"Research Note"

SOME MAXIMAL INEQUALITIES FOR RANDOM VARIABLES AND APPLICATIONS[*]

M. AMINI[**] AND A. BOZORGNIA

Department of Statistics, Ferdowsi University, Mashhad, I. R. of Iran
mamini48@yahoo.com, bozorg@math.um.ac.ir

Abstract – In this paper, we extend some famous maximal inequalities and obtain strong laws of large numbers for arbitrary random variables by use of these inequalities and martingale techniques.

Keywords – Kolmogorove's inequality, Hajek-Renyi inequality, Strong law of large numbers, Martingale

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on the probability space (Ω, F, P). The convergence problem with probability one (*w.p.1*) for the sequence $\{\frac{1}{n}(\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i))\}$ has been studied by many authors for the sequence of independent random variables. The strong laws of large numbers and complete convergence for ND random variables were studied by Amini and Bozorgnia [1], [2]. In [3] we obtained some maximal inequalities under condition $E[X_n \mid F_{n-1}]=0$ where $F_n = \sigma(X_1, X_2, ..., X_n)$ for all $n \geq 1$. In this paper, we extend some famous maximal inequalities by martingale techniques, and then by using these inequalities, we obtain strong laws of large numbers and some strong limit theorems for arbitrary random variables. To prove our main results we need the following lemma.

Lemma 1. ([4]) Let $\{X_n, F_n, n \geq 1\}$ be a submartingale and $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers, then for every $\varepsilon > 0$,

$$P[\max_{1 \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] \leq \frac{1}{\varepsilon}(b_1^{-1}EX_1^+ + \sum_{k=2}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+)),$$

Remark 1. Under the assumption of Lemma 1 for every $1 \leq m \leq n$, we have,

$$P[\max_{m \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] = P[\max_{1 \leq k \leq n-m+1} \frac{|X_{k+m-1}|}{b_{k+m-1}} > \varepsilon] \leq \frac{1}{\varepsilon}(b_m^{-1}EX_m^+ + \sum_{k=2}^{n-m+1} b_{k+m-1}^{-1}(EX_{k+m-1}^+ - EX_{k+m-2}^+))$$

$$= \frac{1}{\varepsilon}(b_m^{-1}EX_m^+ + \sum_{k=m+1}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+)).$$

---

[*]Received by the editor January 13, 2004 and in final revised form January 31, 2007
[**]Corresponding author

2. MAXIMAL INEQUALITIES

Hajek and Renyi [5] proved the following important inequality. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $E(X_n) = 0, E(X_n^2) < \infty, n \geq 1$ and $\{b_n, n \geq 1\}$ is a positive nondecreasing sequence of real numbers, then for every $m < n$ and $\varepsilon > 0$,

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} (\sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + \sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2}),$$

where $\sigma_k^2 = Var(X_k)$.

This inequality has been studied by many authors. The latest literature is given by Liu, et.al. [6] for negative association random variables and Christofides [7] and [8]. We extend this inequality for arbitrary random variables.

Theorem 1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $E(X_n) = 0$, $EX_n^2 < \infty, n \geq 1$, and $\{b_n, n \geq 1\}$ be a sequence of positive nondercreasing real numbers, then for every $\varepsilon > 0$,

i)
$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{m} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}).$$

ii)
$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2} + 2 \sum_{k=2}^{m} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_m^2} + \sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=m+1}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_j^2}).$$

where $\sigma_k = \sqrt{Var(X_k)}$

Proof: Set $S_{1n} = \sum_{k=1}^{n} X_k^+$ and $S_{2n} = \sum_{k=1}^{n} X_k^-$. Since $E[S_{1n} | F_{n-1}] \geq S_{1(n-1)}$. w.p.1. and $E[S_{2n} | F_{n-1}] \geq S_{2(n-1)}$. w.p.1. Hence the sequences $\{S_{1n}, F_n, n \geq 1\}$ and $\{S_{2n}, F_n, n \geq 1\}$ are submartingales, where $F_n = \sigma(X_1, ..., X_n)$ for all $n \geq 1$. In addition, if h is any real convex and nondecreasing function, then $\{h(S_{1n}), F_n, n \geq 1\}$ and $\{h(S_{2n}), F_n, n \geq 1\}$ are also submartingales. Thus for every $\varepsilon > 0$, Lemma 1 implies that

i)
$$P[\max_{1 \leq k \leq n} \frac{S_{1k}}{b_k} \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} \frac{(S_{1k})^2}{b_k^2} \geq \varepsilon^2] \leq \varepsilon^{-2} (b_1^{-2} E(X_1^+)^2 + \sum_{k=2}^{n} b_k^{-2} (E(S_{1k})^2 - E(S_{1(k-1)})^2))$$

$$\leq \frac{1}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}),$$

and similarly we have

$$P[\max_{1 \leq k \leq n} \frac{S_{2k}}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}),$$

Now, by $|S_n| \leq S_{1n} + S_{2n}$, we obtain

$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} \frac{S_{1k}}{b_k} \geq \frac{\varepsilon}{2}] + P[\max_{1 \leq k \leq n} \frac{S_{2k}}{b_k} \geq \frac{\varepsilon}{2}] \leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=2}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}).$$

ii) By Remark 1 and part i for $1 \leq m < n$,

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{m \leq k \leq n} \frac{S_{1k}}{b_k} \geq \frac{\varepsilon}{2}] + P[\max_{m \leq k \leq n} \frac{S_{2k}}{b_k} \geq \frac{\varepsilon}{2}]$$

$$\leq \frac{8}{\varepsilon^2} (\sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2} + 2 \sum_{k=2}^{m} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_m^2} + \sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + 2 \sum_{k=m+1}^{n} \frac{\sigma_k \sum_{i=1}^{k-1} \sigma_i}{b_k^2}).$$

Hence, the proof is complete. In the following we assume $\sum_{i=1}^{0}\sigma_i = 0$ and $S_{1(0)} = S_{2(0)} = 0$.

Corollary 1. Under the assumptions of Theorem 1 we have

i)
$$P[\max_{1\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{k=2}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}).$$

ii)
$$P[\max_{m\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=m+1}^{n}\frac{\sigma_k^2}{b_k^2}+\sum_{k=1}^{m}\frac{\sigma_k^2}{b_m^2}+2\sum_{k=2}^{m}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_k}{b_m^2}+2\sum_{k=1}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}).$$

Remark 2. We have the following inequalities

i)
$$ES_{1k}^2 - ES_{1(k-1)}^2 = E(X_k^+)^2 + 2E(X_k^+\sum_{j=1}^{k-1}X_j^+)\le\sigma_k^2+2\sigma_k\sum_{j=1}^{k-1}\sigma_j;$$

the inequality holds by Cauchy- Schwartz' inequality. Similarly

$$ES_{2k}^2 - ES_{2(k-1)}^2 \le \sigma_k^2 + 2\sigma_k\sum_{j=1}^{k-1}\sigma_j.$$

ii)
$$\sum_{k=1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{k=2}^{n}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}\le\sum_{k=1}^{n}\frac{1}{b_k^2}(\sigma_k^2+2\sigma_k\sum_{i=1}^{k-1}\sigma_i)\le2\sum_{k=1}^{n}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2}.$$

iii) If $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2}$ converges, then series $\sum_{k=1}^{\infty}\frac{\sigma_k^2}{b_k^2}$ and $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}$ are converged.

The following corollary is an extension of Kolmogorov's inequality for arbitrary random variables.

Corollary 2. Let $\{X_n, n\ge 1\}$ be a sequence of random variables with $E(X_n) = 0, EX_n^2 < \infty, n\ge 1$ and $b_k = 1, k\ge 1$. Then for every $\varepsilon > 0$,

$$P[\max_{1\le k\le n}|S_k|\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=1}^{n}\sigma_k^2+2\sum_{k=2}^{n}\sigma_k\sum_{i=1}^{k-1}\sigma_i).)$$

3. SOME STRONG LIMIT THEOREMS

In this section, we use the results of section 2 to prove some strong limit theorems for arbitrary random variables.

Theorem 2. Let $\{X_n, n\ge 1\}$ be a sequence of random variables and $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. If $\sum_{k=1}^{\infty}\frac{\sigma_k\sum_{i=1}^{k}\sigma_i}{b_k^2} < \infty$. then for every $0 < \beta < 2$,

i) $E(\sup_n(\frac{|S_n|}{b_n})^\beta) < \infty$, and ii) if $b_n\to\infty$, as $n\to\infty$, then $\frac{S_n}{b_n}\to 0$, $w.p.1$.

where $S_n = \sum_{k=1}^{n}(X_k - E(X_k))$, $\sigma_n^2 = Var(X_n)$ and $\sigma_n = \sqrt{Var(X_n)}$, $n\ge 1$.

Proof: *i*) Note

$$E(\sup_n(\frac{|S_n|}{b_n})^\beta) < \infty \Leftrightarrow \int_1^{\infty} P[\sup_n(\frac{|S_n|}{b_n}) > t^{\frac{1}{\beta}}]dt < \infty.$$

By Theorem 1, we get

www.SID.ir

$$\int_1^\infty P[\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}]dt \le \wedge\int_1^\infty t^{\frac{-2}{\beta}}(\sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} + 2\sum_{n=1}^\infty \frac{\sigma_n\sum_{i=1}^{n-1}\sigma_i}{b_n^2})dt = C(\sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} + 2\sum_{n=1}^\infty \frac{\sigma_n\sum_{i=1}^{n-1}\sigma_i}{b_n^2}) < \infty.$$

where $0 < C < \infty$. The second inequality holds by Remark 2.

ii) For every $\varepsilon > 0$, Theorem 1 implies that

$$P[\sup_{k\ge m}\frac{|S_k|}{b_k} > \varepsilon] \le \frac{8}{\varepsilon^2}(\sum_{k=1}^m \frac{\sigma_k^2}{b_m^2} + 2\sum_{k=2}^m \frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_m^2} + \sum_{k=m+1}^\infty \frac{\sigma_k^2}{b_k^2} + 2\sum_{k=m+1}^\infty \frac{\sigma_k\sum_{i=1}^{k-1}\sigma_i}{b_k^2}),$$

Therefore $\sum_{k=1}^\infty \frac{\sigma_k\sum_{i=1}^k \sigma_i}{b_k^2} < \infty$, and Kronecker's Lemma yield $\lim_{m\to\infty} P[\sup_{k\ge m}\frac{|S_k|}{b_k} > \varepsilon] = 0$. Hence Proposition 5.6 in [9] completes the proof.

Corollary 3. Under the assumptions of Theorem 2, if $\sup_n(\sigma_n\sum_{k=1}^n \sigma_k) < \infty$, then for every $\alpha > \frac{1}{2}$, $\varepsilon > 0$ *and* $m \ge 1$.

i) $$P[\sup_{k\ge m}\frac{|S_k|}{k^\alpha} > \varepsilon] \le \frac{16}{\varepsilon^2} \times \frac{2\alpha}{2\alpha-1} \times m^{1-2\alpha} \sup_n(\sigma_n\sum_{k=1}^n \sigma_k).$$

ii) $\frac{S_n}{n^\alpha} \to 0$, *w.p.*1. as $n \to \infty$, and for every $0 < \beta < 2$, $E\sup_n(\frac{|S_n|}{b_n})^\beta < \infty$.

Theorem 3. Let $\{X_n, n \ge 1\}$ be a sequence of random variables, if $\sum_{k=1}^\infty \sigma_k\sum_{i=1}^k \sigma_i < \infty$ and $E(X_n) = 0$. Then $\sum_{k=1}^\infty X_k$ converges *w.p.1*.

Proof: By Corollary 2 for every $\varepsilon > 0$,

$$P[\sup_{k\ge 1}|S_{n+k} - S_n| > \varepsilon] = \lim_{m\to\infty} P[\max_{1\le k\le m}|S_{n+k} - S_n| > \varepsilon] \le \frac{8}{\varepsilon^2}(\sum_{k=n+1}^\infty \sigma_k^2 + 2\sum_{k=n+1}^\infty \sigma_k\sum_{i=1}^{k-1}\sigma_i).$$

Since by Remark 2 the left hand side of the above inequality tends to zero when $n \to \infty$, Lemma 7.1 in [9] implies that $\sum_{k=1}^\infty X_k$ converges *w.p.*1.

Example: If $\{X_n, n \ge 1\}$ is a sequence of *i.i.d.* random variables with distribution $U[0,1]$, taking $Y_n = \prod_{k=1}^n X_k$ and $\sigma_n = \sqrt{Var(Y_n)}$, we have $\sigma_n < (\frac{1}{\sqrt{3}})^n$, $\sum_{n=1}^\infty EY_n < \infty$, and

$$\sum_{k=1}^\infty \sigma_k\sum_{i=1}^k \sigma_i < \frac{1}{\sqrt{3}-1} \cdot (\sum_{k=1}^\infty 3^{-\frac{k}{2}} - \sum_{k=1}^\infty 3^{-k}) < \infty.$$

Hence these and Theorem 3 prove that $\sum_{n=1}^\infty \prod_{k=1}^n X_k$, converges *w.p.1*.

The following Corollary, which is an extension of Kolmogorov's Theorem, provides strong law of large numbers for arbitrary random variables.

Corollary 4. Let $\{X_n, n \ge 1\}$ be a sequence of random variables with $E(X_n) = \mu_n$ and $Var(X_n) = \sigma_n^2$ for all $n \ge 1$. If $\sum_{k=1}^\infty \frac{\sigma_k\sum_{i=1}^k \sigma_i}{k^2} < \infty$, then $\frac{1}{n}\sum_{k=1}^n (X_k - \mu_k) \to 0$. *w.p.*1

REFERENCES

1. Amini, M. & Bozorgnia, A. (2000). Negatively dependent bounded random variable probability inequalities and the strong law of large numbers. *Journal of App. Math. stoch. Analysis*, *13*(3), 261-267.

2. Amini, M. & Bozorgnia, A. (2003). Compelete convergence for negative dependence random variables. *Journal of App. Math. Stoch. Analysis*, *16*(2), 121-126.

3. Amini, M. & Bozorgnia, A. (2002). Some strong limit theorems for negatively dependent random variables. *Research bulletin of Isfahan university, 16*(1, 2), 173-178.

4. Laha, R. G. & Rohtgi, V. K. (1979). *Probability Theory*. John Wiley & Sons. New York.

5. Hajek, J. & Renyi, A. (1955). A generalization of an inequality of Kolomogorov. *Acta. Math. Acad. Sci. Hung.*, *6*, 281-284.

6. Liu, J., Gan, S. & Chen, P. (1999) The Hajek-Renyi inequality for the NA random variables and its applications. *Statistics & Probability Letters*, *43*, 99-105.

7. Christofides, T. C. (2000). Maximal inequalities for demimartingales and a strong law of large numbers. *Statistics & Probability Letters*, *50*, 357-363.

8. Christofides, T. C. & Vaggelatou, E. (2004). A connection between supermodular ordering and positive/negative association. *Journal of Multivariate analysis*, *88*, 138-151.

9. Karr, A. F. (1993). *Probability*. Springer-Verlag, New York.

www.SID.ir