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N-ARY POLYGROUPS*

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Abstract – Polygroups are multivalued systems that satisfy group like axioms. In this paper the class of *n*-ary polygroups is introduced. The concepts of *n*-ary normal subpolygroups and strong homomorphisms of *n*-ary polygroups are adopted. With respect to these concepts the isomorphism theorems for *n*-ary polygroups are stated and proved. Finally, we will consider the fundamental relation β^* defined on an *n*-ary polygroup and prove some results in this respect.

Keywords – *n*-ary polygroup, *n*-ary normal subploygroup, isomorphism theorem, fundamental relation

1. INTRODUCTION

This paper deals with certain algebraic systems called *n*-ary polygroups. Marty [1] introduced the basic concepts of hyperstructures and hypergroups, since then several authors have further studied this subject. Applications of hypergroups have mainly appeared in special subclasses. Polygroups, which are certain subclasses of hypergroups, are studied in [2] by Ioulidis and are used to study color algebras [3, 4]. A detailed discussion on the theory of polygroups can be found in [3-7].

The notion of an *n*-ary group which is a natural generalization of the notion of a group, was introduced by Dornte [8] and is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various *n*-ary algebras have appeared in the literature, for example [9, 10, 11].

Davvaz and Vougiouklis in 2006 defined and considered *n*-ary hypergroups [12]. Some generalizations of hyperstructures to *n*-ary hyperstructures and *n*-ary H_v- structures may be found in [13, 14].

In this paper, the *n*-ary polygroups, as a subclass of *n*-ary hypergroups and as a generalization of polygroups are defined and considered. Finally, we consider the fundamental relation β^* defined on an *n*-ary polygroup and prove some new results. Also, we adopt the concepts of *n*-ary normal subpolygroups and strong homomorphisms of *n*-ary polygroups. With respect to these concepts, we shall state and prove the isomorphism theorems for *n*-ary polygroups.

2. BASIC DEFINITIONS AND RESULTS

Let *P* be a non-empty set and $P^*(P)$ be the set of all non-empty subsets of *P* and P^n be the *n*-times Cartesian product of *P*. In general, a mapping $f : P^n \longrightarrow P^*(P)$ is called an *n*-ary hyperoperation on *P* and *n* is called the arity of this hyperoperation. If for all $(x_1,\cdots,x_n)\in P^n$, the set $f(x_1,\cdots,x_n)$ is a

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singleton, then f is called an n-ary operation. For an n-ary hyperoperation f on P and for subsets $A_1,\cdots,A_n$ of P, it is defined:

$$f(A_1,\cdots,A_n)=\bigcup_{\substack{x_i\in A_i\\1\le i\le n}} f(x_1,\cdots,x_n).$$

Also, we shall use the following abbreviated notations:

The sequence $x_i,x_{i+1},\cdots,x_j$ will be denoted by x_i^j. For $j<i$, x_i^j is the empty set. In this convention $f(x_1,\cdots,x_i,y_{i+1},\cdots,y_j,z_{j+1},\cdots,z_n)$ will be written as $f(x_1^i,y_{i+1}^j,z_{j+1}^n)$.

If $m=k(n-1)+1$, then the n-ary hyperoperation g defined by

$$g\left(x_1^{k(n-1)+1}\right)=\underbrace{f(f(\cdots,f(f}_{k}(x_1^n),x_{n+1}^{2n-1}),\cdots),x_{(k-1)(n-1)+2}^{k(n-1)+1}),$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ (depending on additional assumptions), we write $f_{(\cdot)}$ to mean $f_{(k)}$ for some $k=1,2,\cdots$. Also $f(a_1^i,x.)$ means $f\left(a_1^i,\underbrace{x,\cdots,x}_{n-i}\right)$ for $a_1,\cdots,a_i,x\in P$ and $1\le i\le n-1$.

According to [4], a polygroup is a multivalued system $\langle P,\cdot,e,^{-1}\rangle$ where,

$$e\in P,\ ^{-1}:P\longrightarrow P,\ \cdot:P\times P\longrightarrow P^*(P)$$

and the following axioms hold for all $x,y,z\in P$

(*i*) $(x\cdot y)\cdot z=x\cdot(y\cdot z)$, (*ii*) $e\cdot x=x=x\cdot e$, (*iii*) $x\in y\cdot z$ implies $y\in x\cdot z^{-1}$ and $z\in y^{-1}\cdot x$.

According to [12], an n-ary hypergroup (P,f) is a non-empty set P with an n-ary hyperoperation $f:P^n\longrightarrow P^*(P)$ such that, if $i,j\in\{1,\cdots,n\}$ and $a_1,\cdots,a_{2n-1},b\in P$, then:

(*i*) $f(a_1^{i-1},f(a_i^{n+i-1}),a_{n+i}^{2n-1})=f(a_1^{j-1},f(a_j^{n+j-1}),a_{n+j}^{2n-1})$, (*ii*) there exists $x_i\in P$ such that

$$b\in f(a_1^{i-1},x_i,a_{i+1}^n). \tag{1}$$

In this definition if f is an n-ary operation, then the relation (1) will be $b=f(a_1^{i-1},x_i,a_{i+1}^n)$, and (P,f) is an n-ary group.

Definition 2.1. An n-ary polygroup is a multivalued system $M=\langle P,f,e,^{-1}\rangle$, where $e\in P$, $^{-1}$ is an unitary operation on P, f is an n-ary hyperoperation on P and the following axioms hold for all $i,j\in\{1,\cdots,n\}$, $x_1,\cdots,x_{2n-1},x\in P$:

(*i*) $f(x_1^{i-1},f(x_i^{n+i-1}),x_{n+i}^{2n-1})=f(x_1^{j-1},f(x_j^{n+j-1}),x_{n+j}^{2n-1})$,

(*ii*) e is an unique element such that $f\left(\underbrace{e,\cdots,e}_{i-1},x,\underbrace{e,\cdots,e}_{n-i}\right)=x$, and $e^{-1}=e$,

(*iii*) $x\in f(x_1^n)$ implies $x_i\in f(x_{i-1}^{-1},\cdots,x_1^{-1},x,x_n^{-1},\cdots,x_{i+1}^{-1})$.

It is clear that any 2-ary polygroup is a polygroup.

Example 2.2. Let $P=\{e,x,y\}$ be a set with 3-ary hyperoperation f as follows:

$$f(e,e,e)=e,$$
$$f(x,e,e)=f(e,x,e)=f(e,e,x)=x,$$
$$f(y,e,e)=f(e,y,e)=f(e,e,y)=y,$$
$$f(x,x,e)=f(x,e,x)=f(e,x,x)=\{e,y\},$$
$$f(y,y,e)=f(y,e,y)=f(e,y,y)=\{e,x\},$$
$$f(e,x,y)=f(e,y,x)=f(x,e,y)=f(x,y,e)=f(y,x,e)=f(y,e,x)=\{x,y\}$$
$$f(x,x,x)=f(y,y,y)=\{x,y\},$$
$$f(x,x,y)=f(x,y,x)=f(y,x,x)=f(x,y,y)=f(y,x,y)=f(y,y,x)=P.$$

It is easy to verify that for $x_i \in P(i=1,\cdots,5)$, we have

$$f(f(x_1,x_2,x_3),x_4,x_5)=f(x_1,f(x_2,x_3,x_4),x_5)=f(x_1,x_2,f(x_3,x_4,x_5)),$$

this implies that *f* is associative. Suppose ${}^{-1}=I:P\longrightarrow P$ is the identity function on *P,* then $x^{-1}=x$, $y^{-1}=y$, $e^{-1}=e$. Also,

$$t\in f(x_1,x_2,x_3) \text{ implies } x_1\in f(t,x_3^{-1},x_2^{-1}),\ x_2\in f(x_1^{-1},t,x_3^{-1}),\ x_3\in f(x_2^{-1},x_1^{-1},t).$$

Therefore, $\langle P,f,e,{}^I\rangle$ is a 3-ary polygroup.

Lemma 2.3. The following elementary facts about *n*-ary polygroups follow easily from the axioms,

(*i*) $e\in f\left(\underbrace{e,\cdots,e}_{i-1},x,e,\cdots,e,x^{-1},\underbrace{e,\cdots,e}_{n-j}\right)$, where $i,j\in\{1,2,\cdots,n\}$, $i\neq j$,

(*ii*) $(x^{-1})^{-1}=x$,

(*iii*) $f(x_1^i,e.)=f\left(x_1^{i-1},\underbrace{e,\cdots,e}_{k},x_i,e.\right)$, $0\leq k<n-i$,

(*iv*) $f(x_1^n)^{-1}=f(x_n^{-1},\cdots,x_1^{-1})$, where $A^{-1}=\{a^{-1}\mid a\in A\}$.

Proof: (*iv*)

$$e\in f(x_1,x_1^{-1},e,\cdots,e)\subseteq f(f(x_1,x_2,x_2^{-1},x_1^{-1},e,\cdots,e)),\text{ by } (i)$$
$$\cdots$$
$$\subseteq f(f(x_1^n),f(x_n^{-1},\cdots,x_1^{-1}),e.).$$

So ,
$$f(x_n^{-1},\cdots,x_1^{-1})\subseteq f(f(x_1^n)^{-1},e.)=f(x_1^n)^{-1}. \tag{2}$$

On the other hand,
$e\in f(f(x_1^n),f(x_n^{-1},\cdots,x_1^{-1}),e.)$ implies $f(x_1^n)\subseteq f(e,\cdots,e,f(x_n^{-1},\cdots,x_1^{-1})^{-1})=f(x_n^{-1},\cdots,x_1^{-1})^{-1}$.
Then,

$$f(x_1^n)^{-1}\subseteq\left(f(x_n^{-1},\cdots,x_1^{-1})^{-1}\right)=f(x_n^{-1},\cdots,x_1^{-1}),\text{ by } (ii). \tag{3}$$

Therefore, by (*2*) and (3) $f(x_1^n)^{-1}=f(x_n^{-1},\cdots,x_1^{-1})$.

Definition 2.4. A non-empty subset *K* of an *n*-ary polygroup *P* is said to be an *n*-ary subpolygroup of *P* if, with the hyperoperation of *P*, K itself forms an *n*-ary polygroup.

Lemma 2.5. A non-empty subset *K* of the polygroup $\langle P, f, e, ^{-1}\rangle$ is an *n*-ary subpolygroup if and only if, (*i*) $e \in K$, (*ii*) $a_1, \cdots, a_n \in K$ implies $f\left(a_1^{i-1}, a_i^{-1}, a_{i+1}^n\right) \subseteq K$.

Proof: It is obtained exactly from definition 2.1.

Lemma 2.6. Let *K* be an *n*-ary subpolygroup of *P* and $a_2, \cdots, a_n \in K$, then $f\left(K, a_2^n\right) = K$.

Proof: It is clear that $f\left(K, a_2^n\right) \subseteq K$. Suppose $k \in K$ so

$$\begin{aligned}
k = f(k, e.) &\in f_{(2)}\left(k, a_2^{-1}, a_2, e.\right) \\
&\subseteq f_{(3)}\left(k, a_2^{-1}, a_3^{-1}, a_3, a_2, e.\right) \\
&\cdots \\
&\subseteq f_{(\cdot)}\left(k, a_2^{-1}, \cdots, a_n^{-1}, a_2^n, e.\right) \\
&= f\left(f\left(k, a_2^{-1}, \cdots, a_n^{-1}\right), a_2^n\right) \\
&\subseteq f\left(K, a_2^n\right).
\end{aligned}$$

Therefore, $K = f\left(K, a_2^n\right)$.

Theorem 2.7. Let $K_1, \cdots, K_n$ be *n*-ary subpolgroups of *P* such that for every $\sigma \in S_n$, $f\left(K_{\sigma(1)}, \cdots, K_{\sigma(n)}\right) = f\left(K_1^n\right)$. Then $f\left(K_1^n\right)$ is an *n*-ary subpolygroup of *P*.

Proof: It is clear that $e = f(e, \cdots, e) \in f\left(K_1^n\right)$. If $t_1, \cdots, t_n \in f\left(K_1^n\right)$, then for some $x_{ij} \in K_j$ and $1 \le i, j \le n$ we have $t_1 = f\left(x_{11}^n\right), \cdots, t_n = f\left(x_{n1}^n\right)$. Thus

$$\begin{aligned}
f\left(t_1^n\right) &= f\left(f\left(x_{11}^n\right), \cdots, f\left(x_{n1}^n\right)\right) \\
&\subseteq f\left(f\left(K_1^n\right), \cdots, f\left(K_1^n\right)\right) \\
&= f_{(2)}\left(K_1, f\left(K_2^n, K_1\right), \cdots, f\left(K_2^n, K_1\right), K_2^n\right), \text{ by associatively of } f \\
&= f_{(2)}\left(K_1, \underbrace{f\left(K_1, K_2^n\right), \cdots, f\left(K_1, K_2^n\right)}_{n-1}, K_2^n\right), \text{ by hypothesis} \\
&= f_{(3)}\left(K_1, K_1, \underbrace{f\left(K_2^n, K_1\right), \cdots, f\left(K_2^n, K_1\right)}_{n-2}, K_2^n, K_2^n\right) \\
&\cdots \\
&= f_{(n+1)}\left(K_1, \cdots, K_1, K_2^n, \cdots, K_2^n\right) \\
&= f_{(n-1)}\left(f\left(K_1, \cdots, K_1\right), f\left(K_2, e.\right), K_3^n, K_2^n, \cdots, K_2^n\right) \\
&= f\left(K_1, f\left(K_2, K_3^n, e\right), \cdots, f(K_2, K_3^n, e)\right) \\
&\cdots \\
&= f\left(K_1, K_2, \cdots, K_n\right).
\end{aligned}$$

Therefore, *f* is an *n*-ary hyperoperation on $f\left(K_1^n\right)$. If $x \in f\left(K_1^n\right)$, then $x \in f\left(x_1^n\right)$, for some $x_i \in K_i$, $i = 1, \cdots, n$. Thus

$$x^{-1} \in \left(f\left(x_1^n\right)\right)^{-1} = f\left(x_n^{-1}, \cdots, x_1^{-1}\right), \text{ by Lemma 2.3}$$

$$\subseteq f(K_n, \cdots, K_1) = f(K_1^n), \text{ by hypothesis.}$$

This conclude that $f(K_1^n)$ is an n-ary subpolygroup of P.

3. QUOTIENT *N*-ARY POLYGROUPS

Definition 3.1. The n-ary subpolygroup N of P is said to be normal in P if for every $a \in P$

$$f(a^{-1}, N, a, e.) \subseteq N.$$

Lemma 3.2. Let N be an n-ary normal subpolygroup of P, then
(i) $f(a^{-1}, N, a, e.) = N$, for every $a \in P$,
(ii) $f(a, N, e.) = f(N, a, e.)$, for every $a \in P$,
(iii) $f(a_2^i, N, a_{i+1}^n) = f(a_2^j, N, a_{j+1}^n)$, for $i, j \in \{1, \cdots, n\}$,
(iv) $f(N, a, e.) = f(N, b, e.)$, for every $b \in f(N, a, e.)$,
(v) if $b_i \in f(N, a_i, e.)$, for $i = 2, \cdots, n$ then $b \in f(N, a, e.)$,
(vi) $f(f(N, a_{12}^n), \cdots, f(N, a_{n2}^n)) = f_{(\cdot)}(N, a_{12}^n, a_{22}^n, \cdots, a_{n2}^n) = f_{(n)}(N, a_{12}^n, \cdots, a_{n2}^n)$.

Proof: (i)

$$\begin{aligned} N &= f(e, N, e.) \\ &\subseteq f(f(a^{-1}, a, e.), N, f(a^{-1}, a, e.), e.) \\ &= f(a^{-1}, a, N, a^{-1}, a, e.) \\ &= f(a^{-1}, f(a, N, a^{-1}, e.), a, e.) \\ &\subseteq f(a^{-1}, N, a, e.), \text{ since } f(a, N, a^{-1}, e.) \subseteq N. \end{aligned}$$

Therefore, $f(a, N, a^{-1}, e.) = N$.
(ii) From $f(a, N, a^{-1}, e.) = N$ we have $f(a, f(a^{-1}, N, a, e.), e.) = f(a, N, e.)$, and so

$$f(N, a, e.) = f(e, N, a, e.) \subseteq f_{(2)}(a, a^{-1}, N, a, e.) = f(a, N, e.).$$

Similarly, $f(a, N, e.) \subseteq f(N, a, e.)$ and so $f(a, N, e.) = f(N, a, e.)$.
(iii) It is concluded from (ii).
(iv) $b \in f(N, a, e.)$ implies $f(N, b, e.) \subseteq f(N, f(N, a, e.), e.) = f(N, a, e.)$. On the other hand, $b \in f(N, a, e.) \;\Rightarrow\; a \in f(N^{-1}, b, e.) = f(N, b, e.)$. Therefore, $f(N, a, e.) = f(N, b, e.)$.
(v)

$$\begin{aligned} f(N, b_2^n) &= f(f(N, \cdots, N), b_2^n) \\ &= f_{(2)}(N, N, b_2, N, b_3, \cdots, N, b_n), \text{ since } N \text{ is normal} \\ &= f_{(2)}(N, f(N, e.), b_2, f(N, e.), b_3, \cdots, f(N, e.), b_n), \text{ since } e \in N \\ &= f(N, f(N, b_2, e.), \cdots, f(N, b_n, e.)) \\ &= f(N, f(N, a_2, e.), \cdots, f(N, a_n, e.)), \text{ by hypothesis and } (iv) \\ &= f(N, a_2^n), \text{ by a similar argument.} \end{aligned}$$

(vi) Since N is normal it is clear.

Lemma 3.3. Let K and N be n-ary subpolygroups of a polygroup P, where N is normal in P. Then
(i) $N \cap K$ is an n-ary normal subpolygroup of K,

(*ii*) $f(N,K,e.)=f(K,N,e.)$ is an *n*-ary subpolygroup of *P*,
(*iii*) *N* is a normal subpolygroup of $f(N,K,e.)$.

Proof: It is straightforward.

Suppose *K* is an *n*-ary subpolygroup of *P*, we define the relation $\equiv_K$ on $P^{(n-1)}$ by

$$(x_2^n)\equiv_K (y_2^n) \text{ if and only if } f(K,x_2^n)=f(K,y_2^n), \text{ for } (x_2^n),(y_2^n)\in P^{(n-1)}.$$

It is clear that the relation $\equiv_K$ is an equivalence relation on $P^{(n-1)}$. The class of $(x_2,\cdots,x_n)\in P^{(n-1)}$ is denoted by $K[x_2^n]=\{(y_2^n)\mid f(K,y_2^n)=f(K,x_2^n), y_2,\cdots,y_n\in P\}$.
And the set of all these classes is denoted by $P^{(n-1)}/K=\{K[x_2^n]\mid x_2,\cdots,x_n\in P\}$.

Also, we define the relation $\stackrel{K}{=}$ on *P* as follows:
$x\stackrel{K}{=}y$ if and only if there exist $a_2,\cdots,a_n\in P$ such that $x,y\in f(K,a_2^n)$, for $x,y\in P$.

Lemma 3.4. For sequences a_2^n, b_2^n in *P*,

(*i*) $f(K,a_2^n)\cap f(K,b_2^n)\neq\phi$ implies $f(K,a_2^n)=f(K,b_2^n)$, and $K[a_2^n]=K[a_2^n]$,

(*ii*) if $x\in f(K,a_2^n)$ then $f(K,x,e.)=f(K,a_2^n)$, and $K\left[x,\underbrace{e,\cdots,e}_{n-2}\right]=K[a_2^n]$.

Proof: (*i*) Suppose $x\in f(K,a_2^n)\cap f(K,b_2^n)$, then $x\in f(K,a_2^n)=f(K,f(e,a_2^n),e.)$. So by Definition 2.2.

$$\begin{aligned} f(e,a_2^n) &\subseteq f(K^{-1},x,e.)\\ &= f(K,x,e.)\\ &\subseteq f(K,f(K,b_2^n),e.), \text{ since } x\in f(K,b_2^n)\\ &= f(K,b_2^n). \end{aligned}$$

Thus, $f(K,a_2^n)=f(K,f(e,a_2^n),e.)\subseteq f(K,f(e,b_2^n),e.)\subseteq f(K,b_2^n)$. Also $f(K,b_2^n)\subseteq f(K,a_2^n)$ and $f(K,a_2^n)=f(K,b_2^n)$.
(*ii*) If $x\in f(K,a_2^n)$, $\{x\}=f(e,x,e.)\subseteq f(K,x,e.)$. Thus by (*i*)

$$f(K,x,e.)=f(K,a_2^n) \text{ and } K\left[x,\underbrace{e,\cdots,e}_{n-2}\right]=K[a_2^n].$$

Theorem 3.5. (*i*) The relation $\stackrel{K}{=}$ is an equivalent relation in *P*,
(*ii*) If $k[x]$ is the $\stackrel{K}{=}$ class of *x* and P/K is the set of all $\stackrel{K}{=}$ classes, then there exists a one-one correspondence between P/K and $P^{(n-1)}/K$.

Proof: (*i*) It is clear that $\stackrel{K}{=}$ is reflexive and symmetric. Suppose $x\stackrel{K}{=}y$ and $y\stackrel{K}{=}z$, then there exist sequences a_2^n, b_2^n in *P* such that $x,y\in f(K,a_2^n)$, and $y,z\in f(K,b_2^n)$. By Lemma 3.4, $f(K,a_2^n)=f(K,b_2^n)$. Thus $x,z\in f(K,a_2^n)$ and $x\stackrel{K}{=}z$.
(*ii*) We define $\varphi: P/K\longrightarrow P^{(n-1)}/K$ by $\varphi(K[x])=K[x,\underbrace{e,\cdots,e}_{n-2}]$, for $x\in P$. If $K[x],K[y]$ are in $\in P^{(n-1)}/K$, then

$$K[x]=K[y] \iff x,y\in f(k,a_2^n), \text{ for some } a_2,\cdots,a_n\in P$$
$$\iff f(K,x,e.)=f(K,a_2^n)=f(K,y,e.), \text{ for some } a_2,\cdots,a_n\in P, \text{ by Lemma 3.4}$$
$$\iff K[x,\underbrace{e,\cdots,e}_{n-2}]=K[y,\underbrace{e,\cdots,e}_{n-2}], \text{ by Lemma 3.4.}$$

Suppose $K[a_2^n]\in P^{(n-1)}/K$, then by (*ii*) of Lemma 3.4, $\varphi(K[a])=K[a,e,\cdots,e]=K[a_2^n]$ for every $a\in f(K,a_2^n)$. Therefore, φ is onto.

We can show $K[x,\underbrace{e,\cdots,e}_{n-2}]$ by $K[x,e.]$ and so by Theorem 3.5 $P^{(n-1)}/K=\{K[x,e.]\mid x\in P\}$, where $K[x,e.]=\{\{x_2^n\}\in P^{(n-1)}\mid f(K,x_2^n)=f(K,x,e.)\}$.

Lemma 3.6. Let $\langle P,f,e,^{-1}\rangle$ be an *n*-ary polygroup and *N* be an *n*-ary normal subpolygroup of *P*. Then $^{-1}: P^{(n-1)}/N\to P^{(n-1)}/N$, $N[a,e.]\mapsto N[a^{-1},e.]$ is a function and $N[a,e.]^{-1}=N[a^{-1},e.]$.

Proof: Because $^{-1}$ is a function on *P* and by (*ii*) of Lemma 2.3 $(x^{-1})^{-1}=x$, then for $A,B\subseteq P$, we have $A=B \iff A^{-1}=B^{-1}$. So for $N[a,e.]$, $N[a,e.]\in P^{(n-1)}/N$

$$N[a,e.]=N[b,e.] \iff f(N,a,e.)=(N,b,e.)$$
$$\iff f(N,a,e.)^{-1}=f(N,b,e.)^{-1}$$
$$\iff f(N,a^{-1},e.)=f(N,b^{-1},e.), \text{ by (iv) of Lemma 2.3 and since } N \text{ is normal}$$
$$\iff N[a^{-1},e.]=N[b^{-1},e.].$$

Theorem 3.7. If *N* is an *n*-ary normal subpolygroup of $\langle P,f,e,^{-1}\rangle$, and for $N[a_1,e.],\cdots,N[a_n,e.]$ in $P^{(n-1)}/N$, $F(N[a_1,e.],\cdots,N[a_n,e.])=\{N[t,e.]\mid t\in f(a_1^n)\}$, then $\langle P^{(n-1)}/N,F,N,^{-1}\rangle$ is an *n*-ary polygroup.

Proof: Suppose $N[a_1,e.],\cdots,N[a_n,e.],N[b_1,e.],\cdots,N[b_n,e.]\in P^{(n-1)}/N$ and $N[a_i,e.]=N[b_i,e.]$, for $i=1,\cdots,n$. Thus

$$f(N,a_i,e.)=f(N,b_i,e.), \text{ for } i=1,\cdots,n$$
$$f(f(N,a_1,e.),\cdots,f(N,a_n,e.))=f(f(N,b_1,e.),\cdots,f(N,b_n,e.))$$
$$f(N,f(a_1^n),e.)=f(N,f(b_1^n),e.)$$
$$\{N[t,e.]\mid t\in f(a_1^n)\}=\{N[t,e.]\mid t\in f(b_1^n)\}.$$

Therefore, $F(N[a_1,e.],\cdots,N[a_n,e.])=F(N[b_1,e.],\cdots,N[b_n,e.])$ and *F* is well defined.
The associativity of f implies that *F* is also associative. Also, for every $n\in N$ we have

$$N[n,e.]=\{(u_2^n)\mid f(N,u_2^n)=f(N,n,e.)=N=f(n,e.)\}=N[e.],$$
$$F(N[a,e.],N[e.].)=N[f(a,e.),e.]=N[a,e.].$$

So $N[e.]$ is the neutral element of $P^{(n-1)}/N$ and $(N[e.])^{-1}=N[e.]$. Suppose $N[a_1,e.],\cdots,N[a_n,e.]$, $N[a,e.]\in P^{(n-1)}/N$ and $N[a,e.]\in F(N[a_1,e.],\cdots,N[a_n,e.])$. Then

$$F(N,a,e.)\subseteq f(N,f(a_1^n,N),e.)$$
$$=f(a_1^{i-1},f(N,a_i,e.),a_{i+1}^n),$$

$$F(N,a_i,e.) \subseteq f\left(a_{i-1}^{-1},\cdots,a_1^{-1},f(N,a,e.),a_n^{-1},\cdots,a_{i+1}^{-1}\right)$$
$$= f\left(N, f\left(a_{i-1}^{-1},\cdots,a_1^{-1},a,a_n^{-1},\cdots,a_{i+1}^{-1},e.\right)\right),$$

$$N[a_i,e.] \in F\left(N\left[a_{i-1}^{-1},e.\right],\cdots,N\left[a_1^{-1},e.\right],N\left[a^{-1},e.\right],N\left[a_n^{-1},e.\right],\cdots,N\left[a_{i+1}^{-1},e.\right]\right).$$

Definition 3.8. Let $\left\langle P_1,f,e_1,^{-1}\right\rangle$ and $\left\langle P_2,g,e_2,^{-1}\right\rangle$ be *n*-ary polygroups. A mapping φ from P_1 to P_2 is said to be strong homomorphism if for every $a_1,\cdots,a_n \in P_1$,

$$\varphi(e_1)=e_2 \text{ and } \varphi\left(f\left(a_1^n\right)\right)=g(\varphi(a_1),\cdots,\varphi(a_n)).$$

A strong homomorphism φ is said to be an isomorphism if *f* is one to one and onto. Two *n*-ary polygroups P_1, P_2 are said to be isomorphic if there exists an isomorphism from P_1 onto P_2. In this case, we write $P_1 \cong P_2$. Moreover, if φ is a strong homomorphism from P_1 to P_2, then the kernel of φ is the set $ker\,\varphi=\{x\in P_1 \mid \varphi(x)=e_2\}$.

Lemma 3.9. Let φ be a strong homomorphism from $\left\langle P_1,f,e_1,^{-1}\right\rangle$ to $\left\langle P_2,g,e_2,^{-1}\right\rangle$. Then
(*i*) $\varphi(a)^{-1}=\varphi\left(a^{-1}\right)$ for every $a\in P_1$,
(*ii*) φ is injective if and only if $ker\,\varphi=\{e_1\}$,
(*iii*) $\varphi\left(f_{(k)}\left(a_1^{k(n-1)+1}\right)\right)=g_{(k)}\left(\varphi(a_1),\cdots,\varphi\left(a_{k(n-1)+1}\right)\right)$ for all $k\in N$ and $a_1,\cdots,a_{k(n-1)+1}\in P_1$.

Proof: (*i*) we know that $e_1\in f\left(a,a^{-1},e_1.\right)$, then

$$e_2=\varphi(e_1)\in\varphi\left(f\left(a,a^{-1},e_1.\right)\right)=g\left(\varphi(a),\varphi\left(a^{-1}\right),\varphi(e_1).\right)=g\left(\varphi(a),\varphi\left(a^{-1}\right),e_2.\right),$$
$$\varphi\left(a^{-1}\right)\in g\left(\varphi(a)^{-1},e_2.\right)=\varphi(a)^{-1}.$$

Therefore, $\varphi(a)^{-1}=\varphi\left(a^{-1}\right)$.
(*ii*) suppose $ker\,\varphi=\{e_1\}$ and $\varphi(y)=\varphi(z)$ for $y,z\in P_1$. Then

$$\varphi(e_1)=e_2\in g\left(\varphi(y),\varphi(y)^{-1},e_2.\right)=g\left(\varphi(z),\varphi\left(y^{-1}\right),e_2.\right)=g\left(f\left(z,y^{-1},e_1.\right)\right).$$

So there exists $x\in f\left(z,y^{-1},e_1.\right)$ such that $\varphi(x)=\varphi(e_1)$. But $ker\,\varphi=\{e_1\}$ implies $x=e_1$ and

$$e_1\in f\left(z,y^{-1},e_1.\right),\ z\in f(e_1,y,e_1.)=\{y\},\ z=y.$$

If φ is injective and $x\in ker\,\varphi$, then $\varphi(x)=e_2=\varphi(e_1)$ and $x=e_1$. So $ker\,\varphi=\{e_1\}$.
(*iii*) The proof is straightforward by induction on *k*.

It is easy to verify that $ker\,\varphi$ is an *n*-ary subpolygroup of P_1, but in general it is not normal in P_1.

We are now in a position to state and review the fundamental isomorphism theorems in *n*-ary polygroups.

Theorem 3.10. (First isomorphism theorem). Let φ be a strong homomorphism from $\left\langle P_1,f,e_1,^{-1}\right\rangle$ to $\left\langle P_2,g,e_2,^{-1}\right\rangle$ with $K=ker\,\varphi$ such that *K* is an *n*-ary normal subpolygroup of P_1, then $P_1^{(n-1)}/K\cong Im\,\varphi$
.

Proof: We define $\psi: P_1^{(n-1)}/K \longrightarrow \mathrm{Im}\,\varphi$ by $\psi(K[a,e.])=\varphi(a)$. If $K[b,e.]\in P_1^{(n-1)}/K$ then

$$K[a,e.]=K[b,e.] \iff f(K,a,e.)=f(K,b,e.)$$
$$\Rightarrow \varphi(f(K,a,e.))=\varphi(f(K,b,e.))$$

$$\Leftrightarrow \quad g(\varphi(K),\varphi(a),e_2.) = g(\varphi(K),\varphi(b),e_2.)$$
$$\Leftrightarrow \quad g(\varphi(a),e_2.) = g(\varphi(b),e_2.),\ \text{since}\ \varphi(K) = e_2$$
$$\Leftrightarrow \quad \varphi(a) = \varphi(b)$$
$$\Leftrightarrow \quad \psi(K[a,e.]) = \psi(K[b,e.]).$$

If , $K[a_1,e.],\cdots,K[a_n,e.] \in P_1^{(n-1)}/K$ then

$$\begin{aligned}\psi(F(K[a_1,e.],\cdots,K[a_n,e.])) &= \psi(\{K[t,e.] \mid t \in f(a_1^n)\}) \\ &= \{\varphi(t) \mid t \in f(a_1^n)\} \\ &= \varphi(f(a_1^n)) \\ &= g(\varphi(a_1),\cdots,\varphi(a_n)) \\ &= g(\psi(K[a_1,e.]),\cdots,\psi(K[a_n,e.])).\end{aligned}$$

We know $\psi(K[e_1.]) = \varphi(e_1) = e_2$. Also, if $\psi(K[a,e.]) = e_2$, then $\varphi(a) = e_2$, and $a \in ker\,\varphi = K$. So $K[a,e_1.] = K[e_1.]$ and by Lemma 3.9, ψ is injective.

Theorem 3.11. (Second isomorphism theorem). Suppose $J_1,\cdots,J_n$ are *n*-ary subpolygroups of an *n*-ary polygroup $\langle P,f,e,^{-1}\rangle$ such that J_i is normal and $f(J_{\sigma(1)}^{\sigma(n)}) = f(J_1^n)$ for every $\sigma \in S_n$ then $f(J_1^{i-1},e,J_{i+1}^n)/ f(J_1^{i-1},e,J_{i+1}^n) \cap J_i \cong f(J_1^n)/ J_i$.

Proof: We define $\varphi : f(J_1^{i-1},e,J_{i+1}^n) \longrightarrow f(J_1^n)/ J_i$ by $\varphi(x) = J_i[x,e.]$. It is clear that φ is well defined. If $x_1,\cdots,x_n \in f(J_1^{i-1},e,J_{i+1}^n)$, then $f(x_1^n) \subseteq f(J_1^{i-1},e,J_{i+1}^n)$ and

$$\varphi(f(x_1^n)) = J_i[f(x_1^n),e.] = \{J_i[t,e.] \mid t \in f(x_1^n)\} = F(J_i[x_1,e.],\cdots,J_i[x_n,e.]) = F(\varphi(x_1),\cdots,\varphi(x_n)).$$

Therefore, φ is a homomorphism. It is clear that φ is onto. Now suppose $x \in f(a_1^{i-1},e,a_{i+1}^n)$ and $x \in ker\,\varphi$ too, then

$$\varphi(x) = J_i[x,e.] = J_i[e.],$$
$$f(J_i,x,e.) = f(J_i,e.) = J_i,$$
$$x \in f(J_i^{-1},J_i,e.) = J_i.$$

So ker $\varphi \subseteq f(J_1^{i-1},e,J_{i+1}^n) \cap J_i$.
Conversely, if $x \in f(J_1^{i-1},e,J_{i+1}^n) \cap J_i$ then $f(J_i,x,e.) = J_i$ and so $\varphi(x) = J_i[x,e.] = J_i[e.]$. So $x \in ker\,\varphi$ and $ker\,\varphi = f(J_1^{i-1},e,J_{i+1}^n) \cap J_i$. Therefore, by first isomorphism theorem

$$f(J_1^{i-1},e,J_{i+1}^n)/ f(J_1^{i-1},e,J_{i+1}^n) \cap J_i \cong f(J_1^n)/ J_i.$$

Theorem 3.12. (Third isomorphism theorem) Suppose K and N are *n*-ary normal subpolygroups of $\langle P,f,e,^{-1}\rangle$ polygroup such that $N \subseteq K$. Then $\langle K/N,F,N,^{-1}\rangle$ is an *n*-ary normal subpolygroup of $\langle P^{(n-1)}/N,F,N,^{-1}\rangle$ and $(P^{(n-1)}/N)/(K^{(n-1)}/N) \cong P^{(n-1)}/K$.

Proof: It can be easily verified that $K^{(n-1)}/N$ is an *n*-ary subpolygroup of $P^{(n-1)}/N$. If for every $N[x,e.] \in P^{(n-1)}/N$ we show $F(N[x^{-1},e.],K^{(n-1)}/N,N[x,e.],N[e.]) \subseteq K^{(n-1)}/N$, then $K^{(n-1)}/N$ is an *n*-ary normal subgroup of $P^{(n-1)}/N$. Suppose $N[x,e.] \in K^{(n-1)}/N$, then

$$F\left(N\left[x^{-1},e.\right],N\left[K,e.\right],N\left[x,e.\right],N\left[e.\right]\right)=\left\{N\left[t,e.\right]\mid t\in f\left(x^{-1},K,x,e.\right)\right\}\subseteq K^{(n-1)}/N,$$

because K is normal in P and $f\left(x^{-1},K,x,e.\right)\subseteq K$.
Let $\varphi:P^{(n-1)}/N\longrightarrow P^{(n-1)}/K$ and $\varphi\left(N[x,e.]\right)=K[x,e.]$. If $N[x,e.],N[y,e.]\in P^{(n-1)}/N$ and $N[x,e.]=N[y,e.]$, then

$$\begin{gathered}f(N,x,e.)=f(N,y,e.)\\ f(K,f(N,x,e.),e.)=f(K,f(N,y,e.),e.)\\ f(f(K,N,e.),x,e.)=f(f(K,N,e.),y,e.)\\ f(K,x,e.)=f(K,y,e.),\text{ since } N\subseteq K\\ K[x,e.]=K[y,e.]\\ \varphi(K[x,e.])=\varphi(K[y,e.]).\end{gathered}$$

Therefore, φ is well defined. Now suppose $N[x_1,e.],\cdots,N[x_n,e.]\in P^{(n-1)}/N$, then

$$\begin{aligned}\varphi(F(N[x_1,e.],\cdots,N[x_n,e.]))&=\left\{\varphi(N[t,e.])\mid t\in f\left(x_1^n\right)\right\}\\ &=\left\{K[t,e.]\mid t\in f\left(x_1^n\right)\right\}\\ &=F(K[x_1,e.],\cdots,K[x_n,e.])\\ &=F(\varphi(N[x_1,e.]),\cdots,\varphi(N[x_n,e.])).\end{aligned}$$

It is clear that φ is onto and $K^{(n-1)}/N\subseteq ker\,\varphi$.
If $N[a,e.]\in ker\,\varphi$, then $\varphi(N[a,e.])=K[e.]$, $f(K,a,e.)=K$ and $a\in K$, $N[a,e.]\in K^{(n-1)}/N$. Therefore, $ker\,\varphi=K^{(n-1)}/N$ and by Theorem 3.10 $\left(P^{(n-1)}/N\right)/\left(K^{(n-1)}/N\right)\cong P^{(n-1)}/K$.

Definition and Theorem 3.13. Let $\left\langle P_1,f_1,e_1,^{-1}\right\rangle$ to $\left\langle P_2,f_2,e_2,^{-I}\right\rangle$ be two n-ary polygroups, so on $P_1\times P_2$ we can defined an n-ary hyperproduct $f_1\times f_2$ and a unitary function $\left(^{-1},^{-I}\right)$ by

$$f_1\times f_2\left((a_1,a_1'),\cdots,(a_n,a_n')\right)=\left\{(a,a')\mid a\in f_1\left(a_1^n\right),a'\in f_2\left(a'^n_1\right)\right\},\text{ and }(a,a')^{(-1,-I)}=\left(a^{-1},(a')^{-I}\right),$$

where $a,a_i\in P_1,a',a_i'\in P_2$, for $i=1,\cdots,n$. Then
(*i*) $\left\langle P_1\times P_2,f_1\times f_2,(e_1,e_2),\left(^{-1},^{-I}\right)\right\rangle$, is an n-ary polygroup,
(*ii*) $$\begin{aligned}(f_1\times f_2)_{(k)}\left((a_1,a_1'),\cdots,(a_{k(n-1)+1},a'_{k(n-1)+1})\right)&=\left\{(u,v)\mid u\in (f_1)_{(k)}\left(a_1^{k(n-1)+1}\right),v\in (f_2)_{(k)}\left(a'^{k(n-1)+1}_1\right)\right\}\\ &=(f_1)_{(k)}\left(a_1^{k(n-1)+1}\right)\times (f_2)_{(k)}\left(a'^{k(n-1)+1}_1\right).\end{aligned}$$

Proof: It is straightforward.

Corollary 3.14. If N_1, N_2 are n-ary normal subpolygroups of P_1, P_2 respectively, then $N_1\times N_2$ is an n-ary normal subploygroup of $P_1\times P_2$ and

$$(P_1\times P_2)^{(n-1)}/(N_1\times N_2)\cong\left(P_1^{(n-1)}/N_1\right)\times\left(P_2^{(n-1)}/N_2\right)$$

Proof: It is straightforward.

4. THE FUNDAMENTAL *N*-ARY GROUPS OF *N*-ARY POLYGROUPS

Let $\langle P, f, e, ^{-1}\rangle$ be an *n*-ary polygroup. We define the relation β^* as the smallest equivalence relation on *P* such that $\langle P/\beta^*, f/\beta^*, \beta^*(e), ^{-1}\rangle$ is an *n*-ary group, where P/β^* is the set of all equivalence classes of β^*, and when $\beta^*(a_1), \cdots, \beta^*(a_n) \in P/\beta^*$

$f/\beta^*\left(\beta^*(a_1), \cdots, \beta^*(a_n)\right) = \beta^*(a)$, for every $a \in f\left(\beta^*(a_1), \cdots, \beta^*(a_n)\right)$.

Also, P/β^* is called the *n*-ary fundamental group of *P*. This relation is studied by Corsini [15] concerning hypergroups, and by Davvaz and Vougiouklis [12] concerning *n*-ary hypergroups. Since *n*-ary polygroups are a certain subclass of *n*-ary hypergroups, we have the definition of β^* for *n*-ary polygroups as β^* for *n*-ary hypergroups, as follows. Let for $k \in N$, $\beta_k = \{(x, y) \mid x, y \in f_{(k)}(a_1, \cdots, a_{k(n-1)+1})$, for some $a_1, \cdots, a_{k(n-1)+1} \in P\}$, $\beta_0 = \{(x, x) \mid x \in P\}$, $\beta = \bigcup_{k\in N_0} \beta_k$. Then, $x\beta y$ if and only if $x\beta_k y$ for some $k \geq 0$. By Theorem 4.1 of [12], β^* is the transitive closure of the relation β. Also, if for *n*-ary polygroups $\langle P, f, e, ^{-1}\rangle$ define

$$f_{[0]} = \{\{a\} \mid a \in P\},\ f_{[k]} = \{f_{(k)}(a_1, \cdots, a_{k(n-1)+1}) \mid a_1, \cdots, a_{k(n-1)+1} \in P\},\ \mathcal{U} = \bigcup_{k\in N_0} f_{[k]},$$

then $x\beta y$ if and only if $\{x, y\} \in \mathcal{U}$, for some $u \in \mathcal{U}$.

The kernel of the canonical map $\varphi : P \longrightarrow P/\beta^*$ is called the core of *P* and is denoted by ω_P. Here we also denote by ω_P the neutral element of P/β^*. It is easy to prove that:

$$\omega_P = \beta^*(e) \text{ and } \beta^*(x)^{-1} = \beta^*(x^{-1}), \text{ for all } x \in P$$

Theorem 4.1. (See Theorem 4.3 of [12]) Let β_1^*, β_2^* and β^* be the fundamental equivalence relations on *n*-ary polygroups P_1, P_2 and $P_1 \times P_2$, respectively, then

$$(P_1 \times P_2)/\beta^* \cong P_1/\beta_1^* \times P_2/\beta_2^*.$$

Corollary 4.2. If N_1, N_2 are *n*-ary normal subpolygroups of P_1, P_2 respectively, and β^*, β_1^* and β_2^* are the fundamental equivalence relation on $(P_1 \times P_2)^{(n-1)}/N_1 \times N_2$, $P_1^{(n-1)}/N_1$, and $P_2^{(n-1)}/N_2$, respectively, then $\left((P_1 \times P_2)^{(n-1)}/N_1 \times N_2\right)/\beta^* \cong \left(P_1^{(n-1)}/N_1\right)/\beta_1^* \times \left(P_2^{(n-1)}/N_2\right)/\beta_2^*$.

Proof: It is concluded from Corollary 3.12 and Theorem 4.1.

Lemma 4.3. Let $\varphi : \langle P_1, f_1, e_1, ^{-1}\rangle \longrightarrow \langle P_2, f_2, e_2, ^{-1}\rangle$ be a strong homomorphism of *n*-ary polygroups, and β_1^*, β_2^* be the fundamental relation of P_1, P_2, respectively, then $\overline{ker\,\varphi} = \{\beta_1^*(x) \mid x \in P_1, \beta_2^*(\varphi(x)) = \omega_{P_2}\}$ is an *n*-ary normal subgroup of P_1/β_1^*.

Proof: Assume $\beta_1^*(x_1), \cdots, \beta_1^*(x_n) \in \overline{\ker\varphi}$. Then $\beta_2^*(\varphi(x_1)) = \cdots = \beta_2^*(\varphi(x_n)) = \omega_{P_2}$ and

$$\begin{aligned} & f_1/\beta_1^*\left(\beta_1^*(x_1), \cdots, \beta_1^*(x_n)\right) \\ &= \beta_1^*\left(f_1\left(\beta_1(x_1), \cdots, \beta_1(x_n)\right)\right), \text{ since } f\left(\beta_1(x_1), \cdots, \beta_1(x_n)\right) \subseteq f\left(\beta_1^*(x_1), \cdots, \beta_1^*(x_n)\right) \\ &= \beta_1^*\left(f\left(x_1^n\right)\right), \text{ since } f\left(x_1^n\right) \subseteq f\left(\beta_1^*(x_1), \cdots, \beta_1^*(x_n)\right) \\ &= \beta_1^*(z), \text{ for some } z \in f\left(x_1^n\right). \end{aligned}$$

So

$$\varphi(z) \in \varphi\left(f_1\left(x_1^n\right)\right) = f_2(\varphi(x_1), \cdots, \varphi(x_n))$$
$$\beta_2^*(\varphi(z)) = \beta_2^*(f_2(\varphi(x_1), \cdots, \varphi(x_n)))$$

$$= f_2 / \beta_2^*\left(\beta_2^*(\varphi(x_1)),\cdots,\beta_2^*(\varphi(x_n))\right)$$
$$= f_2 / \beta_2^*(\omega_{P_2}.) = \omega_{P_2}.$$

Therefore, $f_1 / \beta_1^*\left(\beta_1^*(x_1),\cdots,\beta_1^*(x_n)\right) \in \overline{\ker\varphi}$.
Let $\beta_1^*(b_1),\cdots,\beta_1^*(b_{i-1}),\beta_1^*(b_{i+1}),\cdots,\beta_1^*(b_n),\beta_1^*(a) \in \overline{\ker\varphi}$, then $b_1,\cdots,b_{i-1},b_{i+1},\cdots,b_n,a \in P_1$ and there exists $x \in P_1$ such that $a \in f_1\left(b_1^{i-1},x,b_{i+1}^n\right)$. Thus,

$$\beta_1^*(a) = \beta_1^*\left(f_1\left(b_1^{i-1},x,b_{i+1}^n\right)\right) = f_1 / \beta_1^*\left(\beta_1^*(b_1),\cdots,\beta_1^*(b_{i-1}),\beta_1^*(x),\beta_1^*(b_{i+1}),\cdots,\beta_1^*(b_n)\right).$$

Also, $\varphi(a) \in \varphi\left(f_1\left(b_1^{i-1},x,b_{i+1}^n\right)\right) = f_2(\varphi(b_1),\cdots,\varphi(b_{i-1}),\varphi(x),\varphi(b_{i+1}),\cdots,\varphi(b_n))$, and

$$\omega_{P_2} = \beta_2^*(\varphi(a)) = \beta_2^*(f_2(\varphi(b_1),\cdots,\varphi(b_{i-1}),\varphi(x),\varphi(b_{i+1}),\cdots,\varphi(b_n)))$$
$$= f_2 / \beta_2^*\left(\beta_2^*(\varphi(b_1)),\cdots,\beta_2^*(\varphi(b_{i-1})),\beta_2^*(\varphi(x)),\beta_2^*(\varphi(b_{i+1})),\cdots,\beta_2^*(\varphi(b_n))\right)$$
$$= f_2 / \beta_2^*(\underbrace{\omega_{P_2},\cdots,\omega_{P_2}}_{i-1},\beta_2^*(\varphi(x)),\omega_{P_2},\cdots,\omega_{P_2}) = \beta_2^*(\varphi(x)).$$

Therefore, $\beta_2^*(x) \in \overline{ker\,\varphi}$. Let $\beta_1^*(a) \in P_1 / \beta_1^*$ and $\beta_1^*(x) \in \overline{ker\,\varphi}$, then

$$f_1 / \beta_1^*\left(\beta_1^*(a),\beta_1^*(x),\beta_1^*(a)^{-1},\omega_{P_1}.\right) = \beta_1^*(f(z)), \text{ for some } z \in f_1\left(a,x,a^{-1},e_1.\right).$$

On the other hand,

$$\beta_2^*(\varphi(z)) = \beta_2^*\left(\varphi\left(f_1\left(a,x,a^{-1},e_1.\right)\right)\right)$$
$$= \beta_2^*\left(f_2\left(\varphi(a),\varphi(x),\varphi(a)^{-1},e_2.\right)\right)$$
$$= f_2 / \beta_2^*\left(\beta_2^*(\varphi(a)),\beta_2^*(\varphi(x)),\beta_2^*(\varphi(a))^{-1},\omega_{P_2}.\right)$$
$$= f_2 / \beta_2^*\left(\beta_2^*(\varphi(a)),\omega_{P_2},\beta_2^*(\varphi(a))^{-1},\omega_{P_2}.\right) = \omega_{P_2}.$$

Therefore, $\overline{ker\,\varphi}$ is an *n*-ary normal subgroup.

Theorem 4.4. Let $\langle P,f,e,^{-1}\rangle$ be an *n*-ary polygroup and *M*, *N* two *n*-ary normal subpolygroups of *P* with $N \subseteq M$, then
(*i*) $f\left(N,x_2^n\right) = f\left(N,y_2^n\right)$ implies $f\left(M,x_2^n\right) = f\left(M,y_2^n\right)$, so if in $P^{(n-1)}/N$, $N[x,e.] = N[y,e.]$ then $M[x,e.] = M[y,e.]$ in $P^{(n-1)}/M$,
(*ii*) if $f\left(N,x_2^n\right) \subseteq f_{(k)}\left(f\left(N,t_{12}^n\right),\cdots,f\left(N,t_{(k(n-1)+1)2}^{\;n}\right)\right)$, then

$$f\left(M,x_2^n\right) \subseteq f_{(k)}\left(f\left(M,t_{12}^n\right),\cdots,f\left(M,t_{(k(n-1)+1)2}^{\;n}\right)\right),$$

so if $N[x,e.] \in F_{(k)}\left(N[t_1,e.],\cdots,N\left[t_{k(n-1)+1},e.\right]\right)$, for some $k \in N_0$, $t_s \in P$, $1 \le s \le k(n-1)+1$, then $M[x,e.] \in F_{(k)}\left(M[t_1,e.],\cdots,M\left[t_{k(n-1)+1},e.\right]\right)$.
(*iii*) if $\varphi : P^{(n-1)}/M \longrightarrow P^{(n-1)}/N$ is canonical map and β_M^*, β_N^* are the fundamental equivalence relations of $P^{(n-1)}/M$, $P^{(n-1)}/N$, respectively, then we have

$$\left(\left(P^{(n-1)}/N\right)/\beta_N^*\right)/\overline{ker\,\varphi} \cong \left(P^{(n-1)}/M\right)/\beta_M^*.$$

Proof: (*i*)

$$f\left(N,x_2^n\right) = f\left(N,y_2^n\right) \;\Rightarrow\; f\left(N,x_2^n\right) \subseteq f\left(N,y_2^n\right)$$

$$\Rightarrow \quad f\left(N, x_2^n\right) \subseteq f\left(M, y_2^n\right), \text{ since } N \subseteq M$$
$$\Rightarrow \quad f\left(N, x_2^n\right)^{-1} \subseteq f\left(M, y_2^n\right)^{-1} = f\left(M, y_n^{-1}, \cdots, y_2^{-1}\right), \text{ by Lemma 3.5}$$
$$\Rightarrow \quad M \subseteq f\left(f\left(N, x_2^n\right)^{-1}, y_2^n\right), \text{ by Definition 2.1 and Lemma 2.3}$$
$$= f\left(x_n^{-1}, \cdots, x_2^{-1}, f\left(N, y_2^n\right)\right)$$
$$\subseteq f\left(x_n^{-1}, \cdots, x_2^{-1}, f\left(M, y_2^n\right)\right)$$
$$\Rightarrow \quad f\left(M, y_2^n\right) \subseteq f\left(M, x_2^n\right), \text{ by Definition 2.1 and Lemma 2.3.}$$

Similarly, we can get $f\left(M, x_2^n\right) \subseteq f\left(M, y_2^n\right)$. Therefore, $f\left(M, x_2^n\right) = f\left(M, y_2^n\right)$.

(*ii*) If $f\left(N, x_2^n\right) \subseteq f_{(k)}\left(f\left(N, t_{12}^{\ n}\right), \cdots, f\left(N, t_{k(n-1)+12}^{\ \ n}\right)\right) = f\left(N, t_{12}^{n-1}, f_{(\cdot)}\left(t_{1n}, t_{22}^{\ n}, \cdots, t_{k(n-1)+12}^{\ \ n}\right)\right)$. Then, $f\left(N, x_2^n\right) = f\left(N, t_{12}^{n-1}, t\right)$, for some $t \in f_{(\cdot)}\left(t_{1n}, t_{22}^{\ n}, \cdots, t_{k(n-1)+12}^{\ \ n}\right)$. So by (*i*),

$$f\left(M, x_2^n\right) = f\left(M, t_{12}^n, t\right), \text{ for } t \in f_{(\cdot)}\left(t_{1n}, t_{22}^{\ n}, \cdots, t_{k(n-1)+12}^{\ \ n}\right).$$

Therefore, $f\left(M, x_2^n\right) \subseteq f\left(M, t_{12}^{n-1}, f_{(\cdot)}\left(t_{1n}, t_{22}^{\ n}, \cdots, t_{k(n-1)+12}^{\ \ n}\right)\right) = f_{(k)}\left(f\left(M, t_{12}^{\ n}\right), \cdots, f\left(M, t_{k(n-1)+12}^{\ \ n}\right)\right)$.

(*iii*) We define the map $\psi : \left(P^{(n-1)} / N\right) / \beta_N^* \longrightarrow \left(P^{(n-1)} / M\right) / \beta_M^*$ by

$$\psi\left(\beta_N^*\left(N[x, e.]\right)\right) = \beta_M^*\left(M[x, e.]\right), \text{ for all } N[x, e.] \in P^{(n-1)} / N.$$

Suppose $N[x, e.], N[y, e.] \in P^{(n-1)} / N$ and $\beta_N^*\left(N[x, e.]\right) = \beta_N^*\left(N[y, e.]\right)$, then there exist $N[x_1, e.], \cdots, N[x_n, e.] \in P^{(n-1)} / N$, $u_1, \cdots, u_m \in \mathcal{U}_{P^{(n-1)}/N}$, such that

$$N[x, e.] = N[x_1, e.],\ N[y, e.] = N[x_n, e.], \tag{4}$$

$$\left\{N[x_i, e.], N[x_{i+1}, e.]\right\} \subseteq u_i \text{ for } i = 1, \cdots, m. \tag{5}$$

We know for each $i = 1, \cdots, m$

$$u_i = F_{(k)}\left(N[t_1, e.], \cdots, N\left[t_{k(n-1)+1}, e.\right]\right), \text{ for some } k \in N_0,\ t_s \in P,\ 1 \le s \le k(n-1)+1.$$

Corresponding to every u_i, $i = 1, \cdots, m$, we set $v_i = F_{(k)}\left(M[t_1, e.], \cdots, M\left[t_{k(n-1)+1}, e.\right]\right) \in \mathcal{U}_{P^{(n-1)}/M}$.
Hence, by (*ii*), equation (5) implies

$$\left\{M[x_i, e.], M[x_{i+1}, e.]\right\} \subseteq v_i, \text{ for } i = 1, \cdots, m. \tag{6}$$

By (*i*) and (4) we have $M[x, e.] = M[x_1, e.],\ M[y, e.] = M[x_{m+1}, e.]$. (*7*)
Therefore, by (*6*) and (*7*), $\beta_M^*\left(M[x, e.]\right) = \beta_M^*\left(M[y, e.]\right)$.

Now, we show that ψ is a homomorphism of *n*-ary groups:

$$\begin{aligned}
&\psi\left(F / \beta_N^*\left(\beta_N^*\left(N[x_1, e.]\right), \cdots, \beta_N^*\left(N[x_n, e.]\right)\right)\right) \\
&= \psi\left(\beta_N^*\left(F\left(N[x_1, e.], \cdots, N[x_n, e.]\right)\right)\right) \\
&= \left\{\psi\left(\beta_N^*\left(N[t, e.]\right)\right) \mid t \in f\left(x_1^n\right)\right\} \\
&= \left\{\beta_M^*\left(M[t, e.]\right) \mid t \in f\left(x_1^n\right)\right\} \\
&= \beta_M^*\left(F\left(M[x_1, e.], \cdots, M[x_n, e.]\right)\right) \\
&= F / \beta_M^*\left(\beta_M^*\left(M[x_1, e.]\right), \cdots, \beta_M^*\left(M[x_n, e.]\right)\right) \\
&= F / \beta_M^*\left(\psi\left(\beta_N^*\left(N[x_1, e.]\right)\right), \cdots, \psi\left(\beta_N^*\left(N[x_n, e.]\right)\right)\right).
\end{aligned}$$

Also, $\psi\left(\omega_{P^{(n-1)}/N}\right) = \psi\left(\beta_N^*\left(N[e.]\right)\right) = \beta_M^*\left(M[e.]\right) = \omega_{P^{(n-1)}/M}$. And

$$\begin{aligned} ker\,\psi &= \{\beta_N^*(N[x,e.]) \mid \psi(\beta_N^*(N[x,e.])) = \omega_{P^{(n-1)}/M}\} \\ &= \{\beta_N^*(N[x,e.]) \mid \beta_N^*(\varphi(N[x,e.])) = \omega_{P^{(n-1)}/M}\} \\ &= \overline{ker\,\varphi}\,. \end{aligned}$$

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