(\forall\Re)\left(K_\beta\left(\Re,A\right)\to\left(\exists\wp\right)\left(\left(\wp\leq\Re\right)\wedge K\left(\wp,A\right)\dot{\wedge}FF(\wp)\right)\right)\right]\dot{\wedge}\left[K'_\beta(\Re,A)\right]$$

$$\leq\left[\left(\exists\mathrm{B}\right)\left(\left(\mathrm{B}\leq\Re\right)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\right)\right].$$ Then

$$\left[(\forall\Re)\left(K_\beta\left(\Re,A\right)\to\left(\exists\wp\right)\left(\left(\wp\leq\Re\right)\wedge K(\wp,\Re)\dot{\wedge}FF(\wp)\right)\right)\right]$$

$$\leq\left[K'_\beta(\Re,A)\to\left(\exists\mathrm{B}\right)\left(\left(\left(\mathrm{B}\leq\Re\right)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\right)\right)\right]$$

$$\leq\inf_{\Re\in\Im(P(X))}\left[K'_\beta(\Re,A)\to\left(\exists\mathrm{B}\right)\left(\left(\mathrm{B}\leq\Re\right)\wedge K(\mathrm{B},A)\dot{\wedge}FF(\mathrm{B})\right)\right]=\Gamma_\beta(A).$$

Theorem 4. 3. Let (X,τ) be a fuzzifying topological space.

$$\pi_1:=(\forall\Re)\left(\Re\in\Im(P(X))\right)\wedge\left(\Re\subseteq F_\beta\right)\dot{\wedge}fI(\Re)\to\left(\exists x\right)\left(\forall A\right)\left(A\in\Re\to x\in A\right);$$

$$\pi_2:=(\forall\Re)\ \left(\exists B\right)\left(\left(\left(\Re\subseteq F_\beta\right)\wedge\left(B\in\tau_\beta\right)\right)\dot{\wedge}\left(\forall\wp\right)\right.$$

$$\left.\left(\left(\wp\leq\Re\right)\dot{\wedge}FF(\wp)\to\neg\left(\cap\wp\subseteq B\right)\right)\to\neg\left(\cap\Re\subseteq B\right)\right)$$. Then $\models\Gamma_\beta(X,\tau)\leftrightarrow\pi_i, i=1,2.$

Proof: (a) We prove $\Gamma_\beta(X,\tau) = [\pi_1]$. For any $\Re \in \Im(P(X))$, we set $\Re^c(X-A) = \Re(A)$. Then

$$[\Re \subseteq \tau_\beta] = \inf_{A\in P(X)} \min(1, 1 - \Re(A) + \tau_\beta(A))$$

$$= \inf_{X-A\in P(X)} \min(1, 1 - \Re^c(X-A) + F_\beta(X-A)) = [\Re^c \subseteq F_\beta],$$

$$FF(\Re) = 1 - \inf\{\alpha \in [0,1] : F(\Re_\alpha)\} = 1 - \inf\{\alpha \in [0,1] : F(\Re^c_\alpha)\} = FF(\Re^c) \text{ and}$$

$$\mathrm{B} \le \Re^c \Leftrightarrow \mathrm{B}(M) \le \Re^c(M) \Leftrightarrow \mathrm{B}^c(X-M) \le \Re(X-M) \Leftrightarrow \mathrm{B}^c \le \Re. \text{ Therefore,}$$

$$\Gamma_\beta(X,\tau) = \left[(\forall \Re)\left(K_\beta(\Re,X) \to (\exists \wp)\left(K(\wp,X) \mathbin{\dot\wedge} FF(\wp)\right)\right)\right]$$

$$= \left[(\forall \Re)\left((\Re \subseteq \tau_\beta) \mathbin{\dot\wedge} K(\Re,X) \to (\exists \wp)\left((\wp \le \Re) \wedge K(\wp,X) \mathbin{\dot\wedge} FF(\wp)\right)\right)\right]$$

$$= \left[(\forall \Re)\left((\Re \subseteq \tau_\beta) \to \left(K(\Re,X) \to (\exists \wp)\left((\wp \le \Re) \mathbin{\dot\wedge} K(\wp,X) \mathbin{\dot\wedge} FF(\wp)\right)\right)\right)\right]$$

$$= \left[(\forall \Re)\left((\Re^c \subseteq F_\beta) \to \left((\forall x)(\exists A)(A \in R \wedge x \in A) \to \quad (\exists \wp)\left((\wp \le \Re) \wedge K(\wp,X) \mathbin{\dot\wedge} FF(\wp)\right)\right)\right)\right]$$

$$= \left[(\forall \Re)\left((\Re^c \subseteq F_\beta) \to \left((\forall x)(\exists A)(A \in R \wedge x \in A) \to\right.\right.\right.$$

$$\left.\left.\left.(\exists \mathrm{B}^c)\left((\mathrm{B}^c \le \Re) \wedge K(\mathrm{B}^c,X) \mathbin{\dot\wedge} FF(\mathrm{B}^c)\right)\right)\right)\right] \quad = \left[(\forall \Re)(\Re^c \subseteq F_\beta) \to \left((\forall x)(\exists A)(A \in R \wedge x \in A) \to\right.\right.$$

$$\left.\left.(\exists \mathrm{B})\left((\mathrm{B} \le \Re^c) \wedge FF(\mathrm{B}) \mathbin{\dot\wedge} (\forall x)(\exists B)(B \in \mathrm{B}^c \wedge x \in B)\right]\right)$$

$$= \left[(\forall \Re)\left((\Re^c \subseteq F_\beta) \to (\neg((\exists \mathrm{B})(\mathrm{B} \le \Re^c \wedge FF(\mathrm{B}) \mathbin{\dot\wedge}\right.\right.$$

$$\left.\left.(\forall x)(\exists B)(B \in \mathrm{B}^c \wedge x \in B)) \to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A))))\right]$$

$$= \left[(\forall \Re)\left((\Re^c \subseteq F_\beta) \to \left(fI(\Re^c) \to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A))\right)\right)\right]$$

$$= \left[(\forall \Re)\left((\Re^c \subseteq F_\beta) \mathbin{\dot\wedge} fI(\Re^c) \to (\exists x)(\forall x)(A \in \Re^c \to x \in A)\right)\right] = [\pi_1].$$

(b) We prove $[\pi_1] = [\pi_2]$. Let $X - B \in P(X)$. For any $\Re \in \Im(P(X))$,

$$[(\Re \subseteq F_\beta) \wedge (B \in \tau_\beta)] = [\Re \subseteq F_\beta \wedge (X - B \in F_\beta)]$$

$$= \inf_{A\in P(X)} \min(1, 1 - \Re(A) + F_\beta(A)) \wedge F_\beta(X-B)$$

$$= \inf_{A\in P(X)} \min(1, 1 - \Re(A) + F_\beta(A)) \wedge \inf_{A\in P(X)} \min(1, 1 - [A \in \{X-B\}] + F_\beta(A))$$

$$= \inf_{A\in P(X)} \min(1, 1 - [\Re \cup \{X-B\}(A)] + F_\beta(A))$$

$$= [(\Re \cup \{X-B\}) \subseteq F_\beta].$$

Therefore, for any $\mathrm{B} \in \Im(P(X))$, let $\wp = \mathrm{B} \setminus \{X - B\} \in \Im(P(X))$.

$$\wp(A)=\begin{cases} \mathrm{B}(A) & , \quad A\neq X-B \\ 0 & , \quad A=X-B \end{cases}$$

Then $\wp\leq\mathrm{B}$, $\wp\cup\{\ X-B\ \}\geq\mathrm{B}$, $[\ FF(\wp)\ ]=[\ FF(\mathrm{B})\ ]$, $[\ \wp\leq\Re\ ]=[\ \mathrm{B}\leq(\ \Re\cup\{\ X-B\ \}\ )\ ]$
and

$$\left[(\ \forall\wp)\left((\ \wp\leq\Re)\underset{\bullet}{\wedge}FF(\wp)\rightarrow(\exists x)(\forall A)(\ A\in(\wp\cup\{\ X-B\ \}\ )\rightarrow(x\in A))\right)\right]$$

$$=\inf_{\wp\leq\Re}\min\left(1,1-\left[\ FF(\wp)\right]+\sup_{x\in X}\inf_{A\in P(X)}\left(\left(\wp\cup\{X-B\}\ \right)(A)\rightarrow A(x)\right)\right)$$

$$\leq\inf_{\mathrm{B}\leq(\ \Re\cup\{X-B\})}\min\left(1,1-\left[\ FF(\wp)\right]+\sup_{x\in X}\inf_{A\in P(X)}\left(\mathrm{B}(A)\rightarrow A(x)\right)\right)=fI\left(\Re\cup\{X-B\}\right).$$

Furthermore, we have

$$\pi_1\underset{\bullet}{\wedge}\left[\left((\Re\subseteq F_\beta)\wedge(B\in\tau_\beta)\right)\underset{\bullet}{\wedge}(\forall\wp)\left((\wp\leq\Re)\underset{\bullet}{\wedge}FF(\wp)\rightarrow\neg\left(\bigcap\wp\subseteq B\right)\right)\right]$$

$$=\pi_1\underset{\bullet}{\wedge}\Big[\ \left(\Re\cup\{\ X-B\ \}\subseteq F_\beta\right)\ \underset{\bullet}{\wedge}\ (\forall\wp)\left(\wp\leq\Re\right)\underset{\bullet}{\wedge}FF(\wp)\rightarrow$$

$$(\exists x\ )\ (\forall A)\ \left(A\in\left(\wp\cup\{X-B\}\right)\rightarrow x\in(A)\right)\Big]$$

$$=\pi_1\underset{\bullet}{\wedge}\left[\ \left(\Re\cup\{\ X-B\ \}\subseteq F_\beta\right)\ \underset{\bullet}{\wedge}\ fI\left(\Re\cup\{X-B\}\ \right)\right]$$

$$\leq\left[\ (\exists x\ )\ (\forall A)\left(A\in\left(\Re\cup\{X-B\}\right)\rightarrow x\in A\ \right)\right]$$

$$=\left[\ \neg\left(\bigcap\Re\subseteq B\ \right)\ \right].$$

Therefore

$$\pi_1\leq\inf_{\Re\in\Im(P(X))}\sup_{B\subseteq X}\Big(\ \left(\Re\subseteq F_\beta\wedge B\in\tau_\beta\right)\underset{\bullet}{\wedge}(\forall\wp)\ \left((\wp\leq\Re)\underset{\bullet}{\wedge}FF(\wp)\rightarrow\right.$$

$$\left.\neg\left(\bigcap\wp\subseteq B\ \right)\right)\rightarrow\neg\left(\ \bigcap\Re\subseteq B\right)\Big)\ \ =\pi_2\,.$$

Conversely,

$$\pi_2\underset{\bullet}{\wedge}\left[\ \left(\Re\subseteq F_\beta\ \right)\underset{\bullet}{\wedge}fI(\Re)\ \right]=\pi_2\ \underset{\bullet}{\wedge}\left[\left(\left(\ \Re\setminus\{B\}\ \right)\cup\{B\}\ \right)\subseteq F_\beta\ \right]\underset{\bullet}{\wedge}\ \ \left[\ fI\left(\left(\Re\setminus\{B\}\right)\cup\{B\}\right)\ \right]$$

$$=\ \pi_2\underset{\bullet}{\wedge}\Big[\ \left(\Re'\subseteq F_\beta\ \right)\wedge\left(X-B\right)\in\tau_\beta\underset{\bullet}{\wedge}(\forall\wp)\left(\left(\ \wp\leq\Re'\right)\underset{\bullet}{\wedge}FF(\wp)\rightarrow\right.$$

$$\left.(\exists x)(\forall A)\left(\ A\in\left(\wp\cup\{B\}\right)\rightarrow x\in A\right)\ \right)\Big]$$

$$=\pi_2\underset{\bullet}{\wedge}\left[\ \left(\Re'\subseteq F_\beta\ \right)\wedge\left(X-B\in\tau_\beta\right)\underset{\bullet}{\wedge}(\forall\wp)\left(\left(\wp\leq\Re'\right)\underset{\bullet}{\wedge}FF(\wp)\rightarrow\neg\left(\bigcap\wp\subseteq X-B\right)\right)\right]$$

$$\leq\left[\ \neg\left(\bigcap\Re'\subseteq X-B\right)\right]=\left[(\exists x)(\forall A)\left(\left(A\in\left(\Re'\cup\{B\}\right)\rightarrow(x\in A)\right)\right]\right.$$

$$=\left[\ (\exists x)(\forall A)\left(A\in\Re\rightarrow(x\in A)\right)\right].$$

Therefore

$$\pi_2 \le \inf_{\Re\in\Im(P(X))}\left[\left(\Re\subseteq F_\beta\right) \underset{\bullet}{\wedge} fI(\Re)\to\left(\exists x\right)(\forall A)\left(A\in\Re\to\left(x\in A\right)\right)\right]=\pi_1 .$$

5. SOME PROPERTIES OF FUZZIFYING β – COMPACTNESS

Theorem 5. 1. For any fuzzifying topological space (X,τ) and $A\subseteq X$,

$$\models \Gamma_\beta(X,\tau) \underset{\bullet}{\wedge} A\in F_\beta \to \Gamma_\beta(A).$$

Proof: For any $\Re\in\Im(P(A))$, we define $\bar{\Re}\in\Im(P(X))$ as follows:

$$\bar{\Re}(B)=\begin{cases}\Re(B) & \text{if } B\subseteq A,\\ 0 & \text{otherwise.}\end{cases}$$

Then $FF\left(\bar{\Re}\right)=1-\inf\left\{\alpha\in[0,1]:F\left(\bar{\Re}_\alpha\right)\right\} =1-\inf\left\{\alpha\in[0,1]:F(\Re_\alpha)\right\}= FF(\Re)$ and

$$\sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\bar{\Re}(B)\right)=\sup_{x\in X}\left(\left(\inf_{x\notin B\subseteq A}\left(1-\bar{\Re}(B)\right)\right)\wedge\left(\inf_{x\notin B\not\subseteq A}\left(1-\bar{\Re}(B)\right)\right)\right)$$

$$=\sup_{x\in X}\left(\inf_{x\notin B\subseteq A}\left(1-\bar{\Re}(B)\right)\right)\wedge\sup_{x\in X}\left(\inf_{x\notin B\not\subseteq A}\left(1-\bar{\Re}(B)\right)\right)$$

$$=\sup_{x\in X}\left(\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)$$

$$=\sup_{x\in A}\left(\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)\vee\sup_{x\notin A}\left(\inf_{x\notin B\subsetneq A}\left(1-\Re(B)\right)\right)$$

If $x\notin A$, then for any $x'\in A$ we have

$$\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)=\inf_{B\subseteq A}\left(1-\Re(B)\right)\le\inf_{x'\notin B\subseteq A}\left(1-\Re(B)\right)$$

Therefore, $\sup_{x\in X}\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)=\sup_{x\in A}\inf_{B\subseteq A}\left(1-\Re(B)\right)$,

$$\left[fI\left(\bar{\Re}\right)\right]=\left[\left(\forall \mathrm{B}\right)\left(\left(\mathrm{B}\le\bar{\Re}\right)\wedge FF\left(\mathrm{B}\right)\to\left(\exists x\right)\left(\forall B\right)\left(\left(B\in\bar{\Re}\right)\to\left(x\in B\right)\right)\right)\right]$$

$$=\inf_{\mathrm{B}\le\Re}\min\left(1,1-FF\left(\mathrm{B}\right)+\sup_{x\in X}\inf_{x\notin B\subseteq X}\left(1-\bar{\Re}(B)\right)\right)$$

$$=\inf_{\mathrm{B}\le\Re}\min\left(1,1-FF(\mathrm{B})+\sup_{x\in X}\inf_{x\notin B\subseteq A}\left(1-\Re(B)\right)\right)=\left[fI(\Re)\right]$$

We want to prove that $F_\beta(A)\underset{\bullet}{\wedge}\left[\Re\subseteq F_\beta\mid_A\right]\le\left[\bar{\Re}\subseteq F_\beta\right]$.
In fact, from Lemma 2.2 (3) we have

$$F_\beta(A)\underset{\bullet}{\wedge}\left[\Re\subseteq F_\beta\mid_A\right]=\max\left(0,F_\beta(A)+\inf_{B\subseteq A}\min\left(1,1-\Re(B)+F_\beta\mid_A(B)\right)-1\right)$$

$$\le\inf_{B\subseteq A}\left(1-\Re(B)+F_\beta(A)+F_\beta\mid_A(B)-1\right)$$

$$\leq \inf_{B\subseteq A}\Big(1-\Re(B)\Big)+\Big(F_\beta(A)\wedge F_\beta|_A(B)\Big)$$

$$\leq \inf_{B\subseteq A}\Big(1-\Re(B)\Big)+\left(F_\beta(A)\wedge \sup_{B'\cap B=B,\,B'\subseteq X}F_\beta(B')\right)$$

$$\leq \inf_{B\subseteq A}\Big(1-\Re(B)\Big)+\sup_{B'\cap A=B,\,B'\subseteq X}\Big(F_\beta(A)\wedge F_\beta(B')\Big)$$

$$\leq \inf_{B\subseteq A}\Big(1-\Re(B)\Big)+\sup_{B'\cap A=B,\,B'\subseteq X}\Big(F_\beta\big(A\cap B'\big)\Big)$$

$$\leq \inf_{B\subseteq A}\Big(1-\Re(B)\Big)+F_\beta(B)$$

$$=\inf_{B\subseteq A}\min\Big(1,1-\Re(B)+F_\beta(B)\Big)$$

$$=\inf_{B\subseteq A}\min\Big(1,1-\overline{\Re}(B)+F_\beta(B)\Big)=\Big[\overline{\Re}\subseteq F_\beta\Big].$$

Furthermore, from Theorem 4.3 we have

$$\Gamma_\beta(X,\tau)\mathbin{\dot\wedge}F_\beta(A)\mathbin{\dot\wedge}\Big[\Re\subseteq F_\beta|_A\Big]\mathbin{\dot\wedge}fI(\Re)\ \leq\Gamma_\beta(X,\tau)\mathbin{\dot\wedge}\Big[\overline{\Re}\subseteq F_\beta\Big]\mathbin{\dot\wedge}fI\big(\overline{\Re}\big)$$
$$\leq\sup_{x\in X}\inf_{x\notin B\subseteq A}\Big(1-\overline{\Re}(B)\Big)=\sup_{x\in X}\inf_{x\notin B\subseteq A}\Big(1-\Re(B)\Big).$$

Then

$$\Gamma_\beta(X,\tau)\mathbin{\dot\wedge}F_\beta(A)\leq\Big[\Re\subseteq F_\beta|_A\Big]\mathbin{\dot\wedge}fI(\Re)\rightarrow\sup_{x\in A}\inf_{x\notin B\subseteq A}\Big(1-\Re(B)\Big)$$

$$\leq\inf_{\Re\in\Im(P(A))}\left(\Re\subseteq F_\beta\ |_A\mathbin{\dot\wedge}fI(\Re)\rightarrow\sup_{x\in A}\inf_{x\notin B\subseteq A}\Big(1-\Re(B)\Big)\right)\ =\Gamma_\beta(A).$$

Theorem 5. 2. Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f\in Y^X$ is surjection. Then $\models\Gamma_\beta(X,\tau)\mathbin{\dot\wedge}C_\beta(f)\rightarrow\Gamma(f(X))$.

Proof: For any $\mathrm{B}\in\Im(P(Y))$, we define as follows:
$\Re(A)=f^{-1}(\mathrm{B})(A)=\mathrm{B}(f(A))$. Then

$$K\big(\Re,X\big)=\inf_{x\in X}\sup_{x\in A}\Re(A)=\inf_{x\in X}\sup_{x\in A}\mathrm{B}(f(A))$$

$$=\inf_{x\in X}\sup_{f(x)\in B}\mathrm{B}(B)=\inf_{y\in f(x)}\sup_{y\in B}\mathrm{B}(B)=K\big(\mathrm{B},f(X)\big),$$

$$\big[\mathrm{B}\subseteq\sigma\big]\mathbin{\dot\wedge}\big[C_\beta(f)\big]=\inf_{B\subseteq Y}\min\big(1,1-\mathrm{B}(B)+\sigma(B)\big)\mathbin{\dot\wedge}\inf_{B\subseteq Y}\min\Big(1,1-\sigma(B)+\tau_\beta\big(f^{-1}(B)\big)\Big)$$

$$=\max\left(0,\inf_{B\subseteq Y}\min\big(1,1-\mathrm{B}(B)+\sigma(B)\big)+\inf_{B\subseteq Y}\min\Big(0,\min\Big(1,1-\sigma(B)+\tau_\beta\big(f^{-1}(B)\big)\Big)\Big)-1\right)$$

$$\leq\inf_{B\subseteq Y}\max\Big(0,\min\big(1,1-\mathrm{B}(B)+\sigma(B)\big)+\min\Big(1,1-\sigma(B)+\tau_\beta\big(f^{-1}(B)\big)\Big)-1\Big)$$

$$\leq \inf_{B \subseteq Y} \min\left(1, 1 - \mathrm{B}(B) + \tau_\beta\left(f^{-1}(B)\right)\right)$$

$$\leq \inf_{A \subseteq X} \inf_{f^{-1}(B) = A} \min\left(1, 1 - \mathrm{B}(B) + \tau_\beta\left(f^{-1}(B)\right)\right)$$

$$\leq \inf_{A \subseteq X} \inf_{f^{-1}(B) = A} \min\left(1, 1 - \mathrm{B}(B) + \tau_\beta(A)\right)$$

$$\leq \inf_{A \subseteq X} \min\left(1, 1 - \sup_{f^{-1}(B) = A} \mathrm{B}(B) + \tau_\beta(A)\right)$$

$$= \inf_{A \subseteq X} \min\left(1, 1 - \Re(A) + \tau_\beta(A)\right) = \left\lfloor \Re \subseteq \tau_\beta \right\rfloor.$$

For any $\wp \leq \Re$, we set $\overline{\wp} \in \Im\left(P(Y)\right)$ defined as follows:

$$\overline{\wp}\left(f(A)\right) = f(\wp)\left(f(A)\right) = \wp(A),\ A \subseteq X.$$

Then $\overline{\wp}\left(f(A)\right) = f(\wp)\left(f(A)\right) \leq f(\Re)\, f((A)) = f\left(\left(f^{-1}(\mathrm{B})\right) f(A)\right) \leq \mathrm{B}\left(f(A)\right)$,

$$F\,F(\wp) = 1 - \inf\left\{\alpha \in [0,1] : F\left(\wp_{[\alpha]}\right)\right\} = 1 - \inf\left\{\alpha \in [0,1] : F\left(f(\wp)_{[\alpha]}\right)\right\}$$

$$= F\,F\left(f(\wp)\right) \leq F\,F\left(\overline{\wp}\right) \text{ and}$$

$$K\left(\overline{\wp}, f(X)\right) = \inf_{y \in f(X)} \sup_{y \in B} \overline{\wp}(B) = \inf_{y \in f(X)} \sup_{y \in B = f(A)} \wp(A)$$

$$\geq \inf_{y \in f(X)} \sup_{f^{-1}(y) \in A} \wp(A) = \inf_{x \in X} \sup_{x \in A} \wp(A) = K(\wp, X)$$

Furthermore,

$$\left\lfloor \Gamma_\beta(X, \tau) \right\rfloor \underset{\cdot}{\wedge} \left\lfloor C_\beta(f) \right\rfloor \underset{\cdot}{\wedge} \left[K'_\circ(\mathrm{B}, f(X)) \right]$$

$$= \left\lfloor \Gamma_\beta(X, \tau) \right\rfloor \underset{\cdot}{\wedge} \left\lfloor C_\beta(f) \right\rfloor \underset{\cdot}{\wedge} \left[K(\mathrm{B}, f(X)) \right] \underset{\cdot}{\wedge} \left[\mathrm{B} \subseteq \sigma\right]$$

$$= \left\lfloor \Gamma_\beta(X, \tau) \right\rfloor \underset{\cdot}{\wedge} \left\lfloor \Re \subseteq \tau_\beta \right\rfloor \underset{\cdot}{\wedge} \left[K(\Re, X) \right]$$

$$= \left\lfloor \Gamma_\beta(X, \tau) \right\rfloor \underset{\cdot}{\wedge} \left\lfloor K_\beta(\Re, X) \right\rfloor$$

$$\leq \left\lfloor (\exists \wp)\left((\wp \leq \Re) \wedge K(\wp, X) \underset{\cdot}{\wedge} F\,F(\wp)\right) \right\rfloor$$

$$\leq \left\lfloor (\exists \wp)\left((\wp \leq \Re) \wedge K\left(\overline{\wp}, f(X)\right) \underset{\cdot}{\wedge} F\,F\left(\overline{\wp}\right)\right) \right\rfloor$$

$$\leq \left\lfloor (\exists \wp')\left((\wp' \leq \Re) \wedge K\left(\wp', f(X)\right) \underset{\cdot}{\wedge} F\,F(\wp')\right) \right\rfloor, \text{ where } K'_\circ \text{ is related to } \sigma.$$

Therefore from Theorem 4.2 we obtain

$$\left\lfloor \Gamma_\beta(X, \tau) \right\rfloor \underset{\cdot}{\wedge} \left\lfloor C_\beta(f) \right\rfloor$$

$$\leq K'_\circ\left(\mathrm{B}, f(X)\right) \rightarrow (\exists \wp')\left((\wp' \leq \Re) \wedge K\left(\wp', f(X)\right) \underset{\cdot}{\wedge} F\,F(\wp')\right)$$

$$\leq \inf_{\mathrm{B} \in \Im(P(X))} \left(K'_\circ\left(\mathrm{B}, f(X)\right) \rightarrow (\exists \wp')\left((\wp' \leq \Re) \wedge K\left(\wp', f(X)\right) \underset{\cdot}{\wedge} F\,F(\wp')\right)\right)$$

$$=\left[\Gamma\left(f(X)\right)\right]$$

Theorem 5. 3. Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f\in Y^X$ is surjection. Then $\models \Gamma_\beta(X,\tau) \dot\wedge I_\beta(f) \to \Gamma(f(X))$.

Proof: From the proof of Theorem 5.2 we have any $\mathrm{B}\in\Im(P(Y))$ we define $\Re\in\Im(P(X))$ as follows: $\Re(A)=f^{-1}(\mathrm{B})=\mathrm{B}(f(A))$. Then $K\left(\Re,X\right)=K\left(\mathrm{B},f(X)\right)$ and $\left[\mathrm{B}\subseteq\sigma_\beta\right]\dot\wedge\left[I_\beta(f)\right]\le\left[\Re\subseteq\tau_\beta\right]$. For any $\wp\le\Re$, we set $\overline{\wp}\in\Im(P(Y))$ defined as follows: $\overline{\wp}(f(A))=f(\wp)(f(A))=\wp(A), A\subseteq X$ and we have $FF(\wp)\le FF(\overline{\wp})$,

$K\left(\overline{\wp},f(X)\right)\ge K\left(\wp,X\right)$. Therefore

$$\left[\Gamma_\beta(X,\tau)\right]\dot\wedge\left[I_\beta(f)\right]\dot\wedge\left[K'_\beta\left(\mathrm{B},f(X)\right)\right]$$

$$=\left[\Gamma_\beta(X,\tau)\right]\dot\wedge\left[I_\beta(f)\right]\dot\wedge\left[K\left(\mathrm{B},f(X)\right)\right]\dot\wedge\left[\mathrm{B}\subseteq\sigma_\beta\right]$$

$$\le\left[\Gamma_\beta(X,\tau)\right]\dot\wedge\left[\Re\subseteq\tau_\beta\right]\dot\wedge\left[K\left(\Re,X\right)\right]$$

$$=\left[\Gamma_\beta(X,\tau)\right]\dot\wedge\left[K_\beta\left(\Re,X\right)\right]$$

$$\le\left[(\exists\wp)\left(\left(\wp\le\Re\right)\wedge K\left(\wp,X\right)\dot\wedge FF(\wp)\right)\right]$$

$$\le\left[(\exists\wp)\left(\left(\wp\le\Re\right)\wedge K\left(\overline{\wp},f(X)\right)\dot\wedge FF(\overline{\wp})\right)\right]$$

$\le\left[(\exists\wp')\left(\left(\wp'\le\beta\right)\wedge K\left(\wp',f(X)\right)\dot\wedge FF(\wp')\right)\right]$, where K'_β is related to σ.

Therefore from Theorem 4. 2 we obtain

$$\left[\Gamma_\beta(X,\tau)\right]\dot\wedge\left[I_\beta(f)\right]$$

$$\le K'_\beta\left(\mathrm{B},f(X)\right)\to(\exists\wp')\left(\left(\wp'\le\mathrm{B}\right)\wedge K\left(\wp',f(X)\right)\dot\wedge FF(\wp')\right)$$

$$\le\inf_{\mathrm{B}\in\Im(P(X))}\left(K'_\beta\left(\mathrm{B},f(X)\right)\to(\exists\wp')\left(\left(\wp'\le\mathrm{B}\right)\wedge K\left(\wp',f(X)\right)\dot\wedge FF(\wp')\right)\right)$$

$$=\left[\Gamma_\beta(f(X))\right]$$

Theorem 5. 4. Let (X,τ) be any fuzzifying $\beta-$topological space and $A,B\subseteq X$. Then

(1) $T_2^\beta(X,\tau)\dot\wedge\left(\Gamma_\beta(A)\wedge\Gamma_\beta(B)\right)\wedge A\cap B=\phi \models^{ws} T_2^\beta(X,\tau)\to$

$$(\exists U)(\exists V)\left(\left(U\in\tau_\beta\right)\wedge\left(V\in\tau_\beta\right)\wedge\left(A\subseteq U\right)\wedge\left(B\subseteq V\right)\wedge\left(U\cap V=\phi\right)\right);$$

(2) $T_2^\beta(X,\tau)\dot\wedge\Gamma_\beta(A)\models^{ws} T_2^\beta(X,\tau)\to A\in F_\beta$.

Proof: (1) Assume $A\cap B=\phi$ and $T_2^\beta(X,\tau)=t$. Let $x\in A$. Then for any $y\in B$ and $\lambda<t$, we have from Corollary 2.1 that

$$\sup\left\{ \tau_\beta(P)\wedge\tau_\beta(Q): x\in P, y\in Q, P\cap Q=\phi \right\}$$
$$=\sup\left\{ \tau_\beta(P)\wedge\tau_\beta(Q): x\in P\subseteq U, y\in Q\subseteq V, U\cap V=\phi \right\}$$
$$=\sup_{U\cap V=\phi}\left\{ \sup_{x\in P\subseteq U}\tau_\beta(P)\wedge \sup_{y\in Q\subseteq V}\tau_\beta(Q): \right\}=\sup_{U\cap V=\phi}\left\{ N_x^\beta(U)\wedge N_y^\beta(V) \right\}$$
$$\geq \inf_{x\neq y}\sup_{U\cap V=\phi}\left\{ N_x^\beta(U)\wedge N_y^\beta(V) \right\}=T_2^\beta(X,\tau)=t>\lambda,$$

i.e., there exist P_y, Q_y such that $x\in P_y, y\in Q_y, P_y\cap Q_y=\phi$ and $\tau_P(P_y)>\lambda$, $\tau_p(Q_y)>\lambda$. Set $\mathrm{B}(Q_y)=\tau_P(Q_y)$ for $y\in B$. Since $\left[\mathrm{B}\subseteq\tau_\beta\right]=1$, we have

$$\left[K_\beta\left(\mathrm{B},B\right)\right]=\left[K\left(\mathrm{B},B\right)\right]=\inf_{y\in B}\sup_{y\in C}\mathrm{B}(C)\geq \inf_{y\in B}\mathrm{B}(Q_y)=\inf_{y\in B}\tau_P(Q_y)\geq\lambda.$$

On the other hand, since $T_2^\beta(X,\tau)\underset{\bullet}{\wedge}\left((\Gamma_\beta(A))\wedge\Gamma_\beta(B)\right)>0$, then 1- t < $\Gamma_\beta(A)\wedge\Gamma_\beta(B)\leq\Gamma_\beta(A)$. Therefore, for any $\lambda\in\left(1-\Gamma_\beta(A),t\right)$, it holds that

$$1-\lambda<\Gamma_\beta(A)\leq 1-\left[K_\beta\left(\mathrm{B},B\right)\right]+\sup_{\wp\leq\mathrm{B}}\left\{ K\left(\wp,B\right)\underset{\bullet}{\wedge}FF(\wp)\right\}$$

$\leq 1-\lambda+\sup_{\wp\leq\mathrm{B}}\left\{ K\left(\wp,B\right)\underset{\bullet}{\wedge}FF(\wp)\right\}$, i.e., $\sup_{\wp\leq\mathrm{B}}\left\{ K\left(\wp,B\right)\underset{\bullet}{\wedge}FF(\wp)\right\}>0$ and there exist $\wp\leq\mathrm{B}$ such that $K\left(\wp,B\right)+FF(\wp)-1>0$, i.e., $1-FF\left(\wp\right)<K\left(\wp,B\right)$. Then, $\inf\left\{\theta: F(\wp_\theta)\right\}<K\left(\wp,B\right)$. Now, there exist θ_1 such that $\theta_1<K\left(\wp,B\right)$ and $F\left(\wp_{\theta_1}\right)$. Since $\wp\leq\mathrm{B}$, we may write $\wp_{\theta_1}=\left\{Q_{y_1},\ldots,Q_{y_n}\right\}$. We put $U_x=\left\{P_{y_1}\cap\ldots\cap P_{y_n}\right\}$, $V_x=\left\{Q_{y_1}\cap\ldots\cap Q_{y_n}\right\}$ and have $V_x\supseteq B$, $U_x\cap V_x=\phi, \tau_\beta(U_x)\geq\tau_\beta(P_{y_1})\wedge\ldots\wedge\tau_\beta(P_{y_n})>\lambda$ because (X,τ) is fuzzifying $\beta-$topological space. Also, $\tau_\beta(V_x)\geq\tau_\beta(Q_{y_1})\wedge\ldots\wedge\tau_\beta(Q_{y_n})>\lambda$. In fact, $\inf_{y\in B}\sup_{y\in D}\wp(D)=K\left(\wp,B\right)>\theta_1$, and for any $y\in B$, there exists D such that $y\in D$ and $\wp(D)>\theta_1, D\in\wp_{\theta_1}$. Similarly, if $\lambda\in\left(1-\left[\Gamma_\beta(A)\wedge\Gamma_\beta(B)\right],t\right)$, then we can find $x_1,\ldots,x_m\in A$ with $U_\circ=U_{x_1}\cup\ldots\cup U_{x_m}\supseteq A$. By putting $V_\circ=V_{x_1}\cup\ldots\cup V_{x_m}$ we obtain $V_\circ\supseteq B$, $U_\circ\cap V_\circ=\phi$ and

$$\left(\exists U\right)\left(\exists V\right)\left(\left(U\in\tau_\beta\right)\wedge\left(V\in\tau_\beta\right)\wedge\left(A\subseteq U\right)\wedge\left(B\subseteq V\right)\wedge\left(U\cap V=\phi\right)\right)\geq$$
$$\tau_\beta\left(U_\circ\right)\wedge\tau_\beta(V_\circ)\ \geq \min_{i=1,\ldots,n}\tau_\beta\left(U_{x_i}\right)\wedge \min_{i=1,2,\ldots,n}\tau_\beta(V_{x_i})>\lambda.$$

Finally, let $\lambda\to t$ and complete the proof.

(2) Assume $\left[T_2^\beta(X,\tau)\underset{\bullet}{\wedge}\Gamma_\beta(A)\right]>0$. For any $x\in X-A$ we have from (1)

$$\sup_{x\in U\subseteq X-A}\tau_\beta(U)\geq\sup\left(\tau_\beta(U)\wedge\tau_\beta(V): x\in U, A\subseteq V, U\cap V=\phi\right)\geq\left[T_2^\beta(X,\tau)\right].$$

From Corollary 2. 1. we obtain

$$F_\beta(A)=\inf_{x\in X-A}N_x^\beta(X-A)=\inf_{x\in X-A}\sup_{x\in U\subseteq X-A}\tau_\beta(U)\geq\left[T_2^\beta(X,\tau)\right].$$

Definition 5. 1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. A unary fuzzy predicate

$Q_\beta \in \Im\left(Y^X\right)$, called fuzzifying $\beta-$closedness, is given as follows:
$Q_\beta(f) := \forall B\left(B\in F_\beta^X \to f^{-1}(B)\in F_\beta^Y\right)$, where F_β^X and F_β^Y are fuzzy families of $\tau,\sigma-\beta-$closed in *X* and *Y* respectively.

Theorem 5. 5. Let (X,τ) be a fuzzifying topological space and (Y,σ) be an $\beta-$ fuzzifying topological space and $f\in Y^X$. Then

$$\models \Gamma_\beta(X,\tau) \dot\wedge T_2^\beta(Y,\sigma) \dot\wedge I_\beta(f) \to Q_\beta(f).$$

Proof: For any $A\subseteq X$, we have the following:
(i) From Theorem 5.1 we have $\left[\Gamma_\beta(X,\tau)\dot\wedge F_\beta^X(A)\right] \le \Gamma_\beta(A)$;
(ii) $I_\beta(F\mid_A) = \inf\limits_{U\in P(Y)} \min\left(1, 1-\sigma_\beta(U)+\tau_\beta\mid_A\left((f\mid_A)^{-1}(U)\right)\right)$

$$= \inf_{U\in P(Y)} \min\left(1,1-\sigma_\beta(U)+\tau_\beta\mid_A\left(A\cap f^{-1}(U)\right)\right)$$

$$= \inf_{U\in P(Y)} \min\left(1,1-\sigma_\beta(U)+\sup_{A\cap f^{-1}(U)=B\cap A}\tau_\beta(B)\right)$$

$$= \inf_{U\in P(Y)} \min\left(1,1-\sigma_\beta(U)+\tau_\beta\left(\left(f^{-1}(U)\right)\right)\right) = I_\beta(f).$$

(iii) From Theorem 5.3, we have $\left[\Gamma_\beta(A)\dot\wedge I_\beta\left(f\mid_A\right)\right] \le \Gamma_\beta\left(f(A)\right)$.
(iv) From Theorem 5.4 (2) we have $T_2^\beta(Y,\sigma)\dot\wedge\Gamma_\beta\left(f(A)\right) \models^{ws} T_2^\beta(Y,\sigma)\to$

$f(A)\in F_\beta^Y$, which implies $\models T_2^\beta(Y,\sigma)\dot\wedge\Gamma_\beta\left(f(A)\right)\to f(A)\in F_\beta^Y$.

By combining (i)-(iv) we have

$$\left[\Gamma_\beta(X,\tau)\dot\wedge T_2^\beta(Y,\sigma)\dot\wedge I_\beta(f)\right] \le \left[\left(F_\beta^X(A)\to\Gamma_\beta(A)\right)\dot\wedge I_\beta(f\mid_A)\dot\wedge T_2^\beta(Y,\sigma)\right]$$

$$\le \left[\left(F_\beta^X(A)\to\left(\Gamma_\beta(A)\right)\dot\wedge I_\beta(f\mid_A)\right)\dot\wedge T_2^\beta(Y,\sigma)\right]$$

$$\le \left[F_\beta^X(A)\to\Gamma_\beta\left(f(A)\right)\dot\wedge T_2^\beta(Y,\sigma)\right].$$

Therefore

$$\left[\Gamma_\beta(X,\tau)\dot\wedge T_2^\beta(X,\tau)\dot\wedge I_\beta(f)\right] \le \left[F_\beta^X(A)\to F_\beta^Y\left(f(A)\right)\right]$$

$$\le \inf_{A\subseteq X}\left(\left[F_\beta^X(A)\to F_\beta^Y\left(f(A)\right)\right]\right) = Q_B(f).$$

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