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"Research Note"

BOUND STATE ENERGY OF DELTA-FUNCTION POTENTIAL: A NEW REGULARIZATION SCHEME*

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Abstract – In this letter we have proposed a new regularization scheme to deal with the divergent integrals occurring in the quantum mechanical problem of calculating the bound state energy of the delta-function potential in two and three dimensions. Based on the Schwinger parameterization technique we argue that there are no infinities even in D dimensions. In this way we were able to compare our proposal with the Schwinger regularization approch.

Keywords – Delta-function potential, bound state energy, smeared propagators

1. INTRODUCTION

There are infinities associated with the bound state energy levels of the delta-function potential in two and three dimensions and one must adopt a regularization-renormalization scheme in order to extract a meaningful result for the energy levels of the delta-function potential [1]. In this letter we demonstrate how to regulate the divergent integrals associated with the energy levels of delta-function potential by multiplying the kernel of integrals with a Gaussian damping factor (smearing the free particle propagator), and then to isolate the divergent part in the vanishing limit of the damping factor. Our main source of motivation in introducing such a regularization approach refers to the fact that multiplying the free field momentum space propagators with a Gaussian damping factor (smearing the propagator) leads to the finite result for the divergent integrals of the quantum field theory (QFT) and one recovers the divergent behavior of the QFT in the limit when the Gaussian damping factor disappears. Such a damping factor may arise as the effect of the gravitational fluctuations on the dynamics of the free particles [2], or as the effect of the noncommutativity of space [3-8]. The usual starting point in the non-commutative models is to replace the product of the fields with the so called star product [8]. By such a replacement the noncommutativity of space will be encoded in the interaction term by changing the local interaction point to the non-local one. Authors in [3] develop a new proposal to study the non-commutative space-time. The main idea is to use the expectation values of the operators between the coherent states of the non-commutative space-time instead of using the star product. In this approach the free particle propagator acquires a Gaussian damping factor, which may act as a probable mechanism to make the whole formalism of the perturbation theory finite [7]. So in the coherent state approach, the noncommutativity of space-time manifests itself by modification of the propagators rather than the interaction vertices, and this is to say that there is no UV/IR mixing problem [8] in the coherent state approach to the noncommutative

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space-time. In the next section we examine our new regularization scheme by applying it to the quantum mechanical problem of calculating the bound state energy of the delta-function potential in two and three dimensions. We also demonstrate that, in our approach one gets regularized results even in D dimensions. This enables us to compare our proposal with the Schwinger well-known regularization technique.

2. BOUND STATE ENERGY OF THE DELTA-FUNCTION POTENTIAL IN TWO AND THREE DIMENSIONS

The bound state energy of the non-relativistic free particle subjected to an attractive delta-function potential is given by [1]

$$\frac{1}{\lambda} - \int \frac{d^D k}{(2\pi)^D} \frac{1}{E - k^2} = 0, \tag{1}$$

or

$$\frac{1}{\lambda} + \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \varepsilon^2} = 0, \tag{2}$$

with $E_B = -\varepsilon^2$. The above integral is divergent except for the case $D = 1$. In particular, it diverges logarithmically in two dimensions and is divergent linearly in three dimensions. So a regularization-renormalization scheme is needed in order to get a finite result for the bound states energy. This can be accomplished by introducing a cut-off parameter to make the divergent integral finite and then to absorb the divergent term into the coupling constant. In two dimensions we have

$$\int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + \varepsilon^2} = \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{\varepsilon^2}\right), \tag{3}$$

where the cut-off parameter Λ is introduced to make the (logarithmically) divergent integral finite. Hence, one obtains for the bound state energy as [1]

$$E_B = -\varepsilon^2 = -\mu^2 \exp\left(\frac{4\pi}{\lambda_R}\right), \tag{4}$$

while the divergent term is absorbed into the new coupling constant λ_R as

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{5}$$

The arbitrary constant μ^2 is introduced to make the argument of the logarithm dimensionless.
In three dimensions equation (2) reads

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2 + \varepsilon^2} = \frac{1}{2\pi^2}\left[\Lambda - \varepsilon \tan^{-1}\left(\frac{\Lambda}{\varepsilon}\right)\right], \tag{6}$$

where the cut-off parameter Λ is introduced once again to get a finite result for the divergent integral. Thus in three dimensions the bound state energy reads

$$E_B = -\varepsilon^2 = \left(\frac{4\pi}{\lambda_R}\right)^2, \tag{7}$$

with a new coupling constant defined as

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{\Lambda}{2\pi^2}. \tag{8}$$

3. NEW REGULARIZATION SCHEME

Having reviewed the problem of the bound state energy of an attractive delta-function in two and three dimensions, we are in a position to introduce the new regularization scheme to regulate the divergent integrals arising in calculating the bound state energy of the delta-function potential in two and three dimensions. To this end we multiply the kernel of integrals (3) and (6) with the Gaussian damping factor $\exp(-\theta k^2)$ in order to make the divergent integrals (3) and (6) convergent. The next step is to isolate the divergent part of the integrals in the limit $\theta \to 0^+$, and then to absorb it into the coupling constant. Let us start from the two dimensional case. By multiplying the kernel of the integral in expression (3) with the Gaussian factor we get

$$\int \frac{d^2k}{(2\pi)^2} \frac{e^{-\theta k^2}}{k^2 + \varepsilon^2} = \frac{e^{\theta\varepsilon^2}}{4\pi} E_i(\theta\varepsilon^2) = \frac{e^{\theta\varepsilon^2}}{4\pi}[-\gamma - \ln(\theta\varepsilon^2) - \sum_{n=1} \frac{(-1)^n}{nn!}(\theta\varepsilon^2)^n], \tag{9}$$

which, in the limit $\theta \to 0^+$ reduces to

$$\int \frac{d^2k}{(2\pi)^2} \frac{e^{-\theta k^2}}{k^2 + \varepsilon^2} = \frac{1}{4\pi}\left(-\ln\frac{\theta}{\mu^2} - \ln\frac{\varepsilon^2}{\mu^2}\right). \tag{10}$$

We have ignored the Euler-Mascheroni constant γ since the term $-\ln(\theta/\mu^2)$ in (10) tends to be a very large number in the limit $\theta \to 0^+$. Hence the coupling constant reads

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{1}{4\pi}\ln\left(\frac{\theta}{\mu^2}\right). \tag{11}$$

In three dimensions the divergent integral (6) will be replaced with the regulated one

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{-\theta k^2}}{k^2 + \varepsilon^2} = \frac{1}{2\pi^2}\left(\frac{1}{4}\sqrt{\frac{\pi}{\theta}} - \frac{\pi}{2}\varepsilon e^{\theta\varepsilon^2}\right). \tag{12}$$

Thus the new coupling constant will be

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{1}{2\pi^2}\left(\frac{1}{4}\sqrt{\frac{\pi}{\theta}}\right). \tag{13}$$

So as the results (9) and (12) imply, multiplying the Green's function by the Gaussain damping factor leads to the finite results for the divergent integrals in two and three dimensions. In the limit when the Gaussain factor disappears, i.e. when $\theta \to 0^+$, the divergent behavior of the integrals is presented once again. It is interesting to note that result (9) coincides with expression (3) if one puts $\Lambda^2 = 1/\theta$ in equation (9). By the same replacement for θ in expression (12), we recover the expression (6) in the limit $\Lambda \to \infty$, provided that one rescales $(\sqrt{\pi}\Lambda)/4 \to \Lambda$. An efficient way of handling divergent integrals occurring in QFT is to use the well-known Schwinger parameterization scheme in which one first exponentiates the denominator of the integrand of the integral via the Schwinger parameter and then after integration over the momentum variable, introduces an effective cut-off on the Schwinger parameter. Here, we follow the Schwinger parameterization scheme to conclude that multiplying the divergent integral (2) by the Gaussian damping factor leads to a regularized result even in $3 < D$ dimensions. This also will enable us to compare our proposal with the well-known Schwinger regularization technique. For the expression (2) one finds

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \varepsilon^2} = \int \frac{d^D k}{(2\pi)^D} \int_0^\infty d\alpha\, e^{-\alpha(k^2+\varepsilon^2)} = \frac{1}{(2\sqrt{\pi})^D} \int_0^\infty d\alpha \frac{e^{-\alpha\varepsilon^2}}{\alpha^{D/2}}, \tag{14}$$

which is clearly a divergent result since the denominator of integrand is singular at $\alpha = 0$. Thus the divergent behavior of (2) arises from the lower limit of the last integral of equation (14). But the presence of the Gaussian damping factor in the integrand modifies (14) to

$$\int \frac{d^D k}{(2\pi)^D} \frac{e^{-\theta k^2}}{k^2 + \varepsilon^2} = \int \frac{d^D k}{(2\pi)^D} \int_0^\infty d\alpha\, e^{-\alpha(k^2+\varepsilon^2)} e^{-\theta k^2} = \frac{e^{\theta\varepsilon^2}}{(2\sqrt{\pi})^D} \int_\theta^\infty d\alpha \frac{e^{-\alpha\varepsilon^2}}{\alpha^{D/2}}, \tag{15}$$

In contrast to expression (14), this is a finite result because of the cut-off induced on the lower limit of the integral. This is the same mechanism which leads to the regularized integrals in the Schwinger regularization scheme. The main difference is the way in which the cut-off is introduced. In our approach the cut-off is induced automatically via the Gaussian damping factor.

4. CONCLUSIONS

We proposed a new regularization scheme to deal with divergent integrals occurring in the quantum mechanical problem of energy levels of the delta-function potential in two and three dimensions. One poses a Gaussian damping factor in the integrand of the divergent integrals of the problem to get a finite result for the integrals. In the limit when the Gaussian damping factor disappears the divergent behavior of the integrals appears and one isolates the infinite term to extract the physical meaningful results.

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