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CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE WITH A SPECIAL LIFT FINSLER METRIC*

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Abstract – On a Finsler manifold, we define conformal vector fields and their complete lifts and prove that in certain conditions they are homothetic.

Keywords – Conformal vector field, complete lift, finsler manifold, lift metric

1. PRELIMINARIES

Let (M,g) be a Riemannian manifold, a vector field V on M is called a *conformal vector field* if its local 1-parameter group of transformations is a local conformal transformation. It is well known that V is a conformal vector field on M if and only if there is a scalar function λ on M such that $L_V g = 2\lambda g$. When λ is a constant, V is called *homothetic,* especially when $\lambda = 0$, V is a *killing vector field or* an *infinitesimal isometry* [1].

On a Finsler manifold (M,F), let V be a vector field with the complete lift V^c, then V is called conformal vector field if there is a scalar function ρ on TM such that $L_{V^c} g = 2\rho g$, where $g = (g_{ij})$ is the corresponding fundamental Finsler tensor defined by $g_{ij}(x,y) = (\frac{1}{2}F^2)_{y^i y^j}(x,y)$.

Let TM be the tangent space with a canonical coordinate system (x^i, y^i), then the vertical tangent bundle of $TM_0 = TM \setminus \{0\}$ is defined by

$$VTM = span\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}.$$

A non-linear connection on TM_0 is a complementary distribution HTM defined by

$$HTM = span\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\},$$

where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^{\ j} \frac{\partial}{\partial y^j}$, and $N_i^{\ j}$ are the connection coefficients. HTM is a vector bundle completely determined by the smooth functions $N_i^{\ j}(x,y)$ on TM [2, 3]. Moreover, we have

$$TTM_0 = VTM \oplus HTM \tag{1}$$

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Let ∇ be a linear connection on VTM, then (HTM, ∇) is called a *Finsler connection* on M. Indeed, a Finsler connection is a triad (N, F, C) where $N(N_j^{\ i})$ is a nonlinear, $F(F_{j\ k}^{\ i})$ is the horizontal part and $C(C_{j\ k}^{\ i})$ is the vertical part of this connection. Now let (M, F) be Finsler manifold then a Finsler connection is called a *metric Finsler connection* if g is parallel with respect to ∇. According to the Miron framework this means g is both horizontally and vertically a metric [4, 5, 6]. The *Cartan connection* is a metric Finsler connection for which the deflection, horizontal, and vertical torsion tensor fields vanish.

The *curvature tensor* of a metric Finsler connection is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

where $X, Y \in \chi(TM)$.

They are called horizontal or vertical according to the choice of X and Y in HTM or VTM. Then we have [5]

$$R_{k\ ji}^{\ h} = \delta_i F_{k\ j}^{\ h} - \delta_j F_{k\ i}^{\ h} + F_{k\ j}^{\ m} F_{m\ i}^{\ h} - F_{k\ i}^{\ m} F_{m\ j}^{\ h} + C_{k\ m}^{\ h} R_{j\ i}^{\ m},$$

$R^h_{ij} = \delta_j N^h_i - \delta_i N^h_j$, where we have put $\partial_i = \dfrac{\partial}{\partial x_i}, \partial_{\bar{i}} = \dfrac{\partial}{\partial y_i}, \delta_i = \partial_i - N_i^{\ m} \partial_{\bar{m}}$. When ∇ is a Cartan connection then $N^h_i = y^m F_{m\ i}^{\ h}$.

Proposition 1. [4] Let M be an n-dimensional Finsler manifold with a Cartan connection, then we have the following equations:

(1) $F_{i\ j}^{\ h} = \dfrac{1}{2} g^{hm} (\delta_i g_{mj} + \delta_j g_{im} - \delta_m g_{ij})$;

(2) $C_{ijk} = \dfrac{1}{2} \dot{\partial}_k g_{ij}$ where $C_{ijk} = C^m_{i\,k} g_{jm}$;

(3) $y^m C_{mij} = 0$;

(4) $R^h_{ji} = y^m R_{m\ ij}^{\ h}$.

The Cartan horizontal and vertical covariant derivative of a tensor field of type $(1,2)$ are locally as follows:

$$\nabla_j T_{k\ i}^{\ h} := T_{k\ i|j}^{\ h} = \delta_j T_{k\ i}^{\ h} + F_{m\ j}^{\ h} T_{k\ i}^{\ m} - F_{k\ j}^{\ m} T_{m\ i}^{\ h} - F_{i\ j}^{\ m} T_{k\ m}^{\ h}; \quad (2)$$

$$\nabla_{\bar{j}} T_{k\ i}^{\ h} := T_{k\ i|\bar{j}}^{\ h} = \partial_{\bar{j}} T_{k\ i}^{\ h} + C_{m\ j}^{\ h} T_{k\ i}^{\ m} - C_{k\ j}^{\ m} T_{m\ i}^{\ h} - C_{i\ j}^{\ m} T_{k\ m}^{\ h}.$$

2. LIFT METRICS AND CONFORMAL VECTOR FIELDS

a) Complete Lift Vector Fields and Lie Derivative

Let $V = v^i \partial_i$ be a vector field on M. Then V induces an infinitesimal point transformation on M. This is naturally extended to a point transformation of the tangent bundle TM which is called *extended point transformation*. Let V be a vector field on M and $\{\varphi_t\}$ the local 1-parameter group of M generated by V. Let $\tilde{\varphi}_t$ be the extended point transformation of φ_t, then $\{\tilde{\varphi}_t\}$ induces a vector field V^c on TM which is called the complete lift of V [7, 8].

It can be shown that the extended point transformation is a transformation induced by the complete lift vector field of V, $V^c = v^i \delta_i + y^j \nabla_j v^i \partial_{\bar{i}}$ with respect to the decomposition (1), where ∇ is a linear connection.

The Lie derivation of an arbitrary tensor, $T_i^{\ k}$, is given locally by [9]:

$$L_V T_i^{\ k} = v^a \nabla_a T_i^{\ k} + v^a \nabla_a v^b \nabla_{\bar{b}} T_i^{\ k} - T_i^{\ a} \nabla_a v^k + T_a^{\ k} \nabla_i v^a$$

or equivalently,

$$L_V T_i^{\ k} = v^a \partial_a T_i^{\ k} + y^a \partial_a v^b \partial_{\bar{b}} T_i^{\ k} - T_i^{\ a} \partial_a v^k + T_a^{\ k} \partial_i v^a.$$

So we have

$$L_V y^i = v^a \partial_a y^i + y^a \partial_a v^b \partial_{\bar{b}} y^i - y^a \partial_a v^i = y^a \partial_a v^i - y^a \partial_a v^i = 0, \tag{3}$$

$$L_V g_{ij} = v^a \partial_a g_{ij} + y^a \partial_a v^b \partial_{\bar{b}} g_{ij} + g_{aj} \partial_i v^a + g_{ia} \partial_j v^a. \tag{4}$$

where ∇ is a linear connection.

In Finsler geometry, L_V is replaced by $L_{\tilde{V}}$, where $\tilde{V}$ is the lift of V. We also have this interchanging formula between Cartan covariant derivatives and Lie derivatives.

$$\nabla_k L_V g_{ij} - L_V \nabla_k g_{ij} = g_{aj} L_V F_{i\ k}^{\ a} + g_{ai} L_V F_{j\ k}^{\ a}. \tag{5}$$

b) A Lift Metric on Tangent Bundle

V. Oproiu introduced a family of Riemannian metrics on the tangent space of Riemannian manifolds and considered locally symmetric, Kählerian and anti-Hermitian conditions with these metrics [10-12]. Then Abbassi-Sarih proved in [13] that the Oproiu metrics form a particular subclass of the so-called *g-natural metrics* on the tangent space [14, 15]. Also in [16], Boeckx-Vanhecke obtained an almost contact metric on the unit tangent space.

In this section we consider a new Riemannian metric on the tangent space, and in the next section obtain some conditions which reduce the conformal vector fields to be homothetic.

Let (M, F) be a Finsler manifold, define a tensor field G on TM by

$$G(x,y) = \alpha h_{ij}(x,y) dx^i dx^j + 2\beta h_{ij}(x,y) dx^i \delta y^j + \gamma h_{ij}(x,y) \delta y^i \delta y^j$$

where α, β and γ are real numbers and $h_{ij}(x, y)$ are components of a generalized Lagrange metric [6, 17]. It is clear that G is nonsingular if $\alpha\gamma - \beta^2 \neq 0$ and positive definite if $\alpha\gamma - \beta^2 > 0$, defining, respectively, a *pseudo-Riemannian* or *Riemannian lift metrics* on $T(M)$.

We are going to consider the metric G with $h_{ij}(x, y)$ of the following special deformation of $g_{ij}(x)$

$$h_{ij}(x, y) = a(F^2) g_{ij}(x, y),$$

where $y_i = g_{ij}(x, y) y^j$ and $a: Im(F^2) \subseteq R_+ \to R_+$ with $a > 0$. For shortness we set $g_1 = h_{ij} dx^i dx^j$, $g_2 = 2h_{ij} dx^i \delta y^j$ and $g_3 = h_{ij} \delta y^i \delta y^j$, therefore $G = \alpha g_1 + \beta g_2 + \gamma g_3$.

3. MAIN RESULTS

Analogous to the Riemannian geometry, by straightforward calculation we have the following results in Finsler geometry [18, 19].

Lemma 1. Let (M, F) be a Finsler manifold with Cartan connection, then we have
(1) $[\delta_i, \delta_j] = R^{\ h}_{\ ij} \partial_{\bar{h}}$;
(2) $[\delta_i, \partial_{\bar{j}}] = \partial_{\bar{j}} N_{\ i}^{\ h} \partial_{\bar{h}}$;

(3) $[\partial_{\bar{i}}, \partial_{\bar{j}}] = 0.$

Lemma 2. Let (M, F) be a Finsler manifold with Cartan connection, then we have
(1) $L_{V^c}\delta_i = -\partial_i v^h \delta_h - L_V N^h_i \partial_{\bar{h}}$;
(2) $L_{V^c}\partial_{\bar{i}} = -\partial_i v^h \partial_{\bar{h}}$;
(3) $L_{V^c} dx^h = \partial_m v^h dx^m$;
(4) $L_{V^c}\delta y^h = L_V N^h_m dx^m + \partial_m v^h \delta y^m$.

Proof: First we give the proof of part (2). By a simple calculation, we have:

$$\begin{aligned}L_{V^c}\partial_{\bar{i}} &= [v^c, \partial_{\bar{i}}]\\ &= [v^h\delta_h + y^m v^h|_m \partial_{\bar{h}}, \partial_{\bar{i}}]\\ &= v^h[\delta_h, \partial_{\bar{i}}] - \partial_{\bar{i}}(v^h)\delta_h + y^m v^h|_m [\partial_{\bar{h}}, \partial_{\bar{i}}] - \partial_{\bar{i}}(y^m v^h|_m)\partial_{\bar{h}}\\ &= \partial_{\bar{i}}(v^h N^r_h - y^m v^r|_m)\partial_{\bar{r}}\\ &= -\partial_i v^r \partial_{\bar{r}};\end{aligned}$$

The proof of part (1) is similar to (2).
Since $(dx^h, \delta y^h)$ is the dual basis of $(\delta_h, \partial_{\bar{h}})$, if we put

$$L_{V^c}\delta y^h = \alpha^h_m dx^m + \beta^h_m \delta y^m,$$

then we have

$$0 = L_{V^c}(\delta y^h(\delta_i)) = (L_{V^c}\delta y^h)\delta_i + \delta y^h(L_{V^c}\delta_i) = \alpha^h_i - L_{V^c}N^h_i,$$

and

$$0 = L_{V^c}(\delta y^h(\partial_{\bar{i}})) = (L_{V^c}\delta y^h)\partial_{\bar{i}} + \delta y^h(L_{V^c}\partial_{\bar{i}}) = \beta^h_i - \partial_i v^h.$$

Thus we get (4). In the same way as the proof of part (4), we can prove (3).

Lemma 3. Let (M, g) be a Finsler manifold with Cartan connection, then we have
(1) $L_{V^c} g_1 = a(F^2)(2\varphi g_{ij} + L_V g_{ij})dx^i dx^j$;
(2) $L_{V^c} g_2 = 2a(F^2)g_{mi}(L_V N^m_j)dx^i dx^j + 2a(F^2)(2\varphi g_{ij} + L_V g_{ij})\delta y^i \delta y^j$;
(3) $L_{V^c} g_3 = 2a(F^2)g_{mi}L_V N^m_j dx^i \delta y^j + a(F^2)(2\varphi g_{ij} + L_V g_{ij})\delta y^i \delta y^j$.
where $\varphi = y^m v^h_{|m} y_h \dfrac{a'(F^2)}{a(F^2)}$.

Proof: From the above lemma, we get

$$\begin{aligned}L_{V^c} g_1 &= L_{V^c}(h_{ij} dx^i dx^j) = V^c(a(F^2)g_{ij})dx^i dx^j + 2a(F^2)\, g_{ij}(L_{V^c} dx^i)dx^j\\ &= ((v^h\delta_h + y^m v^h|_m \partial_{\bar{h}})a(F^2))g_{ij} + ((v^h\delta_h + y^m v^h|_m \partial_{\bar{h}})g_{ij})a(F^2)\\ &\quad + 2a(F^2)g_{ij}(\partial_r v^i dx^r)dx^j\\ &= 2a(F^2)\varphi\, g_{ij} dx^i dx^j + a(F^2)L_V g_{ij} dx^i dx^j.\end{aligned}$$

Thus we have (1). (2) and (3) are easily proof in the same way as the proof of (1).

Definition 1. Let X be a conformal vector field on TM with the associated function ρ. X is called quasi-inessential vector field if $\rho-\varphi$ is a function of (x^h), namely there exists a function Ω of (x^h) such that $\rho=\Omega+\varphi$. If Ω is constant, then X is called quasi-homothetic vector field. Moreover, if $\Omega=0$ then X is called quasi-isometry vector field on TM.

Remark: These classes of vector fields contain the classes of inessential, homothetic and isometry vector fields as special cases, respectively (for $\varphi=0$). Hence, the forthcoming results hold for inessential, homothetic and isometry vector fields**.**

Theorem 1. Let (M,F) be a C^∞ connected Finsler manifold, TM its tangent bundle and G the Riemannian (or pseudo-Riemannian) metric on TM derived from g. Then every complete lift conformal vector field on TM is quasi-homothetic.

Proof: Let V be a vector field on M, V^c the complete lift vector field of V which is conformal, and let G be a pseudo-Riemannian metric on TM derived from g. We have by definition $L_{V^c}G=2\rho G$. The Lie derivative of G gives

$$\begin{aligned}L_{V^c}G&=\alpha a(F^2)(2\varphi g_{ij}+L_V g_{ij})dx^i dx^j+2\beta a(F^2)(2\varphi g_{ij}+L_V g_{ij})dx^i\delta y^j\\&+2\beta a(F^2)g_{ai}L_V N^a_j dx^i dx^j+\gamma a(F^2)(2\varphi g_{ij}+L_V g_{ij})\delta y^i\delta y^j\\&+2\gamma a(F^2)g_{aj}L_V N^a_i dx^i\delta y^j.\end{aligned}\tag{6}$$

So we have

$$\begin{aligned}L_{V^c}G&=a(F^2)[\alpha(2\varphi g_{ij}+L_V g_{ij})+2\beta g_{ai}(L_V N^a_j)]dx^i dx^j\\&+a(F^2)[2\beta(2\varphi g_{ij}+L_V g_{ij})+2\gamma g_{aj}(L_V N^a_i)]dx^i\delta y^j\\&+\gamma a(F^2)(2\varphi g_{ij}+L_V g_{ij})\delta y^i\delta y^j=2\rho\, G.\end{aligned}$$

Comparing with the definition of G, we find

$$\alpha L_V g_{ij}+\beta(g_{ai}L_V N^a_j+g_{aj}L_V N^a_i)=2\alpha\Omega g_{ij};\tag{7}$$

$$\beta L_V g_{ij}+\gamma g_{aj}L_V N^a_i=2\beta\Omega g_{ij};\tag{8}$$

$$\gamma L_V g_{ij}=2\gamma\Omega g_{ij}.\tag{9}$$

Where $\Omega=\rho-\varphi$.

I) If $\gamma\neq 0$, then from (9) we have

$$L_V g_{ij}=2\Omega g_{ij}$$

and from (8) we have

$$L_V N^a_i=0.$$

Using this and $N^h_i=y^m F_m{}^h{}_i$ we get

$$0 = L_V N_i^{\ h} = L_V (y^m F_{m\ i}^{\ h}) = y^m L_V F_{m\ i}^{\ h}, \tag{10}$$

where the last equality follows from equation (3).
II) If $\gamma = 0$, since $\alpha\gamma - \beta^2 \neq 0$ we have $\beta \neq 0$. From (8) we get

$$L_V g_{ij} = 2\Omega g_{ij}$$

and from (7) we have

$$g_{ai} L_V N_j^a + g_{aj} L_V N_i^a = 0.$$

Using this, equation (3) and $N_i^h = y^m F_{m\ i}^{\ h}$, we have

$$y^m (g_{ai} L_V F_{m\ j}^{\ a} + g_{aj} L_V F_{m\ i}^{\ a}) = 0. \tag{11}$$

In each case I and II we have

$$L_V g_{ij} = 2\Omega g_{ij} \tag{12}$$

or from equation (4)

$$v^a \partial_a g_{ij} + g_{aj} \partial_i v^a + g_{ia} \partial_j v^a + y^a \partial_a v^b \partial_{\bar{b}} g_{ij} = 2\Omega g_{ij}.$$

Applying $\partial_{\bar{k}}$ to both sides of the above equation, we find that

$$2v^a \partial_a C_{ijk} + 2C_{ajk} \partial_i v^a + 2C_{iak} \partial_j v^a + 2\partial_k v^a C_{ija} + 2y^a \partial_a v^b \partial_{\bar{k}} C_{ijb} = 2g_{ij} \partial_{\bar{k}} \Omega + 4\Omega C_{ijk}.$$

By using $y^i C_{ijk} = 0$, we obtain $\partial_{\bar{k}} \Omega = 0$. Therefore Ω is a function of x alone. From (5) we have

$$y^k (\nabla_k L_V g_{ij} - L_V \nabla_k g_{ij}) = y^k (g_{aj} L_V F_{i\ k}^{\ a} + g_{ai} L_V F_{j\ k}^{\ a}).$$

By using (10), (11) and (12) in each case I and II we find that

$$y^k \nabla_k \Omega = 0.$$

Since Ω is a function of x alone, we obtain $\partial_i \Omega = 0$. This, together with the connectedness of M, shows that Ω is constant.

Note: In a special case when $a'(F^2) = 0$ *e.g.* $a(t) = (t - F^2)^2 + 1$ follows from lemma 3, that $\varphi = 0$ and hence $L_{V^c} G = 2\rho G$, where ρ depends on x only. Therefore we have:

Corollary 1. Let (M, F) be a C^∞ connected Finsler manifold, TM its tangent bundle and G the Riemannian (or pseudo-Riemannian) metric on TM derived from g with $a'(F^2) = 0$. Then every complete lift conformal vector field on TM is homothetic.

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