

“Research Note”

TOPOLOGICAL RING-GROUPOIDS AND LIFTINGS\*

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**Abstract** – We prove that the set of homotopy classes of the paths in a topological ring is a topological ring object (called topological ring-groupoid). Let  $p : \tilde{X} \rightarrow X$  be a covering map and let  $X$  be a topological ring. We define a category  $UTRCov(X)$  of coverings of  $X$  in which both  $X$  and  $\tilde{X}$  have universal coverings, and a category  $UTRGdCov(\pi_1 X)$  of coverings of topological ring-groupoid  $\pi_1 X$ , in which  $X$  and  $\tilde{R}_0 = \tilde{X}$  have universal coverings, and then prove the equivalence of these categories. We also prove that the topological ring structure of a topological ring-groupoid lifts to a universal topological covering groupoid.

**Keywords** – Fundamental groupoids, topological coverings, topological ring-groupoids

1. INTRODUCTION

Let  $X$  be a connected topological group with zero element  $0$ , and let  $p : \tilde{X} \rightarrow X$  be the universal covering map of the underlying space of  $X$ . It follows easily from classical properties of lifting maps to covering spaces that for any point  $\tilde{0}$  in  $\tilde{X}$  with  $p(\tilde{0}) = 0$ , there is a structure of topological group on  $\tilde{X}$  such that  $\tilde{0}$  is the zero element and  $p : \tilde{X} \rightarrow X$  is a morphism of topological groups. We say that the structure of the topological group on  $X$  lifts to  $\tilde{X}$  [1]. It is less generally appreciated that this result fails for the non-connected case. R. L. Taylor [2] showed that the topological group  $X$  determines an obstruction class  $k_X$  in  $H^3(\pi_0 X, \pi_1(X, 0))$ , and that the vanishing of  $k_X$  is a necessary and sufficient condition for the lifting of the topological group structure on  $X$  to the universal covering so that the projection is a morphism. This result was generalized in terms of group-groupoids and crossed modules [3], and then written in a revised version in [4]. A topological version of that was also given in [5].

The ring version of the above results was proved in [6]. Let  $X$  and  $\tilde{X}$  be connected topological spaces and let  $p : \tilde{X} \rightarrow X$  be a universal covering. If  $X$  is a topological ring with a zero element  $0$ , and  $\tilde{0} \in \tilde{X}$  such that  $p(\tilde{0}) = 0$ , then the ring structure of  $X$  lifts to  $\tilde{X}$  [6]. That is,  $\tilde{X}$  becomes a topological ring with zero element  $\tilde{0} \in \tilde{X}$  such that  $p : \tilde{X} \rightarrow X$  is a morphism of topological rings.

In [6] Mucuk defined the notion of a ring-groupoid. He also proved that if  $X$  is a topological ring, then the fundamental groupoid  $\pi_1 X$ , which is the set of all relative to end points homotopy classes of paths in the topological space  $X$ , becomes a ring-groupoid. In addition to this, he proved that if  $X$  is a topological ring whose underlying space has a universal covering, then the category  $TRCov(X)$  of topological ring coverings of  $X$  is equivalent to the category  $RGdCov(\pi_1 X)$  of ring-groupoid coverings of  $\pi_1 X$ .

In this paper we present a similar result for a topological ring-groupoid. The topological ring-groupoid is a topological ring object in the category of topological groupoids. Let  $R$  be a topological ring-

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groupoid and let  $p : \tilde{R} \rightarrow R$  be a universal covering on underlying groupoids such that both topological groupoids  $R$  and  $\tilde{R}$  are transitive. Let  $0$  be the zero element of  $R_0$  and  $\tilde{0} \in \tilde{R}_0$  such that  $p(\tilde{0}) = 0$ . We prove that the topological ring-groupoid structure of  $R$  lifts to  $\tilde{R}$  with zero element  $\tilde{0}$ .

Here we also prove that if  $X$  is a topological ring, whose underlying space has a universal covering, then the category  $UTRCov(X)$  of topological ring coverings in which  $\tilde{X}$  has a universal covering, is equivalent to the category  $UTRGdCov(\pi_1 X)$  of topological ring-groupoid coverings of  $\pi_1 X$ , in which  $\tilde{R}_0 = \tilde{X}$  has a universal covering.

## 2. TOPOLOGICAL RING-GROUPOIDS

We call a subset  $U$  of  $X$  liftable if it is open, path connected and the inclusion  $U \rightarrow X$  maps each fundamental group  $\pi_1(U, x)$ ,  $x \in X$ , to the trivial subgroup of  $\pi_1(X, x)$ . Remark that if  $X$  has a universal covering, then each point  $x \in X$  has a liftable neighborhood [3].

A groupoid consists of two sets  $R$  and  $R_0$  called respectively the set of morphisms or elements and the set objects of the groupoid together with two maps  $\alpha, \beta: R \rightarrow R_0$ , called source and target maps respectively, a map  $1_{(\cdot)}: R_0 \rightarrow R, x \mapsto 1_x$  called the object map and a partial multiplication or composition  $R_{\alpha \times \beta} R \rightarrow R, (b, a) \mapsto b \circ a$  is defined on the pullback

$$R_{\alpha \times \beta} R = \{(b, a) : \alpha(b) = \beta(a)\} \quad [7].$$

These maps are subject to the following conditions:

1.  $\alpha(b \circ a) = \alpha(a)$  and  $\beta(b \circ a) = \beta(b)$ , for each  $(b, a) \in R_{\alpha \times \beta} R$ ,
2.  $c \circ (b \circ a) = (c \circ b) \circ a$  for all  $c, b, a \in R$  such that  $\alpha(b) = \beta(a)$  and  $\alpha(c) = \beta(b)$ ,
3.  $\alpha(1_x) = \beta(1_x) = x$  for each  $x \in R_0$ , where  $1_x$  is the identity at  $x$ ,
4.  $a \circ 1_{\alpha(a)} = a$  and  $1_{\beta(a)} \circ a = a$  for all  $a \in R$ , and
5. each element  $a$  has an inverse  $a^{-1}$  such that  $\alpha(a^{-1}) = \beta(a)$ ,  $\beta(a^{-1}) = \alpha(a)$  and  $a^{-1} \circ a = 1_{\alpha(a)}$ ,  $a \circ a^{-1} = 1_{\beta(a)}$ .

Let  $R$  be a groupoid. For each  $x, y \in R_0$  we write  $R(x, y)$  as a set of all morphisms  $a \in R$  such that  $\alpha(a) = x$  and  $\beta(a) = y$ . We will write  $St_{Rx}$  for the set  $\alpha^{-1}(x)$ , and  $CoSt_{Rx}$  for the set  $\beta^{-1}(x)$  for  $x \in R_0$ . The object or vertex group at  $x$  is  $R(x) = R(x, x) = St_{Rx} \cap CoSt_{Rx}$ . We say  $R$  is transitive (resp. 1-transitive, simply transitive) if for each  $x, y \in R_0$ ,  $R(x, y)$  is non-empty (resp. a singleton, has no more than one element).

Let  $R$  and  $H$  be two groupoids. A morphism from  $H$  to  $R$  is a pair of maps  $f: H \rightarrow R$  and  $f_0: H_0 \rightarrow R_0$  such that  $\alpha_R \circ f = f_0 \circ \alpha_H$ ,  $\beta_R \circ f = f_0 \circ \beta_H$  and  $f(b \circ a) = f(b) \circ f(a)$  for all  $(b, a) \in H_{\alpha \times \beta} H$ .

We refer to [8] and [9] for more details concerning the basic concepts.

Covering morphisms of groupoids are defined in [8] as follows:

A morphism  $f: H \rightarrow R$  of groupoids is called a *covering morphism* if for each  $x \in H_0$ , the restriction of  $f$  mapping  $f_x: St_{Rx} \rightarrow St_{Rf(x)}$  is bijective. Also, the following definition of pullback is given in [10].

Let  $R_{\alpha \times f_0} H_0$  be the pullback

$$R_{\alpha \times f_0} H_0 = \{(a, x) \in R \times H_0 : \alpha(a) = f_0(x)\}.$$

If  $f: H \rightarrow R$  is a covering morphism, then we have a lifting function  $s_f: R_{\alpha \times f_0} H_0 \rightarrow H$  assigning to the pair  $(a, x)$  in  $R_{\alpha \times f_0} H_0$  the unique element  $b$  of  $St_{Hx}$  such that  $f(b) = a$ . Clearly  $s_f$  is inverse to  $(f, \alpha): H \rightarrow R_{\alpha \times f_0} H_0$ . So it is stated that  $f: H \rightarrow R$  is a covering morphism if and only if  $(f, \alpha): H \rightarrow R_{\alpha \times f_0} H_0$  is bijective.

Let  $f: H \rightarrow R$  be a morphism of groupoids. Then for an object  $x \in H_0$  the subgroup  $f[H(x)]$  of  $R(f(x))$  is called the characteristic group of  $f$  at  $x$ . So if  $f$  is the covering morphism then  $f$  maps  $H(x)$  isomorphically to  $f[H(x)]$ . We say that a covering morphism  $f: H \rightarrow R$  is a universal covering morphism if  $H$  is 1-transitive.

A topological groupoid is a groupoid  $R$  such that the sets  $R$  and  $R_0$  are topological spaces, and source, target, object, inverse and composition maps are continuous. Let  $R$  and  $H$  be two topological groupoids. A morphism of topological groupoids is a pair of maps  $f:H \rightarrow R$  and  $f_0:H_0 \rightarrow R_0$  such that  $f$  and  $f_0$  are continuous. A morphism  $f:H \rightarrow R$  of topological groupoids is called a topological covering morphism if and only if  $(f, \alpha):H \rightarrow R_\alpha \times_{f_0} H_0$  is a homeomorphism.

A topological ring is a ring  $R$  with a topology on the underlying set such that the ring structure maps (i.e., group multiplication, group inverse and ring multiplication) are continuous. A topological ring morphism (topological homomorphism) of a topological ring into another is an abstract ring homomorphism which is also a continuous map.

**Definition 1.** A topological ring-groupoid  $R$  is a topological groupoid endowed with a topological ring structure such that the following ring structure maps are morphisms of topological groupoids:

1.  $m:R \times R \rightarrow R, (a,b) \mapsto a+b$ , group multiplication,
2.  $u:R \rightarrow R, a \mapsto -a$ , group inverse map,
3.  $0:(*) \rightarrow R$ , where  $(*)$  is a singleton.
4.  $n:R \times R \rightarrow R, (a,b) \mapsto ab$ , ring multiplication,

We write  $a+b$  for the group multiplication,  $ab$  for the ring multiplication of  $a$  and  $b$ , and  $b \circ a$  for the composition in the topological groupoid  $R$ . Also, by 3 if  $0$  is the zero element of  $R_0$  then  $1_0$  is that of  $R$ .

**Proposition 2.** In a topological ring-groupoid  $R$ , we have the interchange laws

1.  $(c \circ a) + (d \circ b) = (c+d) \circ (a+b)$  and
2.  $(c \circ a)(d \circ b) = (cd) \circ (ab)$

whenever both  $(c \circ a)$  and  $(d \circ b)$  are defined.

**Proof:** Since  $m$  is a morphism of groupoids,

$$(c \circ a) + (d \circ b) = m[c \circ a, d \circ b] = m[(c, d) \circ (a, b)] = m(c, d) \circ m(a, b) = (c+d) \circ (a+b).$$

Similarly, since  $n$  is a morphism of groupoids we have

$$(c \circ a)(d \circ b) = n[c \circ a, d \circ b] = n[(c, d) \circ (a, b)] = n(c, d) \circ n(a, b) = (cd) \circ (ab).$$

**Example 3.** Let  $R$  be a topological ring. Then a topological ring-groupoid  $R \times R$  with object set  $R$  is defined as follows: The morphisms are the pairs  $(y, x)$ , the source and target maps are defined by  $\alpha(y, x) = x$  and  $\beta(y, x) = y$ , the groupoid composition is defined by  $(z, y) \circ (y, x) = (z, x)$ , the group multiplication is defined by  $(z, t) + (y, x) = (z+y, t+x)$  and ring multiplication is defined by  $(z, t)(y, x) = (zy, tx)$ .  $R \times R$  has product topology. So all structure maps of ring-groupoid  $R \times R$  becomes continuous. Then  $R \times R$  is a topological ring-groupoid.

We know from [6] that if  $X$  is a topological ring, then the fundamental groupoid  $\pi_1 X$  becomes a ring-groupoid. We will now give a similar result.

**Proposition 4.** Let  $X$  be a topological ring whose underlying space  $X$  has a universal covering. Then the fundamental groupoid  $\pi_1 X$  becomes a topological ring-groupoid.

**Proof:** Let  $X$  be a topological ring with the structure maps

$$\begin{aligned} m: X \times X &\rightarrow X, (a,b) \mapsto a+b \\ n: X \times X &\rightarrow X, (a,b) \mapsto ab \\ 0: (*) &\rightarrow R \end{aligned}$$

and the inverse map

$$u: X \rightarrow X, a \mapsto -a.$$

Then these maps give the following induced maps:

$$\begin{aligned}\pi_1 m: \pi_1 X \times \pi_1 X &\rightarrow \pi_1 X, ([a],[b]) \mapsto [a+b] \\ \pi_1 n: \pi_1 X \times \pi_1 X &\rightarrow \pi_1 X, ([a],[b]) \mapsto [ab] \\ \pi_1 u: \pi_1 X &\rightarrow \pi_1 X, [a] \mapsto [-a] \\ \pi_1 0: \pi_1(*) &\rightarrow \pi_1 R.\end{aligned}$$

It is known from [6] that  $\pi_1 X$  is a ring-groupoid. In addition, from [11],  $\pi_1 X$  is a topological group-groupoid. Further, we will prove that the ring multiplication

$$\pi_1 n: \pi_1 X \times \pi_1 X \rightarrow \pi_1 X, ([a],[b]) \mapsto [a][b] = [ab]$$

is continuous.

By assuming that  $X$  has a universal covering [12], each  $x \in X$  has a liftable neighbourhood. Let  $U$  consist of such sets. Then  $\pi_1 X$  has a lifted topology [8]. So the set  $\tilde{U}$ , consisting of all liftings of the sets in  $U$ , forms a basis for the topology on  $\pi_1 X$ . Let  $\tilde{U}$  be an open neighbourhood of  $\tilde{e}$  and a lifting of  $U$  in  $U$ . Since the multiplication

$$n: X \times X \rightarrow X, (a,b) \mapsto ab$$

is continuous, there is a neighborhood  $V$  of 0 in  $X$  such that  $n(V \times V) \subseteq U$ . Using the condition on  $X$  and choosing  $V$  small enough we can assume that  $V$  has a liftable neighbourhood. Let  $\tilde{V}$  be the lifting of  $V$ . Then we have  $\pi_1 n(\tilde{V} \times \tilde{V}) \subseteq \tilde{U}$ . Hence

$$\pi_1 n: \pi_1 X \times \pi_1 X \rightarrow \pi_1 X, ([a],[b]) \mapsto [a][b] = [ab]$$

becomes continuous. So  $\pi_1 X$  is a topological ring-groupoid.

**Proposition 5.** Let  $R$  be a topological ring-groupoid and let  $0 \in R_0$  be the zero element in the ring  $R_0$ . Then the transitive component  $C_R(0)$  of 0 is a topological ring-groupoid.

**Proof:** In [6] it was proved that  $C_R(0)$  is a ring-groupoid. Further, since  $C_R(0)$  is a subset of  $R$ ,  $C_R(0)$  is a topological ring-groupoid with induced topology.

**Proposition 6.** Let  $R$  be a topological ring-groupoid and let  $0 \in R_0$  be the zero element in the ring  $R_0$ . Then the star  $St_R 0 = \{a \in R: \alpha(a) = 0\}$  of 0 becomes a topological ring.

The proof is straightforward.

Let  $R$  and  $H$  be two topological ring-groupoids. A morphism  $f: H \rightarrow R$  from  $H$  to  $R$  is a morphism of underlying topological groupoids preserving the topological ring structure, i.e.,  $f(a+b) = f(a) + f(b)$  and  $f(ab) = f(a)f(b)$  for  $a, b \in H$ . A morphism  $f: H \rightarrow R$  of topological ring-groupoids is called a *topological covering morphism* if it is a covering morphism on the underlying topological groupoids.

**Definition 7.** Let  $R$  be a topological ring-groupoid and let  $X$  be a topological ring. A topological action of the topological ring-groupoid  $R$  on  $X$  consists of a topological ring morphism  $w: X \rightarrow R_0$  and a continuous action of the underlying topological groupoid of  $R$  on the underlying space of  $X$  via  $w: X \rightarrow R_0$  such that the following interchange laws hold

1.  $({}^b y) + ({}^a x) = {}^{b+a} (y+x)$
2.  $({}^b y)({}^a x) = {}^{ba} (yx)$

whenever both sides are defined.

**Example 8.** Let  $R$  be a topological ring-groupoid which acts on a topological ring  $X$  via  $w: X \rightarrow R_0$ . In [13] it is proved that  $R \bowtie X$  is a topological groupoid with object set  $(R \bowtie X)_0 = X$  and morphism set  $R \bowtie X = \{(a, x) \in R \times X : {}^a x = y\}$ . Furthermore, the projection  $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$  becomes a covering morphism of topological groupoids. Also, in [14] it is shown that if a ring-groupoid  $R$  acts on a ring  $X$  via  $w: X \rightarrow R_0$ , then  $R \bowtie X$  becomes a ring-groupoid and the projection  $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$  is a covering morphism of ring-groupoids. Clearly, the ring operations

$$(a, x) + (b, y) = (a + b, x + y) \text{ and} \\ (a, x)(b, y) = (ab, xy)$$

are also continuous since they are defined by the operations of the topological rings  $R$  and  $X$ . Thus  $R \bowtie X$  becomes a topological ring-groupoid and the projection  $p: R \bowtie X \rightarrow R, (a, x) \mapsto a$  is a covering morphism of topological ring-groupoids.

### 3. TOPOLOGICAL COVERINGS

Let  $X$  be a topological space. Then we have a category denoted by  $TCov(X)$  whose objects are covering maps  $p: \tilde{X} \rightarrow X$  and a morphism from  $p: \tilde{X} \rightarrow X$  to  $q: \tilde{Y} \rightarrow X$  is a map  $f: \tilde{X} \rightarrow \tilde{Y}$  (hence  $f$  is a covering map) such that  $p = qf$ . Further, we have a groupoid  $\pi_1 X$  called a fundamental groupoid [8] and have a category denoted by  $GdCov(\pi_1 X)$  whose objects are the groupoid coverings  $p: \tilde{R} \rightarrow \pi_1 X$  of  $\pi_1 X$  and a morphism from  $p: \tilde{R} \rightarrow \pi_1 X$  to  $q: \tilde{H} \rightarrow \pi_1 X$  is a morphism  $f: \tilde{R} \rightarrow \tilde{H}$  of groupoids (hence  $f$  is a covering morphism) such that  $p = qf$ .

We recall the following result from Brown [8].

**Proposition 9.** Let  $X$  be a topological space which has a universal covering. Then the category  $TCov(X)$  of topological coverings of  $X$  and the category  $GdCov(\pi_1 X)$  of covering groupoids of fundamental groupoid  $\pi_1 X$  are equivalent.

Let  $X$  and  $\tilde{X}$  be topological rings. A map  $p: \tilde{X} \rightarrow X$  is called a covering morphism of topological rings if  $p$  is a morphism of rings and  $p$  is a covering map on the underlying spaces. For a topological ring  $X$ , we have a category denoted by  $TRCov(X)$  whose objects are covering morphisms of topological rings  $p: \tilde{X} \rightarrow X$  and a morphism from  $p: \tilde{X} \rightarrow X$  to  $q: \tilde{Y} \rightarrow X$  is a map  $f: \tilde{X} \rightarrow \tilde{Y}$  (hence  $f$  is a covering map) such that  $p = qf$ . For a topological ring  $X$ , the fundamental groupoid  $\pi_1 X$  is a ring-groupoid and so we have a category denoted by  $RGdCov(\pi_1 X)$  whose objects are the ring-groupoid coverings  $p: \tilde{R} \rightarrow \pi_1 X$  of  $\pi_1 X$  and a morphism from  $p: \tilde{R} \rightarrow \pi_1 X$  to  $q: \tilde{H} \rightarrow \pi_1 X$  is a morphism  $f: \tilde{R} \rightarrow \tilde{H}$  of ring-groupoids (hence  $f$  is a covering morphism) such that  $p = qf$ .

Then the following result is given in [6].

**Proposition 10.** Let  $X$  be a topological ring whose underlying space has a universal covering. Then the category  $TRCov(X)$  of the topological ring coverings of  $X$  is equivalent to the category  $RGdCov(\pi_1 X)$  of ring-groupoid coverings of the ring-groupoid  $\pi_1 X$ .

In addition to these results, here we prove Theorem 11.

Let  $UTRCov(X)$  be the full subcategory of  $TRCov(X)$  on those objects  $p: \tilde{X} \rightarrow X$  in which both  $\tilde{X}$  and  $X$  have universal coverings. Let  $UTRGdCov(\pi_1 X)$  be the full subcategory of  $TRGdCov(\pi_1 X)$  on those objects  $p: \tilde{R} \rightarrow \pi_1 X$  in which  $X$  and  $\tilde{R}_0 = \tilde{X}$  have universal coverings. Then we prove the following result.

**Theorem 11.** The categories  $UTRCov(X)$  and  $UTRGdCov(\pi_1 X)$  are equivalent.

**Proof:** Define a functor

$$\pi_1:UTRCov(X) \rightarrow UTRGdCov(\pi_1 X)$$

as follows: Let  $p: \tilde{X} \rightarrow X$  be a covering morphism of topological rings in which both underlying spaces  $\tilde{X}$  and  $X$  have universal coverings. Then the induced morphism  $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$  is a covering morphism of ring-groupoids [6]. Further,  $\pi_1 p$  is a morphism of topological group-groupoids [11]. So  $\pi_1 p$  becomes a morphism of topological ring-groupoids. Since  $\pi_1 p$  is a covering morphism of ring-groupoids,  $(\pi_1 p, \alpha): \pi_1 \tilde{X} \rightarrow \pi_1 X_\alpha \times_{(\pi_1 p)_0} (\pi_1 \tilde{X})_0$  is bijective. On the other hand,  $\pi_1 p$  is a morphism of topological ring-groupoids and  $\alpha$  is source map of topological ring-groupoid  $\pi_1 X$ , so  $(\pi_1 p, \alpha)$  becomes continuous. We prove that  $(\pi_1 p, \alpha)$  is an open mapping.

Let  $[\tilde{a}]$  be a morphism of  $\pi_1 \tilde{X}(\tilde{x}, \tilde{y})$ . Since  $X$  and  $\tilde{X}$  have universal coverings,  $\pi_1 X$  and  $\pi_1 \tilde{X}$  have lifting topology. So we can choose liftable neighbourhoods  $\tilde{V}, \tilde{V}'$  of  $\tilde{x}, \tilde{y}$ , respectively such that  $U=p(\tilde{V}), U'=p(\tilde{V}')$  are liftable neighbourhoods of  $x=p(\tilde{x}), y=p(\tilde{y})$ , respectively. If  $W = \tilde{V}_{\tilde{x}}[\tilde{a}](\tilde{V}'_{\tilde{y}})^{-1}$ , then  $\pi_1 p(W)$  is a basic neighbourhood of  $\pi_1 p([\tilde{a}])$ , while  $(\pi_1 p, \alpha)(W) = \pi_1 p(W)_\alpha \times_{(\pi_1 p)_0} V$ , which is open in  $\pi_1 X_\alpha \times_{(\pi_1 p)_0} \tilde{X}$ . So  $(\pi_1 p, \alpha)$  is a homeomorphism. Hence  $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$  becomes a covering morphism of topological ring-groupoids.

We now define a functor

$$\Gamma:UTRGdCov(\pi_1 X) \rightarrow UTRCov(X)$$

as follows: Let  $q: \tilde{R} \rightarrow \pi_1 X$  be a covering morphism of topological ring-groupoids in which both  $\tilde{R}_0 = \tilde{X}$  and  $X$  have universal coverings. Since  $X$  has a universal covering,  $\tilde{X}$  has lifting topology. Hence we have a covering map  $p: \tilde{X} \rightarrow X$  of topological spaces, where  $p=q_0$  and  $\tilde{R}_0 = \tilde{X}$  [8]. Further, since  $q$  is a covering morphism of topological ring-groupoids,  $q$  and  $p=q_0$  are morphisms of topological rings. So  $p$  becomes a covering morphism of topological rings.

Since the category of topological ring coverings is equivalent to the category of ring-groupoid coverings, by Proposition 10 the proof is completed by the following diagram:

$$\begin{array}{ccc} UTRCov(X) & \xrightarrow{\pi_1} & UTRGdCov(\pi_1 X) \\ \downarrow & & \downarrow \\ TRCov(X) & \xrightarrow{\pi_1} & RGdCov(\pi_1 X) \end{array}$$

Before giving the main theorem we adopt the following definition:

**Definition 12.** Let  $p: \tilde{R} \rightarrow R$  be a covering morphism of groupoids and  $q: H \rightarrow R$  a morphism of groupoids. If there exists a unique morphism  $\tilde{q}: H \rightarrow \tilde{R}$  such that  $q=p\tilde{q}$  then we say that  $q$  lifts to  $\tilde{q}$  by  $p$ .

We recall the following theorem from [8] which is an important result to have the lifting maps on covering groupoids.

**Theorem 13.** Let  $p: \tilde{R} \rightarrow R$  be a covering morphism of groupoids,  $x \in R_0$  and  $\tilde{x} \in \tilde{R}_0$  such that  $p_0(\tilde{x})=x$ . Let  $q: H \rightarrow R$  be a morphism of groupoids such that  $H$  is transitive and  $\tilde{y} \in H_0$  such that  $q_0(\tilde{y})=x$ . Then the morphism  $q: H \rightarrow R$  uniquely lifts to a morphism  $\tilde{q}: H \rightarrow \tilde{R}$  such that  $\tilde{q}_0(\tilde{y})=\tilde{x}$  if and only if  $q[H(\tilde{y})] \subseteq p[\tilde{R}(\tilde{x})]$ , where  $H(\tilde{y})$  and  $\tilde{R}(\tilde{x})$  are the object groups.

Let  $R$  be a topological ring-groupoid and let  $0 \in R_0$  be the zero element in the ring  $R_0$ . Let  $\tilde{R}$  be just a topological groupoid and let  $p: \tilde{R} \rightarrow R$  be a covering morphism of topological groupoids  $\tilde{0} \in \tilde{R}_0$ , such that  $p(\tilde{0})=0$ . We say the topological ring structure of  $R$  lifts to  $\tilde{R}$  if there exists a topological ring structure on  $\tilde{R}$  with the zero element  $\tilde{0} \in \tilde{R}_0$ , such that  $\tilde{R}$  is a topological ring-groupoid and  $p: \tilde{R} \rightarrow R$  is a morphism of topological ring-groupoids.

**Theorem 14.** Let  $\tilde{R}$  be a topological groupoid and let  $R$  be a topological ring-groupoid. Let  $p: \tilde{R} \rightarrow R$  be a universal covering on the underlying groupoids such that both groupoids  $R$  and  $\tilde{R}$  are transitive. Let  $0$  be the zero element in the ring  $R_0$  and  $\tilde{0} \in \tilde{R}_0$  such that  $p(\tilde{0})=0$ . Then the topological ring structure of  $R$  lifts to  $\tilde{R}$  with zero element  $\tilde{0}$ .

**Proof:** Since  $R$  is a topological ring-groupoid, it has the following maps:

$$\begin{aligned} m: R \times R &\rightarrow R, (a, b) \mapsto a + b \\ n: R \times R &\rightarrow R, (a, b) \mapsto ab \\ u: R &\rightarrow R, a \mapsto -a \\ 0: (*) &\rightarrow R. \end{aligned}$$

Since  $\tilde{R}$  is a universal covering, the object group  $\tilde{R}(\tilde{0})$  has one element at most. So by Theorem 13 these maps respectively lift to the maps

$$\begin{aligned} \tilde{m}: \tilde{R} \times \tilde{R} &\rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b} \\ \tilde{n}: \tilde{R} \times \tilde{R} &\rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} \tilde{b} \\ \tilde{u}: \tilde{R} &\rightarrow \tilde{R}, \tilde{a} \mapsto -\tilde{a} \\ \tilde{0}: (*) &\rightarrow \tilde{R} \end{aligned}$$

by  $p: \tilde{R} \rightarrow R$  such that

$$\begin{aligned} p(\tilde{a} + \tilde{b}) &= p(\tilde{a}) + p(\tilde{b}), \\ p(\tilde{a} \tilde{b}) &= p(\tilde{a})p(\tilde{b}), \\ p(\tilde{u}(\tilde{a})) &= -p(\tilde{a}). \end{aligned}$$

Since the multiplication  $m: R \times R \rightarrow R, (a, b) \mapsto a + b$  is associative, we have  $m(m \times 1) = m(1 \times m)$ , where  $1$  denotes the identity map. Then again by Theorem 13 these maps  $m(m \times 1)$  and  $m(1 \times m)$  respectively lift to

$$\tilde{m}(\tilde{m} \times 1), \tilde{m}(1 \times \tilde{m}): \tilde{R} \times \tilde{R} \times \tilde{R} \rightarrow \tilde{R}$$

which coincide on  $(\tilde{0}, \tilde{0}, \tilde{0})$ . By the uniqueness of the lifting we have  $\tilde{m}(\tilde{m} \times 1) = \tilde{m}(1 \times \tilde{m})$ , i.e.,  $\tilde{m}$  is associative. Similarly,  $\tilde{n}$  is associative. In a similar way, we can show that  $\tilde{0}$  is the zero element and  $-\tilde{a}$  is the inverse element of  $\tilde{a}$ . Further, we will prove that the group multiplication

$$\tilde{m}: \tilde{R} \times \tilde{R} \rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b}$$

is continuous.

By assuming that  $R$  has a universal covering, we can choose a cover  $U$  of liftable subsets of  $R$ . Since the topology on  $\tilde{R}$  is the lifted topology, the set consisting of all liftings of the sets in  $U$  forms a basis for the topology on  $\tilde{R}$ . Let  $\tilde{U}$  be an open neighbourhood of  $\tilde{0}$  and a lifting of  $U$  in  $U$ . Since the multiplication

$$m: R \times R \rightarrow R, (a, b) \mapsto a + b$$

is continuous, there is a neighbourhood  $V$  of  $0$  in  $R$  such that  $m(V \times V) \subseteq U$ . Using the condition on  $R$  and choosing  $V$  small enough, we can assume that  $V$  is liftable. Let  $\tilde{V}$  be the lifting of  $V$ . Then  $p \tilde{m}(\tilde{V} \times \tilde{V}) = m(V \times V) \subseteq U$  and so we have  $\tilde{m}(\tilde{V} \times \tilde{V}) \subseteq \tilde{U}$ . Hence

$$\tilde{m}: \tilde{R} \times \tilde{R} \rightarrow \tilde{R}, (\tilde{a}, \tilde{b}) \mapsto \tilde{a} + \tilde{b}$$

becomes continuous. Similarly,  $\tilde{n}$  is continuous. Further, the distributive law is satisfied as follows:

Let  $p_1, p_2: R \times R \times R \rightarrow R$  be the morphisms defined by

$$p_1(a,b,c)=ab, p_2(a,b,c)=bc$$

and

$$(p_1, p_2): R \times R \times R \rightarrow R \times R, (a, b, c) \mapsto (ab, bc)$$

for  $a, b, c \in R$ . Since the distributive law is satisfied in  $R$ , we have  $n(1 \times m) = m(p_1, p_2)$ . The maps  $n(1 \times m)$  and  $m(p_1, p_2)$  respectively lift to the maps

$$\tilde{n}(1 \times \tilde{m}), \tilde{m}(\tilde{p}_1, \tilde{p}_2) : \tilde{R} \times \tilde{R} \times \tilde{R} \rightarrow \tilde{R}$$

coinciding at  $(\tilde{0}, \tilde{0}, \tilde{0})$ . So by Theorem 13 we have  $\tilde{n}(1 \times \tilde{m}) = \tilde{m}(\tilde{p}_1, \tilde{p}_2)$ . That means the distribution law on  $\tilde{R}$  is satisfied. Hence  $\tilde{R}$  becomes a topological ring-groupoid and clearly  $p$  is a morphism of the topological ring-groupoid.

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