Iranian Journal of Science & Technology, Transaction A, Vol. 31, No. A2
Printed in The Islamic Republic of Iran, 2007

ON THE HELICES IN THE GALILEAN SPACE G_3*

A. O. OGRENMIS**, M. ERGUT AND M. BEKTAS

Department of Mathematics, Firat University, 23119, Elazig, Turkey
Emails: aogrenmis@firat.edu.tr, mergut@firat.edu.tr & mbektas@firat.edu.tr

Abstract – T. Ikawa obtained an ordinary differential equation for the circular helix. Recently, the helix have been investigated by many differential geometers such as T. Ikawa, H. Balgetir, M. Bektas, M. Ergut, N. Ekmekci and H. H. Hacısalihoglu. In this paper, making use of this author's methods, we obtained characterizations of helix for a curve with respect to the Frenet frame in 3-dimensional Galilean space G_3.

Keywords – Galilean Space, Helix

1. PRELIMINARIES

The Galilean space is a three dimensional complex projective space, P_3, in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points, $I_1, I_2 \in f$ (the absolute points) [1].

We shall take, as a real model of the space G_3, a real projective space P_3, with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$, and a real line $f \subset w$, on which an elliptic involution ε has been defined.

Let ε be in homogeneous coordinates

$$w....x_0 = 0, \quad f...x_0 = x_1 = 0$$
$$\varepsilon : (0:0:x_2:x_3) \to (0:0:x_3:-x_2).$$

In the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\begin{aligned} x' &= a_{11} + a_{12}x \\ y' &= a_{21} + a_{22}x + a_{23}\cos\varphi\, y + a_{23}\sin\varphi\, z \\ z' &= a_{31} + a_{32}x - a_{23}\sin\varphi\, y + a_{23}\cos\varphi\, z \end{aligned} \quad (1)$$

where a_{ij} and φ are real numbers.

For $a_{12} = a_{23} = 1$, we have the subgroup, B_6, the group of Galilean motions:

$$\begin{aligned} & x' = a + x \\ B_6 \ldots\ & y' = b + cx + y\cos\varphi + z\sin\varphi \\ & z' = d + ex - y\sin\varphi + z\cos\varphi \end{aligned}$$

In G_3 there are four classes of lines:

a) (proper) nonisotropic lines - they do not meet the absolute line f.

---

*Received by the editor October 10, 2006 and in final revised form May 20, 2007
**Corresponding author

b) (proper) isotropic lines - lines that do not belong to the plane *w* but meet the absolute line *f*.
c) unproper nonisotropic lines - all lines of *w* but *f*.
d) the absolute line *f*.
Planes *x* =*const.* are Euclidean and so is the plane *w*. Other planes are isotropic.
In what follows, the coefficients a_{12} and a_{23} will play a special role.
In particular, for $a_{12} = a_{23} = 1$, (1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3.

2. FRENET FORMULAS

For a curve $c : I \to G_3$, $I \subseteq R$ parametrized by the invariant parameter $s = x$, is given in the coordinate form

$$c(x) = (x, y(x), z(x)), \tag{2}$$

the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \sqrt{y''^2(x) + z''^2(x)}, \quad \tau(x) = \frac{\det(c'(x), c''(x), c'''(x))}{\kappa^2(x)} \tag{3}$$

The associated moving trihedron is given by

$$\begin{aligned} T &= c'(x) = (1, y'(x), z'(x)), \\ N &= \frac{1}{\kappa(x)} c''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ B &= \frac{1}{\kappa(x)}(0, -z''(x), y''(x)). \end{aligned} \tag{4}$$

The vectors *T*, *N*, and *B* are called the vectors of the tangent, principal normal and the binormal line, respectively. For their derivatives the following Frenet's formulas hold [2]

$$\begin{aligned} \nabla_T T &= \kappa N, \\ \nabla_T N &= \tau B, \\ \nabla_T B &= -\tau N \end{aligned} \tag{5}$$

3. THE CHARACTERIZATIONS IN THE GALILEAN SPACE G_3

We used the same terminologies as in [3-5], and the following Definitions, Theorems and Corollaries were obtained.

Definition 3. 1. Let α be a curve in 3-dimensional Galilean space G_3, and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α. If κ and τ are positive constants along α, then α is called a circular helix with respect to the Frenet frame.

Definition 3. 2. Let α be a curve in 3-dimensional Galilean space G_3, and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α. A curve α such that

$$\frac{\kappa}{\tau} = const.$$

is called a general helix with respect to Frenet frame.

Theorem 3. 1. Let α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$, if and only if

$$\nabla_T \nabla_T \nabla_T T - K \nabla_T T = 3\kappa' \nabla_T N \qquad (6)$$

where $K = \frac{\kappa''}{\kappa} - \tau^2$.

Proof: Suppose that α is general helix with respect to the Frenet frame $\{T, N, B\}$. Then from (5), we have

$$\nabla_T \nabla_T \nabla_T T = (\kappa'' - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B. \qquad (7)$$

Now, since α is general helix with respect to the Frenet frame

$$\frac{\kappa}{\tau} = const.$$

and this upon the derivation gives rise to

$$\kappa'\tau = \kappa\tau'. \qquad (8)$$

If we substitute the equations (8),

$$N = \frac{1}{\kappa} \nabla_T T, \qquad (9)$$

and

$$B = \frac{1}{\tau} \nabla_T N \qquad (10)$$

in (7), we obtain (6).

Conversely, let us assume that the equation (6) holds. We show that the curve α is a general helix. Covariant differentiating (9), we obtain

$$\nabla_T N = -\frac{\kappa'}{\kappa^2} \nabla_T T + \frac{1}{\kappa} \nabla_T \nabla_T T \qquad (11)$$

and so

$$\nabla_T \nabla_T N = \left(-\frac{\kappa'}{\kappa^2}\right)' \nabla_T T - 2\frac{\kappa'}{\kappa^2} \nabla_T \nabla_T T + \frac{1}{\kappa} \nabla_T \nabla_T \nabla_T T. \qquad (12)$$

If we use (6) in (12) and make some calculations, we have

$$\nabla_T \nabla_T N = \left[\left(-\frac{\kappa'}{\kappa^2}\right)' + \frac{K}{\kappa}\right] \nabla_T T - 2\frac{\kappa'^2}{\kappa^2} N + \frac{\kappa'\tau}{\kappa} B. \qquad (13)$$

Also we obtain

$$\nabla_T \nabla_T N = -\tau^2 N + \tau' B \tag{14}$$

Since (13) and (14) are equal, routine calculations show that α is a general helix.

Corollary 3. 1. Let α be a curve in 3-dimensional Galilean space G_3. α is a circular helix with respect to the Frenet frame $\{T, N, B\}$, if and only if

$$\nabla_T \nabla_T \nabla_T T = -\tau^2 \nabla_T T\ . \tag{15}$$

Proof: From the hypothesis of corollary 3.1 and since α is a circular helix, we can easily show (15).

Theorem 3. 2. Let α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$, if and only if

$$\nabla_T \nabla_T \nabla_T T - K \nabla_T T = 3\lambda \tau' \nabla_T N \tag{16}$$

where $K = \dfrac{\kappa''}{\kappa} - \tau^2$ and $\lambda = \dfrac{\kappa}{\tau} = const.$

Proof: It is similar to the proof of Theorem 3. 1.

Theorem 3. 3. If α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$, then

$$\nabla_T \nabla_T \nabla_T T - \widetilde{K} \nabla_T B = 3\kappa' \nabla_T N \tag{17}$$

where $\widetilde{K} = -\dfrac{\kappa''}{\tau} + \kappa\tau$.

Proof: Suppose that α is a general helix with respect to the Frenet frame $\{T, N, B\}$. Then from (7) and (8), we have

$$\nabla_T \nabla_T \nabla_T T = (\kappa'' - \kappa\tau^2)N + 3\kappa'\tau B\ . \tag{18}$$

If we substitute the equations

$$N = -\frac{1}{\tau} \nabla_T B \tag{19}$$

and (10) in (18), we obtain (17).

Theorem 3. 4. If α be a curve in 3-dimensional Galilean space G_3. α is a general helix with respect to the Frenet frame $\{T, N, B\}$, then

$$\nabla_T \nabla_T \nabla_T T - \widetilde{K} \nabla_T B = 3\lambda \tau' \nabla_T N \tag{20}$$

where $\widetilde{K} = -\dfrac{\kappa''}{\tau} + \kappa\tau$ and $\lambda = \dfrac{\kappa}{\tau} = const.$

Proof: It is similar to the proof of Theorem 3.3.

Corollary 3. 2. Let α be a curve in 3-dimensional Galilean space G_3. α is a circular helix with respect to the Frenet frame $\{T, N, B\}$ if and only if

$$\nabla_T \nabla_T \nabla_T T = \kappa\tau \nabla_T B . \tag{21}$$

Proof: From the hypothesis of corollary 3.2 and since α is a circular helix, we can easily show (21).

REFERENCES

1. Kamenarovic, I. (1991). Existence Theorems for Ruled Surfaces In The Galilean Space G_3. *Rad HAZU Math, 456*(10), 183-196.
2. Pavkovic, B. J. & Kamenarovic, I. (1987). The Equiform Differential Geometry of Curves In The Galilean Space G_3. *Glasnik Matematicki, 22*(42), 449-457.
3. Ikawa, T. (1985). On Curves and Submanifolds in an Indefinite-Riemannian Manifold. *Tsukuba J. Math, 9*, 353-371.
4. Balgetir, H., Bektas, M. & Ergüt, M. (2001). On a Characterization of Null Helix. *Bull. Ins. Math. Aca. Sin, 29*(1), 71-78.
5. Ekmekçi, N. & Hacisalihoglu, H. H. (1996). On Helices of a Lorentzian manifold. *Commun. Fac. Sci., Univ. Ank. Series, A1*, 45-50.