Iranian Journal of Science & Technology, Transaction A, Vol. 32, No. A1
Printed in The Islamic Republic of Iran, 2008

ON ENERGY DECAY OF AN N-DIMENSIONAL THERMOELASTICITY SYSTEM WITH A NONLINEAR WEAK DAMPING[*]

F. TAHAMTANI

Department of Mathematics, Shiraz University, Shiraz, I. R. of Iran
Email: tahamtani@susc.ac.ir

Abstract – We study the exponential decay of global solution for an n-dimensional thermo-elasticity system in a bounded domain of $\Re^n$. By using the multiplier technique and constructing an energy functional well adapted to the system, the exponential decay is proved.

Keywords – Thermo-elasticity system, non-linear weak damping, energy decay rate

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

For $n \geq 1$, let Ω be a domain in $\Re^n$ of a finite measure with a smooth boundary $\Gamma = \partial\Omega$. Let x_0 be any point of $\Re^n$ and define the following partition of boundary Γ:

$$\Gamma_0 = \{x \in \Gamma; m(x) \cdot \nu(x) > 0\}$$

$$\Gamma_1 = \Gamma \setminus \Gamma_0 = \{x \in \Gamma; m(x) \cdot \nu(x) \leq 0\}$$

where $m(x) = x - x_0$, $\nu(x)$ denotes the unit outward normal vector to Ω at $x \in \Gamma$. On this domain with time $t \in \Re^+$, we consider the system

$$u_{tt} - \Delta u + \mu \nabla v = 0 \quad \text{in} \quad \Omega \times \Re^+, \tag{1}$$

$$v_t - \Delta v + \mu \nabla u_t = 0 \quad \text{in} \quad \Omega \times \Re^+, \tag{2}$$

($\mu \neq 0$ is a real number) with initial and boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_1 \times \Re^+, \tag{3}$$

$$v = 0 \quad \text{on} \quad \Gamma \times \Re^+, \tag{4}$$

$$\frac{\partial u}{\partial \nu} = -(m(x) \cdot \nu(x) h(u_t) \quad \text{on} \quad \Gamma_0 \times \Re^+ \tag{5}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad v(x,0) = v_0(x) \tag{6}$$

which can be viewed as an n-dimensional thermo-elasticity system with displacement u and v the temperature deviation from the reference temperature. The considered model has several sources of dissipation. It is not only the thermal dissipation, but also the frictional damping acting in the boundary. It

[*]Received by the editor June 6, 2007 and in final revised form May 17, 2008

is known that the model, without any frictional dissipation in the boundary of the domain, enjoys exponential decay of energy.

In this paper we shall prove an exponential decay of energy for a thermo-elasticity system. Motivated by this problem, we are interested here in the decay property of the couple (u, v) which is the solution of (1)-(6) with $f(s)$ such that

$$-\infty < \lim_{s\to-\infty} f(s) < \lim_{s\to+\infty} f(s) < +\infty, \tag{7}$$

if f satisfies at most (1.7), the dissipative effect by $f(u_t)$ is as weak as $|u_t|$ is large, and for convenience we call such a term weak dissipation.

Hereafter, we consider the most typical example $f(s) = \dfrac{s}{\sqrt{1+s^2}}$

$$\frac{1}{\sqrt{2}}|s| \le |f(s)| \le |s| \quad \text{if} \quad |s| \le 1, \tag{8}$$

$$\frac{1}{\sqrt{2}} \le |f(s)| \le |s| \quad \text{if} \quad |s| > 1, \tag{9}$$

which is increasing, globally Lipshitz continuous, satisfies

$$sf(s) \ge 0 \quad \text{and} \quad \lim_{s\to\pm\infty} f(s) = \pm 1. \tag{10}$$

Before we present our main result, let us dwell a moment on some previous interesting articles.

Since the pioneering work of Dafermos [1] on linear thermo-elasticity, significant progress has been made on the mathematical aspect of thermo-elasticity, see [1-12] among others. More precisely, Dafermos [1] has shown that if

$$(u_0, u_1, v_0) \in H^1 \times L^2 \times L^2,$$

then the energy function of the one-dimensional homogenous thermo-elasticity bar defined as

$$E(t) = \| u_x \|_2^2 + \| u_t \|_2^2 + \| v \|_2^2$$

converges to zero as time goes to infinity. However, no decay rate was given. It is well known that $\|\cdot\|$ denotes the L^2 norm and H^1 is the usual Sobolev space when u and v satisfy the Dirichlet and Neumann boundary conditions, respectively (or vice versa). Hansen [2] in 1992 succeeded establishing an energy estimate of the form

$$E(t) \le ME(0)e^{-\alpha t} \text{ for all } t > 0, \tag{11}$$

where M and α are positive constants. He used the Fourier series expansion method and decoupling technique. We refer to Gibson et. al [3] for another approach that is a combination of the semi-group theory and the energy method.

In recent years, the existence, uniqueness and asymptotic behavior of solutions of the system of thermo-elasticity has been analyzed intensively ([4, 5, 6] and the references cited there in). However, as far as we know, very little is known about the energy estimate of the form (1.11) for thermo-elasticity systems. The object of this paper is to prove that (1.11) holds for solutions of (1)-(6) which is assumed to exist in the class

$$u \in C(\Re^+, H^2(\Omega) \cap H^1(\Omega)) \cap C^1(\Re^+, H^1(\Omega)) \cap C^2(\Re^+, L^2(\Omega)), \tag{12}$$

$$v \in C(\Re^+, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\Re^+, L^2(\Omega)). \tag{13}$$

The function spaces we use are all familiar and we omit their definitions. Our main tool is an integral inequality combined with a multiplier technique.

We define the energy of the solution by the formula

$$E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2 + v^2)dx. \tag{14}$$

If (u,v) is a strong solution, then from (1.10) and by simple combination

$$\frac{d}{dt}E(t) = -\{\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_0} (m\cdot\nu)u_t f(u_t)d\Gamma\} \le 0, \tag{15}$$

and for all $0 \le t_1 < t_2 < +\infty$

$$E(t_1) - E(t_2) = \int_{t_1}^{t_2}\int_\Omega |\nabla v|^2 dxdt + \int_{t_1}^{t_2}\int_\Omega (m\cdot\nu)u_t f(u_t)d\Gamma dt. \tag{16}$$

Hence, the energy is non-increasing and our result is the following:

Theorem: There exist two positive constants M and β such that

$$E(t) \le ME(0)\exp(-\beta t) \text{ for all } t > 0 \tag{17}$$

for all initial data $(u_0, u_1, v_0) \in H^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$.

For the proof of the THEOREM, we need the following useful lemma.

Lemma: ([7]) Let $E:\Re^+ \to \Re^+$ be a non-increasing function and assume that there exists a constant $T > 0$ such that

$$\int_t^{+\infty} E(\tau)d\tau \le TE(t) \quad \text{for all} \quad t \in \Re^+. \tag{18}$$

Then

$$E(t) \le E(0)\exp(1-\frac{t}{T}), \quad \text{for all} \quad t \ge T. \tag{19}$$

2. PROOF OF THE THEOREM

Multiplying the equation (1) by u, we have

$$\int_0^T\int_\Omega (|\nabla u|^2 - u_t^2)dxdt = -[\int_\Omega uu_t dx]_0^T + \int_0^T\int_\Omega u\frac{\partial u}{\partial \nu}d\Gamma dt$$

$$= +\mu\int_0^T\int_\Omega u\nabla v dxdt.$$

Whence

$$\int_0^T\int_\Omega (u_t^2+|\nabla u|^2+v^2)dxdt=\int_0^T\int_\Omega (2u_t^2+v^2-\mu u\nabla v)dxdt$$

$$-[\int_\Omega uu_t dx]_0^T+\int_0^T\int_{\Gamma_1\cup\Gamma_0} u\frac{\partial u}{\partial \nu}d\Gamma dt.$$

That is

$$2\int_0^T E(t)dt=+\int_0^T\int_\Omega (2u_t^2+v^2-\mu u\nabla v)dxdt$$

$$-[\int_\Omega uu_t dx]_0^T+\int_0^T\int_{\Gamma_1\cup\Gamma_0} u\frac{\partial u}{\partial \nu}d\Gamma dt. \qquad (20)$$

Next, multiplying the equation (1) by $2m(x)\cdot\nabla u$, we obtain

$$[\int_\Omega 2u_t(m(x)\cdot\nabla u)]_0^T+\int_0^T\int_\Omega 2\mu\nabla v(m(x)\cdot\nabla u)dxdt$$

$$=\int_0^T\int_\Omega (2u_t(m(x)\cdot\nabla u_t)+2\Delta u(m(x)\cdot\nabla u))dxdt. \qquad (21)$$

Here, we formally see

$$\int_\Omega (2u_t(m(x)\cdot\nabla u_t)+2\Delta u(m(x)\cdot\nabla u))dx$$

$$=\int_\Omega ((m(x)\cdot\nabla(u_t^2))-n|\nabla u|^2)dx+\int_{\Gamma_1\cup\Gamma_0}(m\cdot\nu)|\frac{\partial u}{\partial \nu}|^2\, d\Gamma$$

$$=-n\int_\Omega (u_t^2+|\nabla u|^2)dx+\int_{\Gamma_1\cup\Gamma_0}(m\cdot\nu)(u_t^2+|\frac{\partial u}{\partial \nu}|^2)d\Gamma. \qquad (22)$$

Thus, we have from equations (21) and (22) that

$$[\frac{4}{n}\int_\Omega u_t(m(x)\cdot\nabla u)dx]_0^T+\frac{4\mu}{n}\int_0^T\int_\Omega \nabla v(m(x)\cdot\nabla u)dxdt$$

$$=-2\int_0^T\int_\Omega (u_t^2+|\nabla u|^2)dxdt+\frac{2}{n}\int_0^T\int_{\Gamma_1\cup\Gamma_0}(m\cdot\nu)(u_t^2+|\frac{\partial u}{\partial \nu}|^2)d\Gamma dt. \qquad (23)$$

Combining equations (20), (2.4) and using boundary conditions (4)-(6) we have

$$2\int_0^T E(t)dt$$

$$\le [\int_\Omega (uu_t + \frac{4}{n} u_t(m(x)\cdot\nabla u))dx]_T^0 + \int_0^T\int_\Omega (v^2 - \mu u\nabla v)dxdt$$

$$-\frac{4\mu}{n}\int_0^T\int_\Omega \nabla v(m(x)\cdot\nabla u)dxdt + \frac{2}{n}\int_0^T\int_{\Gamma_0}(m\cdot\nu)(u_t^2 + f^2(u_t))d\Gamma dt$$

$$-\int_0^T\int_{\Gamma_0}(m\cdot\nu)uf(u_t)d\Gamma dt. \qquad (24)$$

First, we derive a bound for the last boundary integral on the right-hand side of (24). Using (20) we get

$$|-\int_0^T\int_{\Gamma_0}(m\cdot\nu)uf(u_t)d\Gamma dt|$$

$$\le [\int_\Omega uu_t dx]_0^T + \int_0^T\int_\Omega (2u_t^2 + v^2 - \mu u\nabla v)dxdt. \qquad (25)$$

We deduce from (24) and (25) that

$$2\int_0^T E(t)dt$$

$$\le [\int_\Omega (2uu_t + \frac{4}{n}u_t(m(x)\cdot\nabla u))dx]_T^0 + 2\int_0^T\int_\Omega (u_t^2 + v^2 - \mu u\nabla v)dxdt$$

$$+\frac{2}{n}\int_0^T\int_{\Gamma_0}(m\cdot\nu)(u_t^2 + f^2(u_t))d\Gamma dt - \frac{4\mu}{n}\int_0^T\int_\Omega \nabla v(m(x)\cdot\nabla u)dxdt. \qquad (26)$$

Next, by using Holder and Poincare inequalities we majorize the right-hand side of inequality (26). Then we obtain

$$|\int_\Omega (2uu_t + \frac{4}{n}u_t(m(x)\cdot\nabla u))dx|$$

$$\le (\frac{1}{\varepsilon} + \frac{2}{\varepsilon}\| m\|_{L_\infty})\int_\Omega u_t^2 dx + (\varepsilon C(\Omega) + \frac{2}{n}\| m\|_{L_\infty})\int_\Omega |\nabla u|^2\ dx$$

$$\le \alpha_1\int_\Omega (u_t^2 + |\nabla u|^2 + v^2)dx$$

$$\le 2\alpha_1 E(0), \qquad (27)$$

with

$$\alpha_1 = \max\{\frac{1}{\varepsilon} + \frac{2}{\varepsilon}\| m\|_{L_\infty}, \varepsilon C(\Omega) + \frac{2}{n}\| m\|_{L_\infty}, 1\}\ \text{and}$$

$$2\int_\Omega (u_t^2 + v^2 - \mu\nabla v)dx$$

$$\leq 2\int_\Omega (u_t^2 + v^2)dx + 2\mu\{\frac{\varepsilon}{2}C(\Omega)\int_\Omega |\nabla u|^2\ dx + \frac{1}{2\varepsilon}\int_\Omega |\nabla u|^2\ dx\}$$

$$\leq \gamma_1 E(t) + \frac{\mu}{\varepsilon}(\int_{\Gamma_0}(m\cdot\nu)u_t f(u_t)d\Gamma, \tag{28}$$

with $\gamma_1 = \max\{2, \mu\varepsilon C(\Omega)\}$. Hence, from (15) we get

$$2\int_0^T\int_\Omega (u_t^2 + v^2 - \mu u\nabla v)dxdt \leq \gamma_1\int_0^T E(t)dt + \frac{\mu}{\varepsilon}E(0), \tag{29}$$

and

$$|-\frac{4\mu}{n}\int_0^T\int_\Omega \nabla v(m(x)\cdot\nabla u)dxdt|$$

$$\leq \frac{2\mu}{n}\|m\|_{L_\infty}\int_0^T(\varepsilon\int_\Omega |\nabla u|^2\ dx + \frac{1}{\varepsilon}\int_\Omega |\nabla v|^2\ dx)dt$$

$$\leq \gamma_2\int_0^T E(t)dt + \frac{2\mu}{\varepsilon n}\|m\|_{L_\infty}\ E(0), \tag{30}$$

with $\gamma_2 = \max\{1, \frac{2\varepsilon\mu}{n}\|m\|_{L_\infty}\}$. We deduce from (27)-(30) that

$$(2-\gamma)\int_0^T E(t)dt \leq \alpha E(0) + \frac{2}{n}\int_0^T\int_{\Gamma_0}(m\cdot\nu)u_t^2 d\Gamma dt. \tag{31}$$

If $|u_t| \leq 1$, then we obtain from (8) and (31)

$$(2-\gamma)\int_0^T E(t)dt$$

$$\leq \alpha E(0) + \frac{2\sqrt{2}}{n}\int_0^T\int_{\Gamma_0}(m\cdot\nu)u_t f(u_t)d\Gamma dt$$

$$\leq (\alpha + \frac{2\sqrt{2}}{n})E(0).$$

Choosing $\gamma < 2$, we obtain the desired result by applying Lemma.

If $|u_t| > 1$, then we obtain from the trace theorem $H^1(\Omega) \subset C(\overline{\Omega}) \subset L_\infty(\Gamma)$ and (9)

$$(2-\gamma)\int_0^T E(t)dt$$

$$\leq \alpha E(0) + \| u_t \|_{L_\infty} \int_0^T \int_{\Gamma_0} (m \cdot \nu) u_t f(u_t) d\Gamma dt$$

$$\leq (\alpha + \| u_t \|_{L_\infty}) E(0),$$

and hence, the choice $\gamma < 2$ with Lemma yield the desired decay estimate.

REFERENCES

1. Dafermos, C. M. (1968). On the existence and asymptotic stability of solution to the equations of linear thermo-elasticity. *Arch. Rational Mech. Anal*. *29*, 241-271.
2. Hansen, S. W. (1992). Exponential energy decay in a linear thermo-elastic rod. *J. Math. Appl*, *167*, 429-442.
3. Gibson, J. S., Rosen, I. G. & Tao, G. (1992). Approximation in control of thermo-elastic systems. *SIAM J. Control option*. *30*, 1163-1189.
4. Lebeau, G. & Zuazua, E. (1999). Decay rates for the three-dimensional linear system of thermo-elasticity. *Arch. Rational Mech. Anal*. *148*, 179-231.
5. Quintanilla, R. & Straughan, B. (2000). Growth and uniqueness in thermo-elasticity. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci*. *456*, 1419-1429.
6. Zhang, X. & Zuazua, E. (2003). Decay of solutions of the system of thermo-elasticity of type III. *Communication in Contemporary Math*. *5*(1), 1-59.
7. Komornik, V. (1994). *Exact controllability and stabilization, the multiplier method*. Paris, Masson.
8. Burns, J., Liu, Z. & Zheng, S. (1993). On the energy decay of a thermo-elastic bar. *J. Anal. Math. Appl*. *179*, 574-591.
9. Jiang, S. (1990). Global existence of smooth solution in one-dimensional nonlinear thermo-elasticity. *Proc. Roy. Soc. Edinburgh*, *115*, 257-274.
10. Jiang, S. (1992). Global solution of the Neumann problem in one-dimensional nonlinear thermo-elasticity. *Nonlinear Analysis*. *19*, 107-121.
11. Kim, J. U. (1992). On the energy decay of a linear thermo-elastic bar and plate. *SIAM J. Math. Anal*. *23*, 889-899.
12. Liu, Z. Y. & Zheng, S. (1993). Exponential stability of semi-group associated with thermo-elasticity system. *Quart. Appl. Math*. *51*, 535-545.