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TRANSITIVITY OF θ-RELATION ON HYPERMODULES*

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Abstract – In this paper we consider a strongly regular relation θ on hypermodules so that the quotient is a module (with abelian group) over a fundamental commutative ring. Also, we state necessary and sufficient conditions so that the relation θ is transitive, and finally we prove that θ is transitive on hypermodules.

Keywords – Hypermodule, hyperring, strongly regular relation, fundamental relation

1. INTRODUCTION

A *hypergroupoid* $(H,+)$ is a non-empty set H equipped with a *hyperoperation* $+$ defined on H, that is a mapping of $H \times H$ into the family of non-empty subsets of H [1]. If $(x,y) \in H \times H$, its image under $+$ is denoted by $x+y$. If A, B are non-empty subsets of H, then $A+B$ is given by $A+B = \bigcup\{x+y \mid x \in A, y \in B\}$. $x+A$ is used for $\{x\}+A$ and $A+x$ for $A+\{x\}$. A hypergroupoid $(H,+)$ is called a *hypergroup* in the sense of Marty if for all $x,y,z \in H$ the following two conditions hold: (i) $x+(y+z)=(x+y)+z$, (ii) $x+H=H+x=H$. The second condition is called the *reproduction axiom*, meaning that for any $x,y \in H$ there exist $u,v \in H$ such that $y \in x+u$ and $y \in v+x$. An exhaustive review updated to 1992 of hypergroup theory appears in [2] also see [3-6]. A recent book [7] contains a wealth of applications.

If H is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, then for all pairs (A,B) of non-empty subsets of H we set $A\overline{\overline{\rho}}B$ if and only if $a\rho b$ for all $a \in A$ and $b \in B$. The relation ρ is said to be strongly regular to the right if $x\rho y$ implies $x+a\ \overline{\overline{\rho}}\ y+a$ for all $(x,y,a) \in H^3$. Analogously, we can define strongly regular to the left. Moreover, ρ is called strongly regular if it is strongly regular to the right and to the left. Let H be a hypergroup and ρ an equivalence relation on H. Let $\rho(a)$ be the equivalence class of a with respect to ρ and let $H/\rho = \{\rho(a) \mid a \in H\}$. A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in \rho(a)+\rho(b)\}$. If ρ is strongly regular then it readily follows that $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in a+b\}$. It is well known for ρ strongly regular that $< H/\rho, \otimes >$ is a group (see Theorem 31 in [2]), that is $\rho(a) \otimes \rho(b) = \rho(c)$ for all $c \in a+b$.

A *hyperring* is a multi-valued system $(R,+,\cdot)$ which satisfies the ring-like axioms in the following way:

(i) $(R,+)$ is a hypergroup in the sense of Marty,
(ii) $(R,\cdot)$ is a semi-hypergroup,
(iii) The multiplication is distributive with respect to the hyperoperation $+$.

The fundamental relation was introduced on hypergroups by Koskas [8], and then studied by Corsini [2]. The fundamental relation on a hyperring was introduced by Vougiouklis at the fourth AHA congress

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(1990) [9], and studied by many authors, for example see [10-14]. The fundamental relation on a hyperring is defined as the smallest equivalence relation so that the quotient would be the (fundamental) ring. Note that the commutativity with respect to both sum and product in the fundamental ring are not assumed. In [15], Davvaz and Vougiouklis introduced a new strongly regular equivalence relation on a hyperring such that the set of quotients is an ordinary commutative ring. We recall the following definition from [15].

Definition 1.1. Let R be a hyperring. We define the relation α as follows:

$$x\alpha y \Leftrightarrow \exists n\in N,\ \exists(k_1,\cdots,k_n)\in N^n,\ \exists\sigma\in S_n \text{ and } \left[\exists(x_{i1},\cdots x_{ik_i})\in R^{k_i},\ \exists\sigma_i\in S_{k_i},\ (i=1,\cdots,n)\right]$$

such that

$$x\in\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i}x_{ij}\right) \text{ and } y\in\sum_{i=1}^{n}A_{\sigma(i)}$$

where $A_i=\prod_{j=1}^{k_i}x_{i\sigma_i(j)}$.

The relation α is reflexive and symmetric. Let α^* be the transitive closure of α, then we have:

Theorem 1.2. [15]. α^* is a strongly regular relation both on $(R,+)$ and $(R,\cdot)$, and the quotient R/α^* is a commutative ring.

2. θ-RELATION ON HYPERMODULES

Let $(R,+,\cdot)$ be a hyperring and $(M,+)$ be a hypergroup. According to [14], M is said to be a left hypermodule over a hyperring R if there exists

$$\cdot : R\times M\rightarrow \wp^*(M)\ ;\ (a,m)\mapsto a\cdot m$$

such that for all $a,b\in$ R and $m_1,m_2,m\in M$ we have

1) $a\cdot(m_1+m_2)=a\cdot m_1+a\cdot m_2$,
2) $(a+b)\cdot m=(a\cdot m)+(b\cdot m)$,
3) $(a\cdot b)\cdot m=a\cdot(b\cdot m)$.

Let R be a hyperring and M be a hypermodule over R. We define the relation ε on M as follows:

$$x\varepsilon y \Leftrightarrow x,y\in\sum_{i=1}^{n}m_i'\ ;\ m_i'=m_i \text{ or } m_i'=\sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}}x_{ijk}\right)z_i\ ,$$

$$m_i\in M\ ,\ x_{ijk}\in R\ ,\ z_i\in M\ .$$

The fundamental relation ε^* on M can be defined as the smallest equivalence relation on M such that the quotient M/ε^* be a module over the corresponding fundamental ring such that M/ε^* as a group is not abelian, see [14]. Moreover, the fundamental ring is not commutative with respect to both sum and product. Now, we would like the fundamental module to be an abelian group and the fundamental ring to be commutative with respect to both sum and product.

Definition 2.1. Let R be a hyperring and M be a hypermodule over R. We define the relation θ as follows:

$$x\theta y \Leftrightarrow \exists n\in N,\ \exists(m_1',\cdots,m_n'),\ \exists(k_1,k_2,\cdots,k_n)\in N^n,$$

$$\exists\sigma\in S_n,\ \exists(x_{i1},x_{i2},\cdots,x_{ik_i})\in R^{k_i},\ \exists\sigma_i\in S_{n_i},\ \exists\sigma_{ij}\in S_{k_{ij}},$$

such that

$$x\in\sum_{i=1}^{n}m'_i;\quad m'_i=m_i \text{ or } m'_i=\sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}}x_{ijk}\right)m_i$$

and

$$y\in\sum_{i=1}^{n}m'_{\sigma(i)}$$

where

$$m'_{\sigma(i)}=m_{\sigma(i)}\quad\text{if}\quad m'_i=m_i,$$

$$m'_{\sigma(i)}=B_{\sigma(i)}m_{\sigma(i)}\quad\text{if}\quad m'_i=\sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}}x_{ijk}\right)m_i,$$

with

$$B_i=\sum_{j=1}^{n_i}A_{i\sigma_i(j)},\quad A_{ij}=\prod_{k=1}^{k_{ij}}x_{ij\sigma_{ij}(k)}.$$

The relation θ is reflexive and symmetric. Let θ^* be the transitive closure of θ. Then θ^* is a strongly regular relation both on the group $(M,+)$ and M as an R - hypermodule. Also, the (abelian group) M/θ^* is an R/α^* - module, where R/α^* is a commutative ring and the relation θ^* is the smallest equivalence relation such that the (abelian) quotient M/θ^* is an R/α^* - module [16].

If M is an R-hypermodule, then we set

$$\theta_0=\{(m,m)\mid m\in M\}$$

and for every integer $n\geq 1$, θ_n is the relation defined as follows:

$$x\theta_n y\Leftrightarrow x\in\sum_{i=1}^{n}m'_i,\quad y\in\sum_{i=1}^{n}m'_{\sigma(i)},\quad \sigma\in S_n.$$

Obviously, for every $n\geq 1$ the relation θ_n is symmetric, and the relation $\theta=\bigcup_{n\geq 0}\theta_n$ is reflexive and symmetric. If M is a hypermodule over a hyperring R and $n\geq 1$ then $\theta_n\subseteq\theta_{n+1}$ [16].

3. TRANSITIVITY CONDITION OF θ

In the following m'_i is the notation that has been defined in Definition 2.1.

Definition 3.1. Let M be an R-hypermodule and H be a non-empty subset of M. We say that H is a θ-part of M if for every $n\in N$, for every $\sigma\in S_n$ and for every $(m'_1,\cdots,m'_n)$

$$\sum_{i=1}^{n}m'_i\cap H\neq\phi\Rightarrow\sum_{i=1}^{n}m'_{\sigma(i)}\subseteq H,$$

H is said to be a complete part of M if σ is identity.

We consider the following notations:

$$[x,z]^{n}_{k_1,\cdots,k_n}=\{(x_{i1},\cdots,x_{ik}),(z_1,\cdots,z_n)\mid x_{ij}\in R, z_i\in M,\ (i=1,\cdots,n)\}$$

$$T_n(u)=\left\{(m'_1,\cdots,m'_n)\mid u\in\sum_{i=1}^{n}m'_i,\ m'_i=m_i\ or\ m'_i=\sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}}x_{ijk}\right)z_i\right\}$$

$$P_n(u)=\bigcup_n\left\{\sum_{i=1}^{n}m'_{\sigma(i)}\mid(m'_1,\cdots,m'_n)\in T_n(u),\sigma\in S_n\right\}$$

$$P_\sigma(u)=\bigcup_n P_n(u)$$

For every $u\in M$, $P_\sigma(u)=\{v\in M\mid u\theta v\}$.

Theorem 3.2. [16]. Let *M* be an *R*-hypermodule, then the following conditions are equivalent:
1) θ is transitive,
2) for every $u\in M$, $\theta^*(u)=P_\sigma(u)$,
3) for every $u\in M$, $P_\sigma(u)$ is θ-part of *M*.

Definition 3.3. Let *M* be an *R*-hypermodule. Then for every $(a,b)\in M^2$ and for every pair (A,B) of non-empty subsets of *M*, we set:

$$m/n=\{x\in M\mid m\in x+n\},\quad m\backslash n=\{x\in M\mid n\in m+x\}$$

$$A/B=\bigcup_{m\in A,n\in B}m/n,\quad A\backslash B=\bigcup_{m\in A,n\in B}m\backslash n$$

Moreover, let D_1, D_2, D denote the sets

$$D_1=\bigcup_{(x,y)\in M^2}(x+y)/(y+x),\ D_2=\bigcup_{(x,y)\in M^2}(x+y)\backslash(y+x),\ D=D_1\cup D_2$$

Let *M* be an *R*-hypermodule and *B* be a non empty subset of *M. W*e say *B* is *invariant* if for every $x\in M$, $x+B=B+x$. Also, *B* is said to be *invertible to the left* if for every $(x,y)\in M^2$, the implication $y\in B+x$ implies $x\in B+y$.

Lemma 3.4. Let *M* be an *R*-hypermodule and the derived *R*-hypermodule $D(M)$ be the intersection of all subhypermodules of *M* that are complete parts and contain *D.* Then
1) $D(M)$ is a complete part of *M*
2) $D(M)$ is an invariant subset of *M*.

Proof: 1) It is clear.
2) Since $D(M)$ is an invariant subset of *M* as a hypergroup [5], and $D(M)$ is a hypermodule, then $D(M)$ is an invariant subset of *M* as a hypermodule.

Let *M* and *N* be *R*-hypermodules. A function $f:M\to N$ is called an *R*-homomorphism, if for every $(x,y)\in M^2$ and $r\in R$

$$f(x+y)=f(x)+f(y)\ \text{and}\ f(r,x)=r\cdot f(x).$$

If *H* is an *R*-module and $f:M\to H$ is an *R*-homomorphism, we let $\ker f=\{m\in M\mid f(m)=0_H\}$. If $\Phi_M:M\to M/\theta^*$ is the canonical *R*-homomorphism, then $\ker\Phi_M$ is called the *heart of M* and it will be denoted by ω_M.

Lemma 3.5. Let *M* be an *R*-hypermodule and *H* be a complete part of *M*. Then *H* is an invertible subset of *M*.

Proof: We know $\Phi_M(H)$ is a submodule of H/θ^*. For every $(x,y)\in M^2$, if $y\in H+x$ there exists $a\in H$ such that $y\in a+x$, then

$$\Phi_M(y)=\Phi_M(a)\otimes\Phi_M(x)\Rightarrow\Phi_M(x)=(\Phi_M(a))^{-1}\otimes\Phi_H(y)$$

$$\Rightarrow\Phi_M(x)\in\Phi_M(H)\otimes\Phi_M(y)=\Phi_M(H+x)$$

Also, $H+y$ is a complete part of *M*. Therefore $x\in\Phi_M^{-1}(\Phi_M(a+y))$ and *H* is invertible.

Lemma 3.6. Let $<D_2>_C$ be the intersection of all *R*-subhypermodules of *M* that are complete parts and contain D_2. Then $D(M)=<D_2>_C$.

Proof: Every complete part of *M* is invertible by Lemma 3.5 and also, for every invertible submodule *H* of *M* we have $D_1\subseteq H$ if and only if $D_2\subseteq H$. Since *D*, $<D_2>_C$ and $<D_1>_C$ are *R*-hypermodules, we have $D(M)=<D_2>_C$.

Lemma 3.7. Let *H* be an *R*-subhypermodule of *M* and $D\subseteq H$. Then the following relation

$$\forall(x,y)\in M^2,\ xR_Hy\Leftrightarrow y\in x+h,\ h\in H$$

is a strongly regular equivalence relation on *M* and $R_H(x)=x+D(H)$.

Proof: Straightforward.

Theorem 3.8. Let $\Phi_M:M\to M/\theta^*$ be the canonical projection. Then for every *R* -hypermodule *M* we have $D(M)=\Phi_M^{-1}(0_{M/\theta^*})$.

Proof: If $a\in D_2$, then by definition a pair $(x,y)\in M^2$ exists such that $a\in(x+y)\setminus(y+x)$. So there exist $u\in x+y$ and $v\in y+x$ such that $a\in u\setminus v$. Now, $a\in u\setminus v$ implies $v\in u+a$, therefore $\theta^*(v)=\theta^*(u)\otimes\theta^*(a)$. Moreover, $u\in x+y$ and $v\in y+x$ imply $u\theta_2v$, thus $\theta^*(u)=\theta^*(v)$ and $\theta^*(a)=0_{M/\theta^*}$. Hence we have $a\in\Phi_M^{-1}(0_{M/\theta^*})$ and $D_2\subseteq\Phi_M^{-1}(0_{M/\theta^*})$.

Moreover, since θ^* is strongly regular [16], $\Phi_M^{-1}(0_{M/\theta^*})$ is a complete part subhypermodule of *M* whence $<D_2>_C\subseteq\Phi_M^{-1}(0_{M/\theta^*})$.

Conversely, $M/D(M)$ is an *R*-subhypermodule of *M* (with commutative hypergroup [5]) and $D(M)$ is an invariant *R*-hypermodule and complete part of *M*. Hence $M/R_M\cong M/(x+D(M))=M/D(M)$ by Lemma 3.7 and Theorem 2.4 of [16], so we have $\theta^*\subseteq R_M$.

Finally, let $\varepsilon\in D(M)$, then for every $x\in\Phi_M^{-1}(0_{M/\theta^*})$ we have $\Phi_M(\varepsilon)=0_{M/\theta^*}=\Phi_M(x)$, since $\varepsilon\in D(M)\subseteq\Phi_M^{-1}(0_{M/\theta^*})$. So we obtain $x\theta^*\varepsilon$, whence xR_M and $x\in R_M(\varepsilon)=\varepsilon+D(M)=D(M)$. Thus $\Phi_M^{-1}(0_{M/\theta^*})\subseteq D(M)$.

Lemma 3.9. Let *M* be an *R*-hypermodule, $B\in\wp^*(M)$ and $\Phi_M:M\to M/\theta^*$ be a canonical projection. Then

$$\omega_M+B=B+\omega_M=\Phi_M^{-1}(\Phi_M(B)).$$

Proof: Let $\Phi_M^{-1}(\Phi_M(B))=\{x\in M\mid\exists b\in B,\Phi_M(x)=\Phi_M(b)\}$. Then for every $x\in\Phi_M^{-1}(\Phi_M(B))$ there exists $b\in B$ such that $\Phi_M(x)=\Phi_M(b)$. Since *M* is an *R*-hypermodule, there exists $u\in M$ such that $x\in b+u$. Now

$\Phi_M(b)=\Phi_M(x)=\Phi_M(b+u)=\Phi_M(b)+\Phi_M(u)\Rightarrow u\in\omega_M$. Hence $x\in B+\omega_M$, and therefore $\Phi_M^{-1}(\Phi_M(B))\subseteq B+\omega_M$.

Conversely, if $z\in B+\omega_M$, then $\Phi_M(z)\in\Phi_M(B)$ and $B+\omega_M\subseteq\Phi_M^{-1}(\Phi_M(B))$. Therefore $B+\omega_M=\Phi_M^{-1}(\Phi_M(B))$.

Lemma 3.10. For every non empty subset *H* of an *R*-hypermodule *M* we have
1) $\Phi_M^{-1}(\Phi(H))=D(M)+H=H+D(M)$.
2) If *H* is a θ-part of *M*, then $\Phi_M^{-1}(\Phi_M(H))=H$.

Proof: 1) For every $x\in D(M)+H$ there exists a pair $(a,b)\in D(M)\times H$ such that $x\in a+b$, so $\Phi_M(x)=\Phi_M(a)\otimes\Phi_M(b)=0_{M/\theta^*}\otimes\Phi_M(b)=\Phi_M(b)$. Therefore $x\in\Phi_M^{-1}(\Phi_M(b))\subseteq\Phi_M^{-1}(\Phi_M(H))$.

Conversely, for every $x\in\Phi_M^{-1}(\Phi_M(H))$, an element $b\in H$ exists such that $\Phi_M(x)=\Phi_M(b)$. By reproducibility $a\in M$ exists such that $x\in a+b$, so $\Phi_M(b)=\Phi_M(x)=\Phi_M(a)\otimes\Phi_M(b)$, hence $\Phi_M(a)=0_{M/\theta^*}$ and $a\in\Phi_M^{-1}(0_{M/\theta^*})=D(M)$. Therefore $x\in a+b\subseteq D(M)+H$. This proves that $\Phi_M^{-1}(\Phi_M(M))=D(M)+H$. In the same way, it is possible to prove that $\Phi_M^{-1}(\Phi_M(M))=H+D(M)$.

2) It is obvious that $H\subseteq\Phi_M^{-1}(\Phi_M(H))$. Moreover, if $x\in\Phi_M^{-1}(\Phi_M(H))$, then there exists an element $b\in H$ such that $\Phi_M(x)=\Phi_M(b)$. Hence $x\in\theta^*(x)=\theta^*(b)\subseteq H$ and $\Phi_M^{-1}(\Phi_M(H))\subseteq H$.

Lemma 3.11. Let *M* be an *R*-hypermodule. Then ω_M is the intersection of all *R*- subhypermodules of *M* that are complete parts.

Proof: By Lemma 3.9 we have $\omega_M+\omega_M=\omega_M$ as a hypermodule. Let $A\subseteq\bigcap M_i$, where every M_i is a complete part subhypermodule of *M*. Then $A+\omega_M=A$. Also, by Lemma 3.5, *A* is an invertible subhypermodule of *M*, hence

$$\forall(a,x)\in A\times\omega_M,\ \exists b\in A: a\in b+x\Rightarrow a\in A+x\Rightarrow x\in A+a=A.$$

Therefore $\omega_M\subseteq A$.

For every element *M* of an *R*-hypermodule *M* we set:

$$P(m)=\left\{A\in\wp^*(M)\mid m\in A,\ \exists n\in N,\ \exists(m'_1,\cdots,m'_n),\ A=\sum_{i=1}^n m'_i\right\};$$

$$M^*(m)=\bigcup_{A\in P(m)}A.$$

In the following, for an *R*-hypermodule *M* let $R\cdot m=M$ for all $m\in M$ and for all $r\in R$, $r\cdot m=m\cdot r$.

Lemma 3.12. For every $u\in M$, $M^*(u)$ is a complete part of *R*-hypermodule *M*.

Proof: Let $(m'_1,\cdots,m'_n)$ and $\sum_{i=1}^n m'_i\cap M^*(u)\neq\phi$. Then

$$\exists a\in\sum_{i=1}^n m'_i\cap M^*(u)\Rightarrow\exists A\in P(u): a\in\sum_{i=1}^n m'_i\cap A.$$

For every $m_i\in M$, there exists $r_i\in R$ such that $m_i\in r_i\cdot u$. On the other hand, $u\in r\cdot a=a\cdot r$ where $r\in R$. Hence

$$\sum_{i=1}^n m'_i=\sum_{i=1}^n\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}}x_{ijk}\right)m_i\subseteq\sum_{i=1}^n\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}}x_{ijk}\right)(r_i\cdot u)$$

$$\subseteq \sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right) r_i \cdot (r \cdot a) \subseteq \sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right)(r_i r)\cdot A$$

and

$$u \in r \cdot a = a \cdot r \subseteq \left(\sum_{i=1}^{n} m_i'\right)\cdot r \subseteq \sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right)(r_i \cdot u)\cdot r$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right) r \cdot (r_i \cdot u) \subseteq \sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right)(r \cdot r_i)\cdot A .$$

So $\sum_{i=1}^{n}\left(\sum_{j=1}^{n_i}\prod_{k=1}^{k_{ij}} x_{ijk}\right)(rr_i)\cdot A \in P(u)$ and $\sum_{i=1}^{n} m_i' \subseteq M^*(u)$. Therefore $M^*(u)$ is a complete part of *M*.

Let *A* be a non empty subset of *R*-hypermodule *M*. The intersection of all complete parts of *M* which contain *A* is called the *complete closure of A* and it will be denoted by $C(A)$.

Corollary 3.13. If $u \in \omega_M$, then $M^*(u) = \omega_M$.

Proof: We know, $\omega_M = C(u) \subseteq M^*(u)$. Also, ω_M is a complete part, so $M^*(u) \subseteq \omega_M$. Therefore $M^*(u) = \omega_M$.

Lemma 3.14. For every element *u* in the heart ω_M, we have $D(M) = P_\sigma(u)$.

Proof: Let $\sum_{i=1}^{n} z_i' \cap P_\sigma(u) \neq 0$, where $z_i' = \sum_{j=1}^{m_i}\left(\prod_{k=1}^{k_{ij}} x_{ijk}\right) z_i$, $y_{ijk} \in R$ and $z_i \in M$. If $x \in \sum_{i=1}^{n} z_i' \cap P_\sigma(u)$, we have $x\theta u$. Thus there exist $n \in N$ $(m_1', \cdots, m_n')$ and $\sigma \in S_n$ such that $u \in \sum_{i=1}^{n} m_i'$ and $x \in \sum_{i=1}^{n} m_{\sigma(i)}'$. Also, there exist $r_i, r \in R$ such that $u \in r \cdot x$ and $z_i \in r_i \cdot u$. Then

$$u \in r \cdot x = x \cdot r \subseteq \sum_{i=1}^{m}\left(\sum_{j=1}^{m_i}\prod_{k=1}^{k_{ij}} y_{ijk}\right) z_i \cdot r \subseteq \sum_{i=1}^{m}\left(\sum_{j=1}^{m_i}\prod_{k=1}^{k_{ij}} y_{ijk}\right)(r_i \cdot u)\cdot r$$

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{m_i}\prod_{k=1}^{k_{ij}} y_{ijk}\right)(rr_i)\cdot u \subseteq \sum_{i=1}^{m}\left(\sum_{j=1}^{m_i}\prod_{k=1}^{k_{ij}} y_{ijk}\right)(rr_i)\cdot \sum_{i=1}^{n} m_i' = \sum_{i=1}^{n} m_i'' ,$$

whence $(m_1'', \cdots, m_n'') \in T_n(u)$.

Moreover, for every $\sigma \in S_n$, we have

$$\sum_{i=1}^{n} z_{\sigma(i)}' = \sum_{i=1}^{m} B_{\sigma(i)} z_{\sigma(i)} \subseteq \sum_{i=1}^{n} B_{\sigma(i)} r_{\sigma(i)} \cdot u \subseteq \sum_{i=1}^{m} B_{\sigma(i)} r_{\sigma(i)} \cdot (r \cdot x)$$

$$\subseteq \sum_{i=1}^{m} B_{\sigma(i)}(rr_{\sigma(i)})\cdot x \subseteq \sum_{i=1}^{m} B_{\sigma(i)}(rr_{\sigma(i)})\cdot \sum_{i=1}^{n} m_{\sigma(i)}' = \sum_{i=1}^{n} m_{\sigma(i)}'' ,$$

where

$$B_i = \sum_{j=1}^{m_i} A_{i\sigma_i(j)} , \quad A_{ij} = \prod_{k=1}^{k_{ij}} y_{ij\sigma_{ij}(k)}$$

with

$$\sigma_i \in S_{n_i} \text{ and } \sigma_{ij} \in S_{k_{ij}} .$$

Since $\sum_{i=1}^{n} m''_{\sigma(i)} \in P_n(u)$, it follows that $\sum_{i=1}^{m} z'_{\sigma(i)} \subseteq P_\sigma(u)$ and $P_\sigma(u)$ is a θ-part of M.

Finally, we have $u \in \omega_M \cap P_\sigma(u) \subseteq D(M) \cap P_\sigma(u)$ and $\theta^*(u) \subseteq D(M) \cap P_\sigma(u)$, since $D(M)$ and $P_\sigma(u)$ are θ-parts of M. By Theorem 3.8, if $x \in D(M)$, then $\Phi_M(x) = 0_{M/\theta^*} = \Phi_M(u)$. Therefore $x \in \theta^*(x) = \theta^*(u) \subseteq P_\sigma(u)$, whence $D(M) \subseteq P_\sigma(u)$.

Conversely, if $x \in P_\sigma(u)$, we have $x\theta u$, hence $x \in \theta^*(u) \subseteq D(M)$, that is $P_\sigma(u) \subseteq D(M)$.

Lemma 3.15. The relation ε is transitive.

Proof: Let M be an R-hypermodule and $\Phi_M : M \to M/\varepsilon^*$ be a canonical projection and $x\varepsilon^* y$. If $x\varepsilon y$, then there exists $(m'_1, \cdots, m'_n)$ such that

$$x, y \in \sum_{i=1}^{n} m'_i;\ m'_i = m_i \text{ or } m'_i = \sum_{j=1}^{n_i} \left(\prod_{k=1}^{k_{ij}} x_{ijk} \right) z_i;$$

$$m_i \in M,\ x_{ijk} \in R,\ z_i \in M.$$

We know $x + \omega_M = \Phi_M^{-1}(\Phi_M(\{x\}))$. On the other hand, $\Phi_M(x) = \Phi_M(y)$. Then $\{x, y\} \subseteq x + \omega_M$ and so there exists $(v, w) \in \omega_M^2$ such that $x \in x + w$, $y \in x + v$. Also, $M^*(u)$ is a complete part of M and by Corollary 3.13, we have $M^*(w) = \omega_M$, hence there exist $A \in P(w)$, $k \in N$ and $(m'_1, \cdots, m'_k)$ such that $v \in A$, $A = \sum_{i=1}^{k} m''_i$. Therefore $\{v, w\} \subseteq \sum_{i=1}^{k} m''_i$, hence $v \in w$ and $x + v\ \overline{\overline{\varepsilon}}\ x + w$. Therefore $x\varepsilon y$ and $\varepsilon = \varepsilon^*$.

Theorem 3.16. θ is a transitive relation on hypermodules.

Proof: By Lemma 3.10 and Lemma 3.14, if $x\theta^* y$ then $x \in \Phi_M^{-1}(\Phi_M(y)) = y + D(M) = y + P_\sigma(u)$. Thus there exist $n \in N$, $(m''_1, \cdots, m''_n)$ and $\sigma \in S_n$ such that $u \in \sum_{i=1}^{n} m''_i$ and $x \in y + \sum_{i=1}^{n} m''_{\sigma(i)}$. Now by reproducibility of M and ω_M, there are $v \in M$ and $w \in \omega_M$ such that $y \in v + u$ and $u \in w + u$. Moreover, since $\{v, w\} \subseteq \omega_M$ and ε is a transitive relation, there exists $(m'_1, \cdots, m'_k)$ such that $\{v, w\} \subseteq \sum_{i=1}^{k} m'_i$. It follows that

$$y \in v + u \subseteq v + w + u \subseteq v + w + w + u \subseteq v + w + \sum_{i=1}^{k} m'_i + \sum_{i=1}^{n} m''_i$$

and

$$x \in y + \sum_{i=1}^{n} m''_{\sigma(i)} \subseteq v + u + \sum_{i=1}^{n} m''_{\sigma(i)} \subseteq v + w + u + \sum_{i=1}^{n} m''_{\sigma(i)} \subseteq v + w + \sum_{i=1}^{k} m'_{\sigma(i)} + \sum_{i=1}^{n} m''_{\sigma(i)}.$$

Therefore $y\theta x$ and $\theta = \theta^*$.

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