Iranian Journal of Science & Technology, Transaction A, Vol. 31, No. A2
Printed in The Islamic Republic of Iran, 2007

ON PROBLEMS REDUCING TO THE GOURSAT PROBLEM FOR A FOURTH ORDER EQUATION*

A. MAHER[1**] AND YE. A. UTKINA[2]

[1]Department of Mathematics, Faculty of Science, Assiut University, 71516, Egypt
Email: a_maher69@yahoo.com
[2]Department of Differential Equations, Kazan State University,
18 Kremlyovskaya St., Kazan, 420008, Russia

Abstract – In this paper, we investigate some problems which can be reduced to the Goursat problem for a fourth order equation. Some results and theorems are given concerning the existence and uniqence for the solution of the suggested problem.

Keywords – Fourth order of partial differential equations, Goursat problem

1. INTRODUCTION

In the domain $D=\{(x,y);\quad x_0<x<x_1,\quad y_0<y<y_1\}$ we consider the equation

$$\begin{aligned} L(u)=u_{xxyy}+a_{21}u_{xxy}+a_{12}u_{xyy}+a_{11}u_{xy}+a_{20}u_{xx}+a_{02}u_{yy}+a_{10}u_x+ \\ +a_{01}u_y+a_{00}u=0\,;\quad a_{ij}\in C^{i+j}(D),\quad (\quad i=0,1,2,\quad j=0,1,2) \end{aligned} \tag{1}$$

where the class C^{k+l} means the existence and continuity for all derivatives

$$\partial^{r+s}/\partial x^r\partial y^s\in C(D)\ (r=0,...k;\qquad s=0,...,l).$$

We will call the solution of the class

$$\partial^{i+j}/\partial x^i\partial y^j\in C(D)\ (i=0,1,2;\qquad j=0,1)$$

as regular. The equation (1) is the generalization of the Bussineska-Lyava equation that describes longitudinal waves in a thin elastic shaft with allowance for the effects of the crosswise inertia. The Goursat problem for (1) consists of finding a solution in D on conditions defined for the characteristics:

$$\begin{gathered} u(x_0,y)=\varphi(y),u_x(x_0,y)=\varphi_1(y),\ y\in p, \\ \varphi,\varphi_1\in C^2(p)\,;u(x,y_0)=\psi(x),u_y(x,y_0)=\psi_1(x)\ \ x\in q\ ,\ \psi,\psi_1\in C^2(q), \\ y\in p=[y_0,y_1],\ x\in q=[x_0,x_1]. \end{gathered} \tag{2}$$

Here, we consider the conditions of function coincidence from (2) on the boundary of their definitions (co-ordination conditions) as satisfied:

*Received by the editor November 11, 2006 and in final revised form April 3, 2007
**Corresponding author

$$\varphi'(y_0) = \psi_1(x_0),\ \varphi(y_0) = \psi(x_0),\ \psi'(x_0) = \varphi_1(y_0). \tag{3}$$

The solution of the mentioned (Goursat) problem is obtained in [1] and its uniqueness is shown. In this paper, we investigate characteristic problems for the equation (1), in which at least one of the Goursat conditions is changed by the value of the next normal derivative. As a result, each time a characteristic that is a carrier of boundary conditions is obtained, the highest order given by a normal derivative is increased by a unit. In this case, we use the results from the work [2]. Here, the problems that are obtained from the equation (1) and the conditions (2), by replacing conditions in Eq. (2), following conditions are concluded:

$$u_{xx}(x_0, y) = \varphi_2(y),\ \varphi_2 \in C^2(p), \tag{4}$$

$$u_{yy}(x, y_0) = \psi_2(x),\ y_2 \in C^2(q). \tag{5}$$

In fact, in this paper we investigate the question of clarification of conditions for coefficients of the equation that provide a definite level of the intentionality of these problems and, naturally, about the acquisition of the decision itself. In this paper the methods of works ([2] and [3]) are developed. For more details see ([4], [5] and [6]).

We have decided to indicate only a class of the desired unknown function u in formulations of all problems suggested below, taking into account that coefficients of the equation (1) are also chosen from the same class.

2. MAIN RESULTS

Problem 1. To find the function

$$u \in C^{2+2}(D) \cap C^{2+0}(D \cup p) \cap C^{0+1}(D \cup q),$$

which is a solution to the equation (1) in D, that satisfies all the conditions (2), except the first condition from (2), which is replaced by the condition (4).

This problem may be reduced to the Goursat problem with the help of a certain integral equation. For this purpose, we integrate equation (1) twice with respect to y in the bound from y_* to y; (y_*, $y \in P$), then, in the obtained relation we direct y_* to y_0 and x_* to x_0.
Taking into account the boundary conditions we obtain

$$a_{02}(x_0, y)\varphi(y) + \int_{y_0}^{y} [(y - \eta)A(\eta) + B(\eta)]\, \varphi(\eta) d\eta = r(y), \tag{6}$$

Where

$$A(y) = a_{02yy}(x_0, y) - a_{01y}(x_0, y) + a_{00}(x_0, y),$$
$$B(y) = -2a_{02y}(x_0, y) + a_{01}(x_0, y),$$

And

$$\begin{aligned}
\mathrm{r(y)} = \int_{\mathrm{y_0}}^{\mathrm{y}} \{ & (y-\eta)[a_{21\eta}(x_0,\eta)\varphi_2(\eta) - a_{12\eta\eta}(x_0,\eta)\varphi_1(\eta) - a_{10}(x_0,\eta)\varphi_1(\eta) + \\
& + a_{11\eta}(x_0,\eta)\varphi_1(\eta) - a_{20}(x_0,\eta)\varphi_2(\eta)] - a_{21}(x_0,\eta)\varphi_2(\eta) - \\
& - \varphi_1(\eta)[-2a_{12\eta}(x_0,\eta) + a_{11}(x_0,\eta)]d\eta\} - \varphi_2(y) + \psi''(x_0) + \\
& + (y-y_0)[\psi_1''(x_0) + a_{21}(x_0,y_0)\psi''(x_0) + a_{12}(x_0,y_0)\psi_1'(x_0) - \\
& - a_{12y}(x_0,y_0)\varphi_1(y_0) + a_{11}(x_0,y_0)\varphi_1(y_0) + a_{02}(x_0,y_0)\psi_1(x_0) - \\
& - a_{02y}(x_0,y_0)\psi\ \ (x_0) + a_{01}(x_0,y_0)\psi\ \ (x_0)] - a_{12}(x_0,y)\varphi_1(y) + \\
& \qquad + a_{12}(x_0,y_0)\varphi_1(y_0) + a_{02}(x_0,y_0)\psi(x_0).
\end{aligned} \tag{7}$$

Here, $\varphi(y)$ is the function from the first condition of (2). For its membership in the class $C^2(P)$ it is sufficient, in addition to the already available conditions of coefficient smoothness, to assume that

$$a_{02}\ ,\quad a_{12} \in C^{0+2}(D \cup P);\quad a_{01}\ ,\quad a_{11}\ ,\ a_{21} \in C^{0+1}(D \cup P).$$

When $a_{02}(x_0,y) \neq 0$, $\varphi(y)$ is uniquely defined from (6) through the resolution of this integral equation. We can easily see the two possibilities to obtain the exact solutions (6) that are provided correspondingly by the identities:

$$A(y) \equiv 0\ , \qquad B(y) - yA(y) \equiv 0. \tag{8}$$

In this case of the first identity

$$\varphi(y) = \frac{r(y)}{a_{02}(x_0,y)} - \frac{1}{a_{02}(x_0,y)} \int_{y_0}^{y} \frac{B(\eta)r(\eta)}{a_{02}(x_0,\tau)}[\exp\int_{y}^{\eta} \frac{B(\tau)}{a_{02}(x_0,\tau)} d\tau]d\eta, \tag{9}$$

while in the case of the second

$$\varphi(y) = \frac{r(y)}{a_{02}(x_0,y)} - \frac{1}{a_{02}(x_0,y)} \int_{y_0}^{y} \frac{A(\eta)r(\eta)}{a_{02}(x_0,\tau)}[\exp\int_{y}^{\eta} \frac{A(\tau)\tau}{a_{02}(x_0,\tau)} d\tau]d\eta. \tag{10}$$

Now, let $a_{02}(x_0,y) \equiv 0$. Here, we also have the two possibilities of the explicit solution (6). When

$$a_{01}(x_0,y) \neq 0$$

and

$$\begin{gathered}
a_{01}\ ,\ a_{11}\ ,\ a_{21} \in C^{0+2}(D \cup P);\quad a_{20}\ ,\ a_{10}\ ,\ a_{00} \in C^{0+1}(D \cup P); \\
a_{12} \in C^{0+3}(D \cup P);\quad \varphi_1\ ,\ \varphi_2 \in C^3(P).
\end{gathered}$$

we find

$$\varphi(y) = \frac{1}{a_{01}(x_0,y)}\{r'(y) - \int_{y}^{y} T(x_0,\eta)r'(\eta)[\exp\int_{y}^{\eta} T(x_0,\tau)d\tau]d\eta\}, \tag{11}$$

Where

$$T(x_0, y) = \frac{-a_{01y}(x_0, y) + a_{00}(x_0, y)}{a_{01}(x_0, y)}.$$

If $a_{01}(x_0, y) \equiv 0$, and when $a_{00}(x_0, y) \neq 0$, while

$$a_{11}, a_{21} \in C^{0+3}(D \cup P); \quad a_{20}, a_{10}, a_{00} \in C^{0+2}(D \cup P);$$
$$a_{12} \in C^{0+4}(D \cup P); \quad \varphi_1, \varphi_2 \in C^4(P),$$

we have

$$\varphi(y) = r''(y)[a_{00}(x_0, y)]^{-1}. \tag{12}$$

Thus, the following take place

Theorem 1. Problem (1) is uniquely solved when the inequality $a_{02}(x_0.y) \neq 0$ is satisfied. The cases of the explicit reduction to the Goursat problem are provided by any of the two of sets of the following conditions:

1) $a_{02}(x_0.y) \neq 0$

and at least one of the identities $A(y) \equiv 0, \quad B(y) - yA(y) \equiv 0$ is satisfied;

2) $a_{02}(x_0, y) \equiv 0, \quad a_{01}^2(x_0, y) + a_{00}^2(x_0, y) \neq 0.$

Problem 2. Its formulation from the previous one is in a way that (4) is changed for the second condition, not the first one in (2). Here the solution is found in the class as in problem (1).

In this case the relation (6) should be written with a provision for (7), like the equation for defining $\varphi_1(y)$. Since the formulation of the Goursat problem presupposes the realization of the coordination (3), the right part of the equation will be completely known.

$$a_{12}(x_0, y)\varphi_1(y) + \int_{y_0}^{y} \left[(y - \eta)A_1(\eta) + B_1(\eta) \right] \varphi_1(\eta) d\eta = r_1(y), \tag{13}$$

Where

$$\begin{aligned} r_1(y) = \int_{y_0}^{y} \{ & (y - \eta)[a_{21\eta}(x_0, \eta)\varphi_2(\eta) - a_{20}(x_0, \eta)\varphi_2(\eta)] - a_{21}(x_0, \eta)\varphi_2(\eta) - \\ & - \varphi(\eta)[A(\eta)(y - \eta) + B(\eta)]\} d\eta - \varphi_2(y) + \psi''(x_0) + \\ & + (y - y_0)[\psi_1''(x_0) + a_{21}(x_0, y_0)\psi''(x_0) + a_{12}(x_0, y_0)\psi_1'(x_0) - \\ & - a_{12y}(x_0, y_0)\varphi_1(y_0) + a_{11}(x_0, y_0)\varphi_1(y_0) + a_{02}(x_0, y_0)\psi_1(x_0) - \\ & - a_{02y}(x_0, y_0)\psi(x_0) + a_{01}(x_0, y_0)\psi(x_0)] - a_{02}(x_0, y_0)\varphi(y) + \\ & + a_{12}(x_0, y_0)\varphi_1(x_0) + a_{02}(x_0, y_0)\psi(x_0) \end{aligned}$$

and

$$A_1(y) = a_{12yy}(x_0, y) - a_{11y}(x_0, y) + a_{10}(x_0, y),$$
$$B_1(y) = -2a_{12y}(x_0, y) + a_{11}(x_0, y).$$

The role of the inequality $a_{02}(x_0, y) \neq 0$, is played here by $a_{12}(x_0, y) \neq 0$. It also provides, for the record, $\varphi_1(y)$ through the resolution of the equation. Here, we should apply the conditions on the coefficients like in the similar case in problem (1). The analogs of the identities (8) are

$$A_1(y) \equiv 0, \qquad B_1(y) - yA_1(y) \equiv 0. \tag{14}$$

When $A_1(y) \equiv 0$, then

$$\varphi_1(y) = \frac{r_1(y)}{a_{12}(x_0, y)} - \frac{1}{a_{12}(x_0, y)} \int_{y_0}^{y} \frac{B_1(\eta) r_1(\eta)}{a_{12}(x_0, \tau)} [\exp \int_{y}^{\eta} \frac{B_1(\tau)}{a_{12}(x_0, \tau)} d\tau] d\eta,$$

while in the case of the second identity (14)

$$\varphi_1(y) = \frac{r_1(y)}{a_{12}(x_0, y)} - \frac{y}{a_{12}(x_0, y)} \int_{y_0}^{y} \frac{A_1(\eta) r_1(\eta)}{a_{12}(x_0, \tau)} [\exp \int_{y}^{\eta} \frac{A_1(\tau)\tau}{a_{12}(x_0, \tau)} d\tau] d\eta.$$

If $a_{12}(x_0, y) \equiv 0$, $a_{11}(x_0, y) \neq 0$ and

$$a_{01}, a_{11}, a_{21} \in C^{0+2}(D \cup P); \quad a_{20}, a_{10}, a_{00} \in C^{0+1}(D \cup P);$$
$$a_{02} \in C^{0+3}(D \cup P); \quad \varphi_1, \varphi_2 \in C^3(P).$$

then

$$\varphi_1(y) = \frac{1}{a_{11}(x_0, y)} \{ r_1'(y) - \int_{y}^{y} T_1(x_0, \eta) r_1'(\eta) [\exp \int_{y}^{\eta} T_1(x_0, \tau) d\tau] d\eta \},$$

where

$$T_1(x_0, y) = \frac{-a_{11y}(x_0, y) + a_{10}(x_0, y)}{a_{11}(x_0, y)}.$$

If $a_{11}(x_0, y) \equiv 0$, and when $a_{10}(x_0, y) \neq 0$, while

$$a_{01}, a_{21} \in C^{0+3}(D \cup P); \quad a_{20}, a_{10}, a_{00} \in C^{0+2}(D \cup P);$$
$$a_{12} \in C^{0+4}(D \cup P); \quad \varphi_1, \varphi_2 \in C^4(P),$$

we have

$$\varphi_1(y) = r_1''(y)[a_{10}(x_0, y)]^{-1}.$$

As a result of the conduced considerations the following theorem can be written:

Theorem 2. The problem (2) is uniquely solved when the inequality $a_{12}(x_0.y) \neq 0$ is satisfied. The explicit solvability takes place when any of the two conditions requirement group is realized satisfied:
1) $a_{12}(x_0.y) \neq 0$
and at least one of the identities $A_1(y) \equiv 0, \quad B_1(y) - yA_1(y) \equiv 0$ is satisfied;
2) $a_{12}(x_0, y) \equiv 0, \quad a_{11}^2(x_0, y) + a_{10}^2(x_0, y) \neq 0.$

Since in the equation (1) variables *x* and *y* occur equally independent, problems on the change of the third and fourth Goursat conditions (2) for (5) are symmetrical to the previous two. Then, we will be limited by writing conditions for the coefficients only.

Problem 3. It will be obtained by the change of the third Goursat condition (2) for (5). We will be searching for the solution in the class

$$u \in C^{2+2}(D) \cap C^{1+0}(D \cup p) \cap C^{0+2}(D \cup q),$$

During the investigation of this problem, we do like we did in the problem (1), only we integrate (1) twice, not by *y*, but by *x*. For $\psi(x)$ the analog of equations (4) and (5) is obtained:

$$a_{20}(x, y_0)\psi(x) + \int_{x_0}^{x} [(x-\eta)K(\eta) + M(\eta)]\psi(\eta)d\eta = r(x), \tag{15}$$

where

$$K(x) = a_{20xx}(x, y_0) - a_{10x}(x, y_0) + a_{00}(x\ , y_0),$$
$$M(x) = -2a_{20x}(x\ , y_0) + a_{10}(x\ , y_0),$$

and

$$\begin{aligned} \mathrm{r(x)} = \int_{x_0}^{x} \{ &(x-\eta)[a_{12\eta}(\eta, y_0)\psi_2(\eta) - a_{21\eta\eta}(\eta, y_0)\psi_1(\eta) - a_{01}(\eta, y_0)\psi_1(\eta) + \\ &+ a_{11\eta}(\eta, y_0)\psi_1(\eta) - a_{02}(\eta, y_0)\psi_2(\eta)] - a_{12}(\eta, y_0)\psi_2(\eta) - \\ &- \psi_1(\eta)[-2a_{21\eta}(\eta, y_0) + a_{11}(\eta, y_0)]\}d\eta - \psi_2(x) + \varphi''(y_0) + \\ &+ (x - x_0)[\varphi_1''(y_0) + a_{12}(x_0, y)\varphi''(x_0) + a_{21}(x_0, y_0)\varphi_1'(y_0) - \\ &- a_{21x}(x_0, y_0)\psi_1(x_0) + a_{11}(x_0, y_0)\psi_1(y_0) + a_{20}(x_0, y_0)\varphi_1(x_0) - \\ &- a_{20x}(x_0, y_0)\varphi\ (y_0) + a_{10}(x_0, y_0)\varphi\ (y_0)] - a_{21}(x, y_0)\psi_1(x) + \\ &+ a_{21}(x_0, y_0)\psi_1(x_0) + a_{20}(x_0, y_0)\varphi(y_0). \end{aligned}$$

With the help of which, we come to the following results

Theorem 3. The problem (3) is uniquely reduced to the Goursat problem when any set of conditions is satisfied:

1) $a_{20}(x.y_0) \neq 0$ and $a_{20}\ , a_{21} \in C^{2+0}(D \cup q);\ \ a_{11}, a_{12}\ , a_{10} \in C^{1+0}(D \cup q)$;

2) $a_{20}(x.y_0) \neq 0$ and $a_{20}\ , a_{21} \in C^{2+0}(D \cup q);\ \ a_{11}, a_{12}\ , a_{10} \in C^{1+0}(D \cup q)$;

any of the following identities is satisfied $K(x) \equiv 0, \quad M(x) - yK(x) \equiv 0$

3) $a_{20}(x.y_0) \equiv 0$, $a_{10}(x.y_0) \neq 0$, *and* $a_{11}, a_{12}\ , a_{10} \in C^{2+0}(D \cup q);$

$$a_{21} \in C^{3+0}(D \cup q);\ \ a_{02}, a_{01}\ , a_{00} \in C^{1+0}(D \cup q); \psi_1\ , \psi_2 \in C^3(q).$$

4) $a_{20}(x.y_0) \equiv 0$, $a_{10}(x.y_0) \equiv 0$, $a_{00}(x, y_0) \neq 0$, and $a_{11}, a_{12} \in C^{3+0}(D \cup q);$

$$a_{21} \in C^{4+0}(D \cup q);\ \ a_{02}, a_{01}\ , a_{00} \in C^{2+0}(D \cup q); \psi_1\ , \psi_2 \in C^4(q).$$

Here, the realization of the first group of conditions leads to the fact that $\psi(x)$ will be written in resolution terms, while the realization of the groups (2), (3), (4) and (5) will provide for writing it in the explicit form with the help of the formulas of (9)-(12) types correspondingly.

Problem 4. The fourth condition (2) is changed for (5). The solution class is the same as in problem (3). In this case, for $\psi_1(x)$, the analog of the equation (13) is obtained:

$$a_{21}(x, y_0)\psi_1(x) + \int_{x_0}^{x}[(x-\eta)K_1(\eta) + M_1(\eta)]\psi_1(\eta)d\eta = r_1(x),$$

where

$$\begin{aligned}\mathrm{r}_1(\mathrm{x}) = \int_{\mathrm{x}_0}^{\mathrm{x}} \{(x-\eta)[a_{12\eta}(\eta, y_0)\psi_2(\eta) - a_{02}(\eta, y_0)\psi_2(\eta)] - a_{12}(\eta, y_0)\psi_2(\eta) - \\ -\psi(\eta)[K(\eta)(x-\eta) + M(\eta)]\}d\eta - \psi_2(x) + \varphi''(y_0) + \\ +(x-x_0)[\varphi_1''(y_0) + a_{12}(x_0, y_0)\varphi''(y_0) + a_{21}(x_0, y_0)\varphi_1'(y_0) - \\ - a_{21x}(x_0, y_0)\psi_1(x_0) + a_{11}(x_0, y_0)\psi_1(x_0) + a_{20}(x_0, y_0)\varphi_1(y_0) - \\ - a_{20x}(x_0, y_0)\varphi\ (y_0) + a_{10}(x_0, y_0)\psi(x_0)] - a_{20}(x, y_0)\psi(x) + \\ + a_{21}(x_0, y_0)\psi_1(x_0) + a_{20}(x_0, y_0)\psi(x_0),\end{aligned}$$

and

$$\begin{aligned}K_1(x) = a_{21xx}(x, y_0) - a_{11x}(x, y_0) + a_{01}(x\ , y_0), \\ M_1(x) = -2a_{21x}(x, y_0) + a_{11}(x, y_0).\end{aligned}$$

With the help of which, we come to the following results.

Theorem 4. The problem (4) is uniquely reduced to the Goursat problem when any of the following sets of conditions is satisfied:

1) $a_{21}(x.y_0) \neq 0$ and $a_{20}\ , a_{21} \in C^{2+0}(D \cup q);\ a_{11}, a_{12}\ , a_{10} \in C^{1+0}(D \cup q);$

2) $a_{21}(x.y_0) \neq 0$ and $a_{20}\ , a_{21} \in C^{2+0}(D \cup q);\ a_{11}, a_{12}\ , a_{10} \in C^{1+0}(D \cup q)$;

when at least one of the following additional identities is satisfied

$$K_1(x) \equiv 0, \quad M_1(x) - xK_1(x) \equiv 0,$$

3) $a_{21}(x.y_0) \equiv 0$, $a_{11}(x.y_0) \neq 0$, *and* $a_{11}, a_{12}\ , a_{10} \in C^{2+0}(D \cup q);$

$$a_{20} \in C^{3+0}(D \cup q);\ a_{02}, a_{01}\ , a_{00} \in C^{1+0}(D \cup q); \psi_1\ , \psi_2 \in C^3(q).$$

4) $a_{21}(x.y_0) \equiv 0$, $a_{11}(x.y_0) \equiv 0$, $a_{01}(x, y_0) \neq 0$, *and* $a_{10}\ , a_{12} \in C^{3+0}(D \cup q);$

$$a_{20} \in C^{4+0}(D \cup q);\ a_{02}, a_{01}\ , a_{00} \in C^{2+0}(D \cup q);\ \psi_1\ , \psi_2 \in C^4(q).$$

Here, the realization of the first group of conditions leads to the fact that $\psi_1(x)$ will be written in resolution terms, while the realization of the groups (2), (3), (4) and (5) will provide for writing it in the explicit form with the help of the formulas of (9)-(12) types correspondingly.

REFERENCES

1. Utkina, Ye. A. & Maher, A. (2005). On a fourth order pseudoparabolic equation, *Proc. Pakistan Acd. Sci. 42*(3), 187-195.

2. Zhegalov, V. I. (1994). "Relation between the Boundary Values of Goursat Problem and the Normal Derivatives" Conditionally Well-Posed Problem. *TVP Sci. Publ*., 346-349.
3. Kotukhov, M. P. (1996). On some differential properties of the solution of partial differential equations, *Izvestia VUZov: Matematika*., *5*, 59-62.
4. Bitsadze, A. V. (1981). *Some classes of partial differential equations*. Moscow, Nauka.
5. Muenz, G. (1934). *Integral equations*: Vol. 1, Moscow, Russia: State Technical and Theoretical Press.
6. Zhegalov, V. I. (1999). The three-dimensional analog of the Goursat problem. (Non-classical and mixed type equations, Novosibirisk), 73-76. [in Russian].