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# PULLBACK CROSSED MODULES OF ALGEBROIDS*

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**Abstract –** In this paper, we present algebroids and crossed modules of algebroids. We also define pullback crossed module of algebroids.

**Keywords –** Crossed module, algebroids, pullback, action

## 1. INTRODUCTION

The term of crossed module was introduced by J. H. C. Whitehead in his work on combinatorial homotopy theory [1]. So many mathematicians and many areas of mathematics have used crossed modules such as the homotopy theory, homology and cohomology of groups, Algebra, K-theory, etc. Brown and Mosa replaced algebras by algebroids and defined crossed module of algebroids in [2]. Actor crossed module of algebroids was defined by Alp in [3].

In section 1 basic facts on algebroids were recalled as in [2]. In section 2, we presented algebroid and its examples. In section 3, the action of algebroids and the definition of crossed module of algebroids according to Brown and Mosa in [2] were presented. In section 4, we define an action and verified the action axioms holds for algebroids. We also defined pullback crossed module of algebroids using the definition of pullback crossed module according to Brown and Wensley in [4].

## 2. ALGEBROIDS

Let *R* be a commutative ring. An *R*-category $A$ is a category equipped with an R-module structure each homomorphism set such that the composition is *R*-bilinear. An *R*-algebroid $A$ is a small *R*-category. More precisely, an *R*-algebroid $A$ on a set of objects $A_0$ is a directed graph over $A_0$ such that for each $x, y \in A_0$, $A(x, y)$ has an *R*-module structure and there is an *R*-bilinear function $\circ : A_0(x,y) \times A(y,z) \to A(x,z)$, $(a,b) \mapsto a \circ b$ called composition, satisfying the associativity condition and the existence of identities. A pre-*R*-algebroid has the same structure as an algebroid and the same axioms except that the existence of identities $I_n \in A(x,x)$ is not assumed [2]. We can give algebroid examples as follows [2]:

1. If $A_0$ has exactly one object, then an *R*-algebroid $A$ over $A_0$ is just an *R*-algebra. An ideal in $A$ is then an example of a pre-algebroid.
2. If $A$ is an *R*-algebroid over $A_0$ and $x \in A_0$, then $A(x,x)$ inherits the structure of an *R*-algebra A morphism $f : B \to C$ of *R*-algebroids $B, C$ is a functor of the underlying categories which is also *R*-linear on each $B(x,y) \to C(fx, fy)$. The set of morphisms $B \to C$ of algebroids is written $A\lg(B,C)$.

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## 3. ACTIONS AND CROSSED MODULES

Let $A$ be an $R$-algebroid over $A_0$ and let $M$ be a pre-algebroid over $A_0$. It is convenient to write the compositions in $A$ and in $M$ as juxtaposition [2].

A left action of $A$ on $M$ assigns to each $m \in M(x,y)$ and $a \in A(w,x)$, an element ${}^{a}m \in M(w,y)$ satisfying the axioms:

**LAct1:** ${}^{c}\left({}^{a}m\right) = {}^{(ca)}m$, ${}^{1}m = m$
**LAct2:** ${}^{c}\left({}^{a}m\right) = {}^{(ca)}m$, ${}^{1}m = m$
**LAct3:** ${}^{a}\left(m + m_1\right) = {}^{a}m + {}^{a}m_1$
**LAct4:** ${}^{a+b}m = {}^{a}m + {}^{b}m$
**LAct5:** ${}^{a}(rm) = r\left({}^{a}m\right)$

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a,b \in A(w,x)$, $c \in A(u,w)$ and $r \in R$.

A right action of $A$ on $M$ assigns to each $m \in M(x,y)$ and $a \in A(y,z)$, an element $m^a \in M(x,z)$ satisfying the axioms:

**RAct1:** $\left(m^a\right)^c = m^{ca}, m^1 = m$
**RAct2:** $(mn)^a = mn^a$
**RAct3:** $\left(m + m_1\right)^a = m^a + m_1^a$
**RAct4:** $m^{a+b} = m^a + m^b$
**RAct5:** $(rm)^a = r\left(m^a\right)$

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a,b \in A(y,u)$, $c \in A(u,v)$ and $r \in R$. Left and right actions of $A$ on $M$ commute if $\left({}^{a}m\right)^b = {}^{a}\left(m^b\right)$, $m \in M(x,y)$, $a \in A(w,x)$ and $b \in A(y,u)$.

**Definition 3.1.** A crossed module of algebroids consists of an R-algebroid $A$, a pre-R-algebroid $M$, both over the same set of objects, and commuting left and right actions of $A$ on $M$, together with a pre-algebroid morphism $\mu : M \to A$ over the identity on $A_0$. These must satisfy the axioms [2]:

**CM1:** $\mu\left(m^b\right) = (\mu m)b$, $\mu\left({}^{a}m\right) = a(\mu m)$,
**CM2:** $mn = m^{(\mu n)} = {}^{(\mu m)}n$

for all $m \in M(x,y)$, $n \in M(y,z)$, $a \in A(w,x)$ and $b \in A(y,u)$.

A morphism of crossed modules $(\alpha,\beta) : (A, M, \mu) \to (A', M', \mu')$ is an algebroid morphism $\alpha : A \to A'$ and a pre-algebroid morphism $\beta \in M \to M'$ over the same map on objects such that $\alpha\mu = \mu'\beta$ and $\beta\left({}^{a}m\right) = {}^{\alpha a}(\beta m)$, $\beta\left(m^b\right) = (\beta m)^{\alpha a}$ for all $a,b \in A$, $m \in M$.

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "\mu"'] & M' \arrow[d, "\mu'"] \\ A \arrow[r, "\alpha"] & A' \end{tikzcd}$$

Thus we get a category of crossed modules of algebroids. The examples of crossed modules of algebroids as are follows [2]:

1. Let $A$ be an $R$-algebroid over $A_0$ and suppose $I$ is a two-sided ideal in $A$. Let $i : I \to A$ be the inclusion morphism and let $A$ operate on $I$ by $a^c = ac$, ${}^{b}a = ba$ for all $a \in I$ and $b,c \in A$ such that these products $ac, ba$ are defined. Then $i : I \to A$ is a crossed module.
2. A two sided module over the algebroid $A$ is defined as a crossed module $\mu : M \to A$, in which $\mu m = 0_{xv}$ for all $m \in M(x,y)$, $x,y \in A_0$.
3. Let $K \to M \to P$ be morphisms of pre-algebroids over the identity on objects and such that: $A$ is an

algebroid; $p$ is surjective. $K = \ker p$; and $i$ is the inclusion. Assume that $K$ is central in $M$, i. e. that $km$ and $mk$ are zeros for all $k$ in $K$, $m$ in $M$. Then $p: M \to A$ can be given the structure of a crossed module with actions ${}^{a}m = a'm, m^{b} = mb'$, where $a \mapsto a'$ is a set-theoretic section of $p$.

## 4. PULLBACK CROSSED MODULES OF ALGEBROIDS

Let $(A, M, \mu)$ be a crossed module of algebroids and $\iota: Q \to A$ be pre-algebroid. Then $(Q, \iota^{**}M, \mu^{**}))$ is the pullback of $(A, M, \mu)$ by $\iota$ where

$$\iota^{**}M = \{(q,m) \in Q \times M \,|\, \iota q = \mu m\}$$

and $\mu^{**}(q,m) = q.$

$$\begin{tikzcd} \iota^{**}M \arrow[r] \arrow[d, "\mu^{**}"'] & M \arrow[d, "\mu"] \\ Q \arrow[r, "\iota"] & A \end{tikzcd}$$

The right action of $Q$ on $\iota^{**}M$ is given by

$$(q_1, m)^{q} = (q_1 q, m\iota q)$$

and the left action of $Q$ on $\iota^{**}M$ is given by

$${}^{q}(q_1, m) = (q q_1, \iota q m)$$

where multiplication in $\iota^{**}M$ is componentwise.

**Theorem 4.1.** There is an algebroid right action of $Q$ on $\iota^{**}M$ given by

$$(q_1, m)^{q} = (q_1 q, m\iota q)$$

**Proof:** We must show that the right action of $Q$ on $\iota^{**}M$ assigns to each $m \in \iota^{**}M(x,y)$, $q \in Q(y,z)$ an element $(q,m)^{q} \in \iota^{**}M(x,z)$ satisfying the right action axioms RAct1-RAct5:

**RAct1:**

$$\begin{aligned} \left((q_1, m)^{q_2}\right)^{q_4} &= (q_1 q_2, m\iota q_2)^{q_4} \\ &= (q_1 q_2 q_4, m\iota q_2 \iota q_4) \\ &= (q_1 q_2 q_4, m\iota(q_2 q_4)) \\ &= (q_1, m)^{q_2 q_4} \end{aligned}$$

$$(q_1, m)^{1} = (q_1, m)$$

**RAct2:**

$$\begin{aligned}\left((q_1,m)(q_2,n)\right)^{q_3} &= (q_1q_2,mn)^{q_3}\\ &= (q_1q_2q_3,mn\imath q_3)\\ (q_1,m)(q_2,n)^{q_3} &= (q_1,m)(q_2q_3,n\imath q_3)\\ &= (q_1q_2q_3,mn\imath q_3)\end{aligned}$$

**RAct3:**

$$\begin{aligned}\left((q_1,m)+(q_2,m_1)\right)^{q_3} &= \left((q_1+q_2,m+m_1)\right)^{q_3}\\ &= \left((q_1+q_2)q_3,(m+m_1)\imath q_3\right)\\ (q_1,m)^{q_3}+(q_2,m_1)^{q_3} &= (q_1q_3,m\imath q_3)+(q_2q_3,m_1\imath q_3)\\ &= (q_1q_3+q_2q_3,m\imath q_3+m_1\imath q_3)\\ &= \left((q_1+q_2)q_3,(m+m_1)\imath q_3\right)\end{aligned}$$

**RAct4:**

$$\begin{aligned}(q_1,m)^{q_1+q_3} &= \left(q_1(q_2+q_3),m\imath(q_2+q_3)\right)\\ (q_1,m)^{q_2}+(q_1,m)^{q_3} &= (q_1q_2,m\imath q_2)+(q_1q_3,m\imath q_3)\\ &= \left(q_1q_2+q_1q_3,m\imath(q_2+q_3)\right)\\ &= \left(q_1(q_2+q_3),m\imath(q_2+q_3)\right)\end{aligned}$$

**RAct5:**

$$\begin{aligned}(rq_1,rm)^{q_2} &= (rq_1q_2,rm\imath q_2)\\ &= r(q_1q_2,m\imath q_2)\\ &= r(q_1,m)^{q_2}\end{aligned}$$

$$\forall (q_1m),(q_2,m_1)\in \imath^{**}M(x,y),\ (q_2,n)\in \imath^{**}M(y,z),\ q_2,q_3\in Q(y,u),\ q_4\in Q(u,v) \text{ and } r\in R.$$

**Theorem 4.2.** There is an algebroid left action of $Q$ on $\imath^{**}M$ given by

$$^{q}(q_1,m)=(qq_1,\imath qm)$$

**Proof:** Similarly, it can be easily shown that defining the left action satisfies the left action conditions of algebroids.

Left and right actions of $Q$ on $\imath^{**}M$ commute if $\left({}^{q_2}(q_1,m)\right)^{q_3} = {}^{q_2}\left((q_1,m)^{q_3}\right)$ for all $(q_1,m)\in \imath^{**}M(x,y)$, $q_2\in Q(w,x)$ and $q_3\in A(y,u)$.

$$\begin{aligned}\left({}^{q_2}(q_1,m)\right)^{q_3} &= (q_2q_1,\imath q_2m)^{q_3}\\ &= (q_2q_1q_3,\imath q_2m\imath q_3)\end{aligned}$$

$$\begin{aligned}{}^{q_2}\left((q_1,m)^{q_3}\right) &= {}^{q_2}(q_1q_3,m\imath q_3)\\ &= (q_2q_1q_3,\imath q_2m\imath q_3)\end{aligned}$$

**Theorem 4.3.** $(Q,\imath^{**}M,\mu^{**})$ has the structure of a crossed module.

**Proof:** We must show that axioms of the crossed module of algebroids are satisfied:

**CM1:**

$$\begin{aligned}\mu^{**}(q_1,m)^{q_3} &= \mu^{**}(q_1q_3, m\iota q_3)\\ &= q_1q_3\\ &= \mu^{**}(q_1,m)q_3\end{aligned}$$

$$\begin{aligned}\mu^{**}\left({}^{q_2}(q_1,m)\right) &= \mu^{**}(q_2q_1, \iota q_2 m)\\ &= q_2q_1\\ &= q_2\mu^{**}(q_1,m)\end{aligned}$$

**CM2:**

$$\begin{aligned}(q_1,m)^{\mu^{**}(q_2,n)} &= (q_1,m)^{q_2}\\ &= (q_1q_2, m\iota q_2)\\ &= (q_1q_2, mn) \quad by\ \iota q_2 = \mu n = n\end{aligned}$$

$$\begin{aligned}{}^{\mu^{**}(q_1,m)}(q_2,n) &= {}^{q_1}(q_2,n)\\ &= (q_1q_2, \iota q_1 n)\\ &= (q_1q_2, mn) \quad by\ \iota q_1 = \mu m = m\end{aligned}$$

A morphism of crossed modules $(\alpha,\beta):(Q,\iota^{**}M,\mu^{**})\to(Q',\iota^{**}M',\mu^{**'})$ is an algebroid morphism $\alpha: Q\to Q'$ and a pre-algebroid morphism $\beta\in\iota^{**}M\to\iota^{**}M'$ over the same map on objects such that $\alpha\mu^{**}=\mu^{**'}\beta$ and $\beta\left({}^{q_2}(q_1,m)\right)={}^{aq_2}\left(\beta(q_1,m)\right)$, $\beta\left((q_1,m)^{q_3}\right)=\left(\beta(q_1,m)\right)^{aq_3}$, $\forall q_2,q_3\in Q \quad (q_1,m)\in\iota^{**}M$.

$$\begin{tikzcd} \iota^{**}M \arrow[r, "\beta"] \arrow[d, "\mu^{**}"'] & \iota^{**}M' \arrow[d, "\mu^{**'}"] \\ Q \arrow[r, "\alpha"] & Q' \end{tikzcd}$$

Then we get the category of pullback crossed module of algebroids.

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