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LIGHTLIKE RULED AND REVOLUTION SURFACES IN $\mathbb{R}^3_1$*

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Abstract – In this paper lightlike ruled surfaces in $\mathbb{R}^3_1=(\mathbb{R}^3,-dx^2+dy^2+dz^2)$ are studied with respect to whether ruling curves are spacelike or null. It is seen that, in the first case the Gaussian curvature of the ruled surfaces vanishes. In the second case the Gaussian curvature of the ruled surfaces are negative. In the second case lightlike ruled surfaces are totally umbilical. Furthermore, lightlike surfaces of revolution are shown to be only cones, and the second type lightlike ruled surface.

Keywords – Lightlike surface, lightlike ruled surface, lightlike revolution surface

1. INTRODUCTION

From the point of view of physics, lightlike surfaces are of importance because they are models of different types of horizons (event horizons, Cauchy's horizons, Kruskal horizons) studied in relativity theory [1-3]. Lightlike hypersurfaces are also studied in the theory of electromagnetism ([4] chapter 8). In [5], Kılıc and Karadag study the geometric properties of two classes of ruled surfaces (M, g_{ab}) in $\mathbb{R}^4_1$. They classify the surface M as type I and type II according to $g_{11}=0$ or $g_{22}=0$ respectively. Then they prove that M of type I (respectively M of type II when it is cylindrical ruled surface) admit an induced metric connection. In this paper we use the classical notation of surface theory; for this purpose we can give [6] as a general reference. Let $\varphi(s, t)$ be a local parameterization of surface M in Minkowski space $\mathbb{R}^3_1$. For the space M to be lightlike we showed that one of the φ_s , φ_t has to be spacelike vector and the other is null vector. The general theory of lightlike submanifolds [4] uses a non degenerate screen distribution which (due to the degenerate induced metric) is not unique. Therefore, the induced objects of submanifolds depend upon the choice of a screen. We choose screen distribution of M as $Sp\{\varphi_s\}$ or as $Sp\{\varphi_t\}$. Noting that a ruled surface in $\mathbb{R}^3_1$ can be expressed as $\varphi(s, t)=\alpha(s)+t\beta(s)$ in terms of a directrix curve α and a vector field β pointing along the ruling. We then considered two cases; one of which, φ_s, is a null vector φ_t is a spacelike vector, and in the second case φ_t is a null vector and φ_s is a spacelike vector. In the first case we proved that the Gaussian curvature of M vanishes, in the second case we showed that the Gaussian curvature of M is negative and lightlike ruled surfaces are totally umbilical.

Let M be a hypersurface of a (m+2)-dimensional semi-Riemannian manifold ($\overline{M}, \overline{g}$) of index $q\in\{1,2,...,m+1\}$, m>0. For $u\in M$ $T_u M$ is a hyperplane of the semi-Euclidean space ($T_u \overline{M}$, $\overline{g}_u$). $T_u M^{\perp}$ and $RadT_uM$ are defined as follows

$$T_uM^{\perp}=\{V_u\in T_u\overline{M}:\overline{g}_u(V_u,W_u)=0,\ \forall W_u \in T_u M\} \text{ and } RadT_uM=T_uM \cap T_uM^{\perp}\}$$

We say that M is a lightlike (null, degenerate) hypersurface of $\overline{M}$ if $RadT_uM\neq \{0\}$ at every

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u∈M. The semi-Riemannian metric $\bar{g}$ on $\bar{M}$ induces on M a symmetric tensor field g of type (0, 2).

Proposition 1. Let (M, g) be a hypersurface of a (m+2)-dimensional semi-Riemannian manifold ($\bar{M}, \bar{g}$). M is a lightlike hypersurface of $\bar{M}$ if and only if

$$RadTM=TM^{\perp}\neq\{0\}.$$

The complementary vector bundle $S(TM)$ of $RadTM$ in TM is called screen distribution on M. i.e.,

$$TM=RadTM\perp S(TM) \quad (1)$$

$S(TM)$ is a non-degenerate distribution, and the rank of $RadTM$ is 1. Note that $S(TM)$ is not unique [4].

Theorem 1. Let (M, g, $S(TM)$) be a lightlike hypersurface of a semi-Riemannian manifold ($\bar{M}, \bar{g}$). Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $U\subset M$, there exists a unique lightlike section N of $tr(TM)$ on U satisfying

$$\bar{g}(N, \xi)=1 \text{ and } N\perp S(TM) \quad (2)$$

It follows from (2) that $tr(TM)$ is a lightlike vector bundle such that

$$tr(TM)_u\cap T_uM=\{0\} \text{ for any } u\in M.$$

Moreover, we have the following decompositions of $T\bar{M}|_M$

$$T\bar{M}|_M=S(TM)\perp(TM^{\perp}\oplus tr(TM))=TM\oplus tr(TM). \quad (3)$$

We called $tr(TM)$ in theorem 1, the lightlike transversal vector bundle of M with respect to $S(TM)$.

Let (M, g) be a lightlike hypersurface of a (m+2)-dimensional semi-Riemannian manifold ($\bar{M}, \bar{g}$) and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, with respect to $\bar{g}$. Because of the decomposition in (3), we obtain

$$\bar{\nabla}_X Y=\nabla_X Y+h(X, Y) \text{ for any } X,Y\in \Gamma(TM) \quad (4)$$

where $\nabla_X Y$ belong to $\Gamma(TM)$ and h (X, Y) belong to Γ(tr(TM)). It can readily be seen that ∇ is a torsion-free linear connection on M, and h is a Γ(tr(TM))-valued symmetric $F(M)$-bilinear form on $\Gamma(TM)$. Let {ξ, N} be a pair of section on $U\subset M$ in theorem 1. We may define a symmetric $F(M)$-bilinear form B given by

$$B(X, Y)=\bar{g}(h(X, Y), \xi), \forall X, Y\in \Gamma(TM|_U). \quad (5)$$

For u ∈ M, if we choose a null plane H of T_uM directed by $\xi_u\in T_u M^{\perp}$, we define the null sectional curvature of H with respect ξ_u and ∇ as

$$K_{\xi_u}(H)=\frac{g_u(R(W_u, \xi_u)\xi_u, W_u)}{g_u(W_u, W_u)} \quad (6)$$

where W_u is an arbitrary non-null vector in H. We define the curvature tensor field R of type (1, 3) given by

$$R(X, Y)=\nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X, Y]}Z \text{ for any } X, Y, Z\in \Gamma(TM) \quad (7)$$

here [,] is the bracket product [4].

Definition 1. Let $v=(v_1,v_2,v_3)$ and $w=(w_1,w_2,w_3)$ be two vectors in $\mathbb{R}^3_1$. The cross product of these vectors is [7].

$$vxw=(v_2w_3-v_3w_2,\ v_1w_3-v_3w_1,\ v_2w_1-v_1w_2)$$

Proposition 2. The following facts for the standard cross product in $\mathbb{R}^3_1$ hold.
i) $\bar{g}$(uxv, u)=0 and $\bar{g}$(uxv, v)=0.
ii) $\bar{g}$(uxv, uxv)=-$\bar{g}$(u, u)$\bar{g}$(v, v) +($\bar{g}$(u, v))^2 ,
iii) Let u be a spacelike vector, v be a null vector. $\bar{g}$(u, v)≠0 if and only if uxv is spacelike, $\bar{g}$(u, v)=0 if and only if uxv is null [8].
iv) If u and v are null vectors then uxv is a spacelike vector.
v) If u is a timelike vector, v is a null vector, then uxv is a spacelike vector.

Proposition 3. Let u and v be two null vectors in $\mathbb{R}^3_1$. Then $\bar{g}$(u, v)=0 if and only if {u, v} is linearly dependent.

2. LIGHTLIKE SURFACES

We use the classical notation of surface theory; for this purpose we can give [6] as a general reference. Let φ(s, t) be a local parameterization of surface *M* in Minkowski space $\mathbb{R}^3_1$. Then the coefficients *E, F, G* of the first fundamental form of this surface are given by

$$E=\bar{g}(\varphi_s, \varphi_s),\ F=\bar{g}(\varphi_s, \varphi_t),\ G=\bar{g}(\varphi_t, \varphi_t). \tag{8}$$

We know that *TM*=Sp(φ_s, φ_t) and {φ_s, φ_t} are linearly independent. From Proposition 2 we have

$$\bar{g}(\varphi_s x\varphi_t, \varphi_s x\varphi_t)=-\bar{g}(\varphi_s, \varphi_s)\bar{g}(\varphi_t, \varphi_t)+\bar{g}(\varphi_s, \varphi_t)^2=-EG+F^2.$$

In order for the surface to be lightlike it is necessary that φ_sxφ_t be a null vector. This means that -$EG+F^2$=0. Since φ_sxφ_t is perpendicular to both φ_s and φ_t, we have φ_sxφ_t∈$TM^\perp$. Note that $TM^\perp$=*RadTM* and the rank of *RadTM* is 1. Hence *RadTM*=Sp{ φ_sxφ_t}. Since φ_sxφ_t is null, from proposition 2, then $\bar{g}$(φ_s, φ_t)=0 and one element of the set {φ_s, φ_t} is null and the other is spacelike.
(i) Suppose that φ_s is a null vector. From proposition 3, φ_s and φ_sxφ_t are linearly dependent. Therefore, *RadTM*=Sp{φ_s} and *TM*=Sp{φ_s}⊥Sp{φ_t}. Thus we may choose *S*(*TM*)=Sp{φ_t} as a screen distribution of *M*.
(ii) If φ_t is a null vector then for the same reasons *RadTM*=Sp{φ_t} and *S*(*TM*)=Sp{φ_s}.

3. LIGHTLIKE RULED SURFACES

M stands for the ruled surface in $\mathbb{R}^3_1$. A ruled surface in $\mathbb{R}^3_1$ can be expressed as φ(s, t)=α(s)+tβ(s) in terms of a directrix curve α and a vector field β pointing along the ruling. According to Section 2, one element of the set {φ_s, φ_t} has to be null and the other spacelike.
If φ_s is a null vector and φ_t is a spacelike vector or vice versa then

$$\bar{g}(\varphi_s, \varphi_t)=0 \text{ implies that } \bar{g}(\alpha', \beta)=0\ . \tag{9}$$

Case 1. Let φ_s be a null vector and φ_t be a spacelike vector. We may take φ_t as a unit vector. Hence $\bar{g}$(φ_t, φ_t)=$\bar{g}$(β, β)=1. From Section 2 we have

$$RadTM=\text{Sp}\{\varphi_s\},\ S(TM)=\text{Sp}\{\varphi_t\}=\text{Sp}\{\beta\}.$$

Since α' ∈ *Γ*(*TM*) then α'=aφ_s+bφ_t, where a and b are real numbers. From (9) we get, $\bar{g}$(α', β)=b=0 i.e,

$$\alpha'=a\varphi_s=a(\alpha'+t\beta') \tag{10}$$

Here, either {α', β'} is linearly dependent or β'=0.
In the case of β'=0, we have β=constant, and then the surface is a cylinder.

In the case $\{\alpha', \beta'\}$ is linearly dependent: From (10) we may take $RadTM$=Sp$\{\alpha'\}$.
From theorem 1, there exists a basis $\{\alpha', N, \beta\}$ of $T\,\mathbb{R}^3_1|_M$ such that

$$\bar{g}(\alpha', \alpha')=\bar{g}(N, N)=\bar{g}(N, X)=0 \text{ for all } X\in \Gamma(S(TM)) \text{ and } \bar{g}(\alpha', N)=1$$

From (6), the Gaussian curvature of M with respect to α' and ∇ is

$$K_{\alpha'} = \frac{g(R(\beta,\ \alpha')\,\alpha',\ \beta\,)}{g(\beta,\ \beta)}\,. \tag{11}$$

(7) leads to

$$R(\beta, \alpha')\alpha'=\nabla_\beta(\nabla_{\alpha'}\alpha')-\nabla_{\alpha'}(\nabla_\beta\,\alpha')-\nabla_{[\beta,\ \alpha']}\,\alpha'. \tag{12}$$

Readily we obtain that

$$\nabla_{\alpha'}\,\alpha'=\Gamma^1_{11}\alpha',$$

$$\nabla_\beta\,\alpha'=\Gamma^1_{12}\alpha'+\Gamma^2_{12}\beta,$$

$$\nabla_{\alpha'}\beta=\Gamma^1_{21}\alpha'+\Gamma^2_{21}\beta,$$

$$\nabla_\beta(\nabla_{\alpha'}\,\alpha')=\Gamma^1_{11}(\Gamma^1_{12}\alpha'+\Gamma^2_{12}\beta),$$

$$\nabla_{\alpha'}(\nabla_\beta\,\alpha')=(\Gamma^1_{12}\Gamma^1_{11}+\Gamma^2_{12}\Gamma^1_{12})\alpha'+(\Gamma^2_{12})^2$$

$$\nabla_{[\beta,\ \alpha']}\alpha'=\frac{\partial s}{\partial\alpha'}\nabla_{\alpha'}\,\alpha'=\frac{\partial s}{\partial\alpha'}\Gamma^1_{11}\alpha'$$

where $\{\Gamma^1_{11}, \Gamma^1_{12}, \Gamma^2_{12}, \Gamma^1_{21}, \Gamma^2_{21}\}$ are the coefficients of the induced linear connection ∇ on M with respect to the frame field $\{\alpha', \beta\}$; Here $\Gamma^2_{12}=\Gamma^2_{21}$, $\Gamma^1_{12}=\Gamma^1_{21}+\frac{\partial s}{\partial\alpha'}$, [4]. Inserting these values in (11) we get

$$K_{\alpha'}=\Gamma^2_{12}(-\Gamma^2_{12}+\Gamma^1_{11}). \tag{13}$$

Note that $\bar{\nabla}_{\alpha'}\beta=\beta'$. Since $\{\alpha', \beta'\}$ is linearly dependent (or β'=0), we have $\bar{\nabla}_{\alpha'}\beta=\beta'=\lambda\alpha'$, $\lambda\in\mathbb{R}$. On the other hand, $\bar{\nabla}_{\alpha'}\beta=\nabla_{\alpha'}\beta+h(\alpha', \beta)=\lambda\alpha'$, i.e. $h(\alpha', \beta)=0$ and since $\nabla_{\alpha'}\beta=\Gamma^1_{21}\alpha'+\Gamma^2_{21}\beta=\lambda\alpha'$ we have $\Gamma^2_{21}=0$ and $\Gamma^1_{21}=\lambda$. Plugging this value in (13) we obtain $K_{\alpha'}=0$.

Case 2. Let $\varphi_t=\beta$ is a null vector, φ_s is a spacelike vector. In this case M is called a second type lightlike surface. From Section 2, we have $RadTM$=Sp$\{\beta\}$, $S(TM)$=Sp$\{\varphi_s\}$. We know that TM=Sp$\{\beta, \varphi_s\}$. Since $\alpha'\in\Gamma(TM)$ we write $\alpha'=c_1\beta+c_2\varphi_s$, c_1, $c_2\in\mathbb{R}$. Note that $g(\alpha', \alpha')=c^2g(\varphi_s, \varphi_s)$. Since φ_s is a spacelike vector, then α' is a spacelike vector. Hence we may choose $S(TM)$=Sp$\{\alpha'\}$ as a screen bundle. We may take α' as a unit vector. Therefore there exist an N such that $\bar{g}(\beta, N)=1$, $\bar{g}(N, N)=\bar{g}(N, X)=0$ for all $X\in\Gamma(S(TM))$ and $\{\beta, N, \alpha'\}$ is a basis for $\mathbb{R}^3_1|_M$. The Gaussian curvature of M with respect to β and ∇ is

$$K_\beta = \frac{g(R(\alpha',\ \beta)\,\beta,\ \alpha')}{g(\alpha',\ \alpha')}\,. \tag{14}$$

Since β is a geodesic curve, $\bar{\nabla}_\beta\beta=\nabla_\beta\beta+h(\beta, \beta)=0$. Thus

$$\nabla_\beta\beta=0 \text{ and } h(\beta, \beta)=0. \tag{15}$$

We easily obtain the following results

$$\nabla_{\alpha'}\beta=\Gamma^2_{21}\beta+\Gamma^1_{21}\alpha',\ \nabla_\beta\,\alpha'=\Gamma^2_{12}\beta+\Gamma^1_{12}\alpha',\ \nabla_\beta(\nabla_{\alpha'}\beta)=\Gamma^1_{21}\Gamma^2_{12}\beta+(\Gamma^1_{21})^2\alpha'. \tag{16}$$

Inserting this equality in (7) we get

$$R(\alpha', \beta)\beta=\nabla_{\alpha'}(\nabla_\beta \beta)-\nabla_\beta(\nabla_{\alpha'} \beta)-\nabla_{[\alpha', \beta]}\beta$$

and

$$R(\alpha', \beta)\beta=-\Gamma^1_{21}\Gamma^2_{12}\beta-(\Gamma^1_{21})^2\alpha'. \quad (17)$$

Inserting (17) into (14) we obtain $K_\beta=-(\Gamma^1_{21})^2$

Now let us calculate the second fundamental form of second type lightlike ruled surfaces.

From (5) for the pair $\{\beta, N\}$ we have

$$B(X, Y)=\bar{g}(h(X,Y), \beta), \forall X, Y \in \Gamma(TM).$$

Since X, Y$\in \Gamma(TM)$, then $X=\lambda_1 \beta+\lambda_2\alpha'$, $Y=\mu_1 \beta+\mu_2\alpha'$, where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. Hence

$$B(X, Y)=\lambda_1\mu_1\bar{g}(h(\beta, \beta), \beta)+\lambda_2\mu_1\bar{g}(h(\alpha', \beta), \beta)+\lambda_1\mu_2\bar{g}(h(\beta, \alpha'), \beta)+\lambda_2\mu_2\bar{g}(h(\alpha', \alpha'), \beta) \quad (18)$$

On the other hand, we have

$$\bar{\nabla}_{\alpha'}\beta=\nabla_{\alpha'}\beta+h(\alpha', \beta),\ \bar{\nabla}_{\alpha'}\beta=\beta' \text{ and } \bar{g}(\beta', \beta)=0$$

Using (16) we get

$$\bar{g}(h(\alpha', \beta), \beta)=\bar{g}(h(\beta, \alpha'), \beta)=0 \quad (19)$$

Setting (15) and (19) in (18) we get,

$$B(X, Y)=\lambda_2\mu_2\,\bar{g}(h(\alpha', \alpha'), \beta) \quad (20)$$

Note that

$$\bar{g}(X, Y)=\lambda_2\mu_2 \quad (21)$$

Note also that $\rho=\bar{g}(h(\alpha', \alpha'), \beta)$ is a smooth function of M. From (20) and (21) we have

$$B(X, Y)=\rho\bar{g}(X,Y), \text{ forall } X, Y\in \Gamma(TM).$$

This shows that second type lightlike ruled surfaces are totally umbilical.

4. SURFACES OF REVOLUTION

Let the profile curve of the revolution surface M be γ.

1) We can suppose that γ is represented by

$$\gamma(u)=(f(u), 0, g(u)), u \in I$$

Where f and g are real functions on the open interval I.

(i) We suppose that the revolution axis of M is the x–axis of our coordinate system. A parameterization of the surface is given by

$$\varphi(s, u)=(f(u), g(u)\sin s, g(u)\cos s), 0\leq s<2\pi, u \in I$$

Hence

$$\bar{g}(\varphi_s, \varphi_s)=g^2, \bar{g}(\varphi_s, \varphi_u)=0, \bar{g}(\varphi_u, \varphi_u)=-(f')^2+(g')^2$$

Since M is lightlike and since φ_s is a spacelike vector, then, From Section 2, φ_u is a null vector. Since φ_u is a null vector then $(f')^2=(g')^2$ implies that

$$g=\pm f+c,$$

where c=constant. Hence γ is a straight line in the xoz plane. Therefore, M is a cone.
If c=0, then M is a null cone.
If c≠0, then

$$\varphi(s, u)=(0, c\sin s, c\cos s)+f(u)\ (1, \sin s, \cos s)$$

Let f(I)=J. Hence we have

$$\varphi(s, t)=(0, c\sin s, c\cos s)+t(1, \sin s \cos s),\ t \in J$$

This is a second type lightlike ruled surface with

$$\alpha(s)=(0, c\sin s, c\cos s),\ \beta(s)=(1, \sin s, \cos s).$$

(*ii*) We suppose the revolution axis of M is the z–axis of our coordinate system. A parameterization of surface is given by,

$$\varphi(s, u)=(f(u) \cosh s, f(u) \sinh s, g(u))$$

After similar calculations to that of (i) we obtain a second type lightlike ruled surface with

$$\alpha(s)=(c\cosh s, c\sinh s, 0),\ \beta(s)=(\cosh s, \sinh s, 1).$$

2) We may suppose that is γ represented by $\gamma(u)=(f(u), g(u), 0),\ u \in I$.
Similar to the previous examinations. Supposing the revolution axis of M is the x–axis or y–axis of our coordinate system we obtain a second type lightlike ruled surface.
3) Let $\gamma\ (u)=(0, f(u), g(u))$. Suppose the axis of revolution of M is the y–axis of our coordinate system. Hence we have

$$\varphi(s,u)=(g(u)\sinh s, f(u), g(u)\cosh s)$$

Therefore,

$$\bar{g}(\varphi_s, \varphi_s)=-g^2\cosh^2 s+g^2\sinh^2 s=-g^2,\ \bar{g}(\varphi_s, \varphi_u)=0,\ \bar{g}(\varphi_u, \varphi_u)=(f')^2+(g')^2>0,$$

From Section 2 there is no lightlike surface.

If the revolution axis of M is the z–axis of our coordinate system, then there exists no lightlike revolution surface.

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