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# POSITIVE LAGRANGE POLYNOMIALS*

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**Abstract –** In this paper we demonstrate the existence of a set of polynomials $P_i$, $1 \le i \le n$, which are positive semi-definite on an interval $[a, b]$ and satisfy, partially, the conditions of polynomials found in the Lagrange interpolation process. In other words, if $a = a_1 < \cdots < a_n = b$ is a given finite sequence of real numbers, then $P_i(a_j) = \delta_{ij}$ ($\delta_{ij}$ is the Kronecker delta symbol); moreover, the sum of $P_i$'s is identically 1.

**Keywords –** Positive polynomials, Lagrange polynomials

## 1. INTRODUCTION

All polynomials referred to in this paper belong to $\mathbb{R}[x]$. Let $S$ be a subset of $\mathbb{R}$ and $P$, a non-identically zero polynomial. $P$ is said to be positive semi-definite (positive or "psd" for short) on $S$ if $P(x) \ge 0$ for all $x$ in $S$. It is called positive definite ("pd" for short) on $S$ if $P(x) > 0$ on $S$. Representations of psd and pd polynomials when $S$ is an interval exist in literature, see for example [1-3]. However, as we are interested in this paper to consider positive polynomials on an interval $[a, b]$ from a different angle, there is no need for any such representations here. Suppose that the real numbers $a = a_1 < \cdots < a_n = b$ belong to the interval $[a, b]$. The polynomials

$$P_i = \prod_{j \ne i} (x - a_j) / \prod_{j \ne i} (a_i - a_j),$$

where $1 \le i \le n$ is an integer, used in the Lagrange interpolation formula [4], satisfying the following conditions

$(I)$ $P_i(a_j) = \delta_{ij}$, $\quad 1 \le i, j \le n$,

$(II)$ $P_i$ is of degree $n-1$ for each $i$.

Actually, the set of polynomials $\{P_i\}$ is uniquely determined by the conditions $(I)$ and $(II)$ above. If $n > 2$, then the polynomials $P_i$ are not psd on $[a, b]$. Therefore, if we try to impose the condition of being psd on $[a, b]$, we have to somehow modify $(I)$ and $(II)$. As these conditions imply $(II')$ below, we replace $(II)$ by

$(II')$ $\sum_{i=1}^{n} P_i = 1$.

Using the transformation $x \mapsto (b-a)x/2 + (b+a)/2$, we may just focus on $[-1, 1]$ instead of working on $[a, b]$. Denote by $\mathcal{A}$ the set of psd polynomials on $[-1, 1]$. Suppose that $-1 = a_1 < a_2 < \cdots < a_n = 1$ and $\{P_i\}$, $P_i \in \mathcal{A}$, $1 \le i \le n$ is a set of polynomials. Then we say $\{P_i\}$ is a set of positive Lagrange (PL for short) polynomials corresponding to $(a_1, a_2, \ldots, a_n)$ if $\{P_i\}$ satisfies $(I)$ and $(II')$. For example,

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$P_1 = x^2(1-x)/2$, $P_2 = 1-x^2$, and $P_3 = x^2(1+x)/2$ form a set of PL polynomials corresponding to $(-1, 0, 1)$.

## 2. EXISTENCE OF POSITIVE LAGRANGE POLYNOMIALS

As in section 1, let $-1 = a_1 < \cdots < a_n = 1$ be a finite sequence of real numbers. Our purpose is to prove the following:

**Theorem 2.1**. Corresponding to $(a_1, \ldots, a_n)$ there exists at least one set of PL polynomials.

In order to prove the theorem we need two lemmas and some notation.

**Lemma 2.1.** Let $a, b \in [-1, 1], a \neq b$. Then there exists a polynomial $g \in \mathcal{A}$ which attains a local maximum with value 1 at $a$ and has a local minimum with value 0 at $b$.

**Proof:** $g$ must be of the form $g(x) = (x-b)^2 g_1(x)$, where $g_1 \in \mathcal{A}$ is a non-constant polynomial. Now it is possible to find constants $c$ and $d$ such that $g(x) = c(x-b)^2(x-d)^2$ satisfies the desired properties. A direct calculation shows that $c = (a-b)^{-4}$ and $d = 2a - b$, that is,

$$g(x) = \left((x-a)^2 - (b-a)^2\right)^2 / (b-a)^4.$$

**Remark 2.1.** In the above lemma the obtained polynomial was of degree 4. In some cases we can find a polynomial $g$ such that it has the stated properties and $\deg(g)$ equals 3. In fact, if $\deg(g) = 3$ then we write $g$ as:

$$g(x) = (x-b)^2[m(1-x) + n(1+x)].$$

As $g \in \mathcal{A}$, we have $g(1) \geq 0$ and $g(-1) \geq 0$. So $g \in \mathcal{A}$ iff $m, n \geq 0$. Imposing the conditions at $a$, we obtain:

$$m = \frac{3a - b + 2}{2(a-b)^3},$$
$$n = \frac{3a - b - 2}{2(a-b)^3}.$$

Therefore, $m \geq 0$ and $n \geq 0$ iff in Fig. 1 the point $(a, b)$ lies on the shaded area.

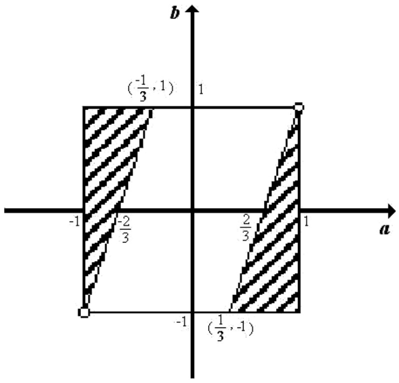


Fig. 1. Acceptable values of (m, n) lie on the shaded area

**Lemma 2.2.** For each $i, 1 \le i \le n$, there exists $p_i \in \mathcal{A}$ having a local maximum $p_i(a_i) = 1$ and a local minimum with value 0 at each $a_j$, $1 \le j \le n$, $j \ne i$.

**Proof:** For each $i$ and $j$, $1 \le i, j \le n$, $j \ne i$, let $g_{ij} \in \mathcal{A}$ be a polynomial such that $g_{ij}$ has a local maximum with value 1 at $a_i$ and a local minimum with value 0 at $a_j$. Define:

$$p_i = \prod_{\substack{j=1 \\ j \ne i}}^{n} g_{ij}.$$

Then $p_i$ has the required properties.

**Remark 2.2.** Note that the polynomials $g$ and $p_i$'s in the lemma 2.1 and the lemma 2.2 might be quite large on $[-1, 1]$. For example, if $a = \frac{1}{4}$, $b = \frac{1}{2}$, then $g(x) = (16x^2 - 8x)^2$ which attains the value 576 at -1. Therefore, in the following proof of the theorem 2.1, we will multiply each $p_i$ by a suitable power of the polynomial $h_i$ as defined below.

For each $i, 1 \le i \le n$, let $h_i = 1 - A_i(x - a_i)^2$, where $A_i = (1 + |a_i|)^{-2}$. Note that $h_i$ has a maximum value of 1 at $a_i$ and $0 \le h_i(x) < 1$ for any $x \in [-1, 1]$, $x \ne a_i$.

**The proof of Theorem 2.1.** If $n = 2$ then $P_1 = (1 - x)/2, P_2 = (1 + x)/2$ have the required properties. So assume $n > 2$. For each non-negative integer $k$ and each integer $i$, $1 \le i < n$, let $p_{i,k} = p_i h_i^k$, where $p_i$ is as given in the lemma 2.2 and $h_i$ as defined just before proof of the theorem. Define the polynomial $q_k$ by

$$q_k = \sum_{i=1}^{n-1} p_{i,k}.$$

Note that $q_k(a_j) = 1$ for each integer $j$, $1 \le j < n$.
**Claim**. If $k = k_0$ is sufficiently large, then $q_{k_0}$ has a relative maximum with value 1 at each $a_j$.

**Proof of the claim:** We have $p_i'(a_j) = 0, p_i(a_j) = \delta_{ij}$ for all $1 \le i, j \le n$, and $h_i'(a_i) = 0$, for all $1 \le i \le n$. Therefore, $q_k'(a_j) = 0$ for each $1 \le j \le n$. Calculating $q_k''(a_j)$, we obtain

$$q_k''(a_j) = \sum_{i=1}^{n-1} p_i''(a_j) h_i^k(a_j) + k\, h_j''(a_j).$$

But $0 \le h_i(a_j) \le 1$ and $h_j''(a_j) < 0$ for each $1 \le i, j \le n$. So if $k = k_0$ is sufficiently large, then $q_k''(a_j) < 0$ for each $1 \le j \le n$. Thus the claim is proved.

Therefore, for each integer $j$, $1 \le j < n$, there exist real numbers $u_j$, $v_j$ with $u_j < a_j < v_j$ such that $q_{k_0}(a_j) = 1$ is the maximum of $q_{k_0}|_{(u_j, v_j)}$.

For each $x \in [-1, 1]$ with $x \ne a_j$, $1 \le j < n$, the sequence $\{q_k(x)\}$ decreasingly converges to 0. On the other hand, the set

$$A = [-1, 1] \setminus \bigcup_{j=1}^{n-1} (u_j, v_j)$$

is compact. Therefore, the sequence $\{q_k\}$ of polynomials converges uniformly to 0 on $A$ ([5]). This means that there exists an integer $k_1 \ge k_0$ such that $q_{k_1}$ is bounded above by 1 on $A$ and hence on $[-1, 1]$.

Now, for each $i$, $1 \le i < n$, let $P_i = p_{i,k_1}$. Then $P_i(a_j) = \delta_{ij}$ for each $i, j$ with $1 \le i < n$ and $1 \le j \le n$. Define $P_n$ by:

$$P_n = 1 - q_{k_1} = 1 - \sum_{i=1}^{n-1} P_i.$$

Thus $\{P_i\}_{1 \le i \le n}$ is a required set of polynomials.

**Example 2.1.** Let $n = 3$, $a_1 = -1$, $a_2 = 1/2$, and $a_3 = 1$. Then we find the polynomials $p_1 = g_{12}g_{13}$, $p_3 = g_{31}g_{32}$ which satisfy the properties stated in the lemma 2.2. Moreover, we can use the remark 2.1 so that each of $g_{12}$, $g_{13}$, $g_{31}$, and $g_{32}$ has degree 3. In other words,

$$p_1 = g_{12}g_{13} = \frac{1}{27}(2x-1)^2(7+4x).\frac{1}{4}(x-1)^2(2+x),$$
$$p_3 = g_{31}g_{32} = \frac{1}{4}(x+1)^2(2-x).(2x-1)^2(5-4x).$$

Furthermore,

$$h_1 = 1 - \frac{1}{4}(x+1)^2,$$
$$h_3 = 1 - \frac{1}{4}(x-1)^2.$$

Then the least non-negative integer $k$ for which we have $p_1h_1^k + p_3h_3^k \le 1$ on $[-1, 1]$ is 4 (see Figs. 2 and 3).

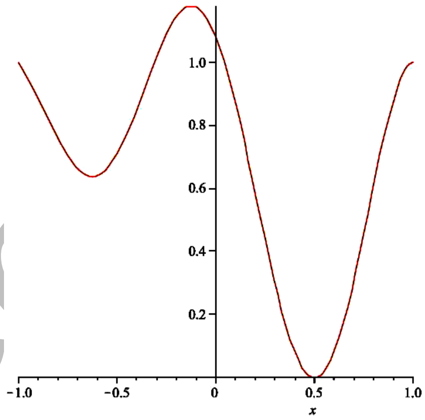


Fig. 2. The graph of $p_1h_1^3 + p_3h_3^3$

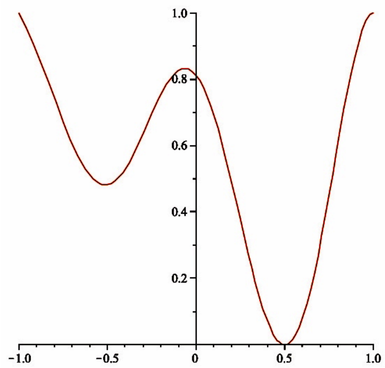


Fig. 3. The graph of $p_1h_1^4 + p_3h_3^4$

Therefore, if we let $P_1 = p_1 h_1^4$, $P_3 = p_3 h_3^4$, and $P_2 = 1 - P_1 - P_3$, then $\{P_1, P_2, P_3\}$ is a set of PL polynomials corresponding to $(-1, 1/2, 1)$.

Note that corresponding to some given $(a_1 = -1, a_2, \ldots, a_n = 1)$, it might be possible to find a set of PL polynomials with degrees less than those of the polynomials found in the theorem 2.1. For example, corresponding to $(-1, 0, 1)$ we had a set of PL polynomials in section 1 with the degrees 2 and 3. However, if we try by using the theorem 2.1 to find a set $\{P_i\}$ of PL polynomials, then the degrees of these polynomials would be at least 6.

**Remark 2.3.** The proof of theorem 2.1 implies that there are infinitely many sets of PL polynomials corresponding to a given $(a_1, \ldots, a_n)$. It is possible to find a kind of minimal set: we choose the least non-negative integer $d$ so that we have $\deg(P_i) = d$, for all $i$, $1 \le i \le n$. However, as the polynomials $p_i$ and $h_i$, for example, are not unique, there exist more than one such set $\{P_i\}_{1 \le i \le n}$ of polynomials. It is an open problem to the authors as how to make a unique choice of such a set of PL polynomials in some reasonable way.

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## REFERENCES


1. Polya, G. & Szego, G. (1976). *Problems and Theorems in Analysis II.* New York, Springer Verlag.
2. Powers, V. & Reznick, B. (2000). Polynomials that are positive on an interval. *Trans. Amer. Math. Soc.*, *352*(10), 4677-4692.
3. Schmudgen, K. (1991). The K-moment problem for compact semi-algebraic sets. *Math. Ann*. *289*, 203-206.
4. Hoffman, K. & Kunze, R. (1961). *Linear Algebra*, Prentice-Hall, Inc., Englewood Cliffs, N. J.
5. Rudin, W. (1976). *Principles of Mathematical Analysis*. 3rd. edition, McGraw Hill Inc.