

TWO-DIMENSIONAL MAGMA FLOW*

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Abstract – Exact solution for steady two-dimensional flow of an incompressible magma is obtained. The magmatic flow is studied by considering the magma as a second grade fluid. The governing partial differential equations are transformed to ordinary differential equations by symmetry transformations. Results are discussed through graphs to understand the rheology of the flowing magma.

Keywords – Magma flow, second-grade fluid, symmetry analysis, exact solution, controlling eruption

1. INTRODUCTION

The study of dynamics of magma fluids is an important area of research in the field of geochemistry. This is due its unique participation in the most important processes responsible for controlling the geophysical evolution of the earth. In geochemistry magmatic fluids have an important role in carrying metals from a source region to ultimate deposition and preservation in an ore deposit. The ensolution of magmatic fluids from granitic magmas at their rise to the surface and crystallization is one of the most important factors in lithospheric rocks alteration and generation of metalliferous hydrothermal systems genetically related to granitic magmas.

The dynamics of the volcanic eruption from a volcanic source is directly related to the rheological behavior of the magmatic fluid [1]. It is therefore, necessary to have a better understanding of the rheology of such materials in order to assess the volcanic hazard. Magma fluids are usually composed of different liquids, metals and gasses such as silicate liquids, Fe, Cu, Pb, etc. [2]. Due to the presence of different species, such a magmatic mixture (magma fluid) may not be completely described by the Navier-Stokes theory. It is well known that the magmatic fluids do not obey the Newton's law of viscosity and hence fall in the category of non-Newtonian fluids [3]-[6]. Unlike Newtonian fluids there is not a single flow model which can completely describe the non-Newtonian fluids. Therefore, many empirical and semi empirical non-Newtonian fluid models have been proposed. The earliest class of non-Newtonian models is usually known as Rivlin-Ericksen fluids (fluids of differential type) [7]. These fluid models are basically called the order fluid models; second-grade, third-grade and fourth-grade models fall into this category. In such fluid models, the infinitesimal part of the history of the deformation gradient has an influence on the stress, and because of the fading memory the stress becomes a pure pressure [8]. Amongst all these models, the second-grade model has received great attention from researchers to understand the rheology of the non-Newtonian fluids. We therefore approximate our magmatic fluid by the second grade model in this study.

The constitutive assumption for the fluid of the second-grade is usually written in the following form:

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$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where \mathbf{T} is the Cauchy stress tensor, $p\mathbf{I}$ is the spherical stress due to the constraint of incompressibility, μ , α_1 and α_2 are the material moduli, and \mathbf{A}_1 and \mathbf{A}_2 are the first two Rivlin-Ericksen tensors [9] defined by:

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T,$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_1.$$

The Claussius -Duhem inequality and the assumption that the Helmholtz free energy is minimum at equilibrium provide the following restrictions [10]:

$$\mu \geq 0, \alpha_1 \geq 0, \text{ and } \alpha_1 + \alpha_2 = 0. \tag{2}$$

A comprehensive discussion on the restrictions on μ , α_1 and α_2 can be found in the work by Dunn and Rajagopal [8].

2. FLOW ANALYSIS

a) Problem formulation

We consider a steady two-dimensional flow of an incompressible magma fluid by assuming the second-grade fluid as magma and a plate over it. The plate is located at $y=0$ and positive y goes deep into the fluid beneath the plate. The flow is generated due to the uniform motion of the plate, and far from the plate the longitudinal component of velocity is zero. The plate is assumed to be uniformly porous subjected to constant suction in order to allow the rising magma to pass through the controlling surface. Deep from the plate the magma has a constant pressure. The governing equations for an incompressible second-grade fluid are the law of conservation of mass and the law of conservation of momentum:

$$\text{div}\mathbf{V} = 0, \tag{3}$$

and

$$\rho \frac{d\mathbf{V}}{dt} = \text{div}\mathbf{T}, \tag{4}$$

in the absence of body forces, where \mathbf{V} is the velocity vector, ρ is density of the fluid, and \mathbf{T} is the Cauchy stress tensor given in (1). Due to our flow assumptions the velocity vector \mathbf{V} must be of the form $\mathbf{V}=[u(x,y),v(x,y),0]$. Therefore, equations (3) and (4) in nondimensional form are given by:

$$u_x + v_y = 0, \tag{5}$$

$$\text{Re}(uu_x + vu_y) = -p_x + (u_{xx} + u_{yy}) + \delta \begin{pmatrix} 5u_xu_{xx} + u_xu_{yy} + uu_{xxx} \\ +vu_{yyy} + uu_{yyx} + 2v_xv_{xx} \\ +u_yu_{yx} + u_yv_{xx} + vu_{yyx} \end{pmatrix}, \tag{6}$$

$$\text{Re}(uv_x + vu_x) = -p_y + (v_{xx} - u_{xy}) + \delta \begin{pmatrix} 5u_xu_{xy} - u_xv_{xx} - vu_{yyx} \\ +uv_{xxx} - vu_{xxx} + 2u_yu_{yy} \\ -v_xu_{xx} + v_xu_{yy} - uu_{yxx} \end{pmatrix}, \tag{7}$$

where the dimensionless quantities are defined by

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{L}, \quad u = \frac{\bar{u}}{U}, \quad v = \frac{\bar{v}}{U}, \quad p = \frac{L\bar{p}}{\mu U}, \quad Re = \frac{\rho UL}{\mu}, \quad \delta = \frac{\alpha_1 U}{L\mu}, \tag{8}$$

in which Re is the Reynolds number and δ is the dimensionless second-grade (viscoelastic) parameter. The boundary conditions for the present problem are given by

$$\begin{aligned} u(x, 0) = 1, \quad u(x, \infty) = 0, \quad \left. \frac{\partial u(x, y)}{\partial x} \right|_{x=0} = 0, \\ v(x, 0) = -V_0, \quad p(x, \infty) = p_0, \end{aligned} \tag{9}$$

where V_0 is the constant normal velocity at the plate, and p_0 is the constant pressure of the fluid far away from the plate.

b) Exact solution

We use the standard Lie symmetry approach [11-13] to reduce the governing system of partial differential equations into a system of ordinary differential equations. Assume that our governing system (5)-(7) and (9) are invariant under the infinitesimal Lie point transformations

$$\left. \begin{aligned} x^* &= x + \epsilon \xi^1(x, y, u, v, p) + O(\epsilon^2) \\ y^* &= y + \epsilon \xi^2(x, y, u, v, p) + O(\epsilon^2) \\ u^* &= u + \epsilon U(x, y, u, v, p) + O(\epsilon^2) \\ v^* &= v + \epsilon V(x, y, u, v, p) + O(\epsilon^2) \\ p^* &= p + \epsilon P(x, y, u, v, p) + O(\epsilon^2) \end{aligned} \right\}, \tag{10}$$

where ξ^1, ξ^2, U, V and P are the infinitesimals to be determined. Lie himself proposed a systematic procedure to find these infinitesimals. A lengthy but systematic algebra yields the following infinitesimals

$$\xi^1 = a, \quad \xi^2 = b, \quad U = 0, \quad V = 0 \quad \text{and} \quad P = c, \tag{11}$$

where a, b and c are arbitrary constants.

The corresponding infinitesimal generator is given by

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial p}.$$

We use the translational symmetries in the x - and y -directions to transform the governing partial differential equations into ordinary differential equations. Therefore, we have the following change of variables

$$\xi = y - mx, \quad u = f(\xi), \quad v = g(\xi) \quad \text{and} \quad p = h(\xi), \tag{12}$$

where m is an arbitrary constant. Equation (12) transforms the system (5)-(7) in the following form:

$$g' - mf' = 0, \tag{13}$$

$$Re(g - mf)f' = mh' + \delta(1 + m^2) \left[\begin{matrix} -2m(1 + m^2)f'f'' \\ -mf f''' + gf'''' \end{matrix} \right] + (1 + m^2)f'', \tag{14}$$

$$mRe(g - mf)f' = h' + \delta(1 + m^2) \left[\begin{matrix} 2(1 + m^2)f'f'' \\ -m^2 f f''' + m g f'''' \end{matrix} \right] + m(1 + m^2)f'', \tag{15}$$

where primes denote the differentiation with respect to ξ .

On integration, eq. (13) gives

$$g = mf + \alpha, \tag{16}$$

where α is an arbitrary constant of integration.

Eliminating h from eqs. (14) and (15) one can easily obtain

$$f''' + \frac{1}{\alpha\delta} f'' - \frac{Re}{\delta(1+m^2)} f' = 0, \tag{17}$$

which admits the general solution

$$f(\xi) = c_1 e^{d_1 \xi} + c_2 e^{d_2 \xi} + c_3, \tag{18}$$

containing c_1, c_2 and c_3 as constants of integration and $d_{1,2}$ are calculated as

$$d_{1,2} = -\frac{1}{2\alpha\delta} \pm \frac{1}{2} \sqrt{\frac{1}{\alpha^2\delta^2} + \frac{4Re}{\delta(1+m^2)}}. \tag{19}$$

Making use of eq. (18) in eq. (16) we get

$$g(\xi) = mc_1 e^{d_1 \xi} + mc_2 e^{d_2 \xi} + mc_3 + \alpha. \tag{20}$$

Substitution of eqs. (18) and (20) in eq. (15) gives

$$h(\xi) = -m\alpha Re [c_1 e^{d_1 \xi} + c_2 e^{d_2 \xi} + c_3] + m(1+m^2) [c_1 d_1 e^{d_1 \xi} + c_2 d_2 e^{d_2 \xi}] + \delta(1+m^2) \left[(1+m^2)(c_1 d_1 e^{d_1 \xi} + c_2 d_2 e^{d_2 \xi})^2 + m\alpha(c_1 d_1^2 e^{d_1 \xi} + c_2 d_2^2 e^{d_2 \xi}) \right] + c_4, \tag{21}$$

where c_4 is the constant of integration.

Making use of back substitution, the solution in the form of original variables reads

$$u(x, y) = c_1 e^{d_1(y-mx)} + c_2 e^{d_2(y-mx)} + c_3, \tag{22}$$

$$v(x, y) = mc_1 e^{d_1(y-mx)} + mc_2 e^{d_2(y-mx)} + mc_3 + \alpha, \tag{23}$$

$$p(x, y) = -m\alpha Re [c_1 e^{d_1(y-mx)} + c_2 e^{d_2(y-mx)} + c_3] + m(1+m^2) [c_1 d_1 e^{d_1(y-mx)} + c_2 d_2 e^{d_2(y-mx)}] + \delta(1+m^2) \left[(1+m^2)(c_1 d_1 e^{d_1(y-mx)} + c_2 d_2 e^{d_2(y-mx)})^2 + m\alpha(c_1 d_1^2 e^{d_1(y-mx)} + c_2 d_2^2 e^{d_2(y-mx)}) \right] + c_4. \tag{24}$$

The constants involved in the system (22)-(24) can be found using the boundary data (9). Therefore, the final form of the solution is given by

$$u(x, y) = e^{d_2 y}, \tag{25}$$

$$v(x, y) = -V_0, \tag{26}$$

$$p(x, y) = \delta \hat{d}_2^2 e^{2\hat{d}_2 y} + p_0, \tag{27}$$

where

$$\hat{d}_2 = \frac{1}{2V_0\delta} - \frac{1}{2} \sqrt{\frac{1}{V_0^2\delta^2} + \frac{4Re}{\delta(1+m^2)}}. \tag{28}$$

c) Discussion on results

In order to understand the physics of the considered magma flow, the graphs are plotted in figures 1-6. Figures 1 and 2 are plotted to see the effects of the material parameter δ and the Reynolds number Re on the velocity component $u(y)$ respectively. It is interesting to note that the velocity u increases by increasing δ , but decreases by increasing Re . The vorticity diffuses in the magma deposit by increasing the viscoelastic property parameter δ . Thus this is the viscoelastic nature of the ore deposits which causes them to shift from one place to the other as a consequence of drastic seismic events such as earthquake or volcanic activity. However, in the case of Reynolds number the situation is reversed. A decrease in

penetration depth with the increase in Reynolds number shows the strong dependence of vorticity diffusion on the viscosity of the magma fluid. In the presence of seepage at the bounding surface the velocity gradient increases rapidly at the porous surface, but it decreases for the cases when magma possesses large values of δ (Fig. 3). However, for large values of Re the velocity gradient increases at the porous boundary. This fact is shown in Fig. 4. Since large values of Re correspond to small viscosity, the magmas with low viscosity have the tendency to break the earth surface in the form of an outburst. In Figs. 5 and 6 the pressure is plotted for different values of δ and Re respectively. It is noted that by increasing δ the pressure at the plate decreases. From here we infer that the presence of viscoelastic property in the magma fluids helps to sustain the controlling surface against the lava from the volcanic source. In Fig. 6 it is shown that by increasing the Reynolds number, Re , the pressure at the plate also increases. Thus, for a large Reynolds number there would be great pressure at the wall which may cause the magma to break the earth surface resulting in the birth of a volcano. Thus the formation of the volcanic system strongly depends upon the viscosity of the bounded magma. Hence the eruption of the volcanic fluid can be controlled through the viscosity of the erupting magma. In magma fluids the viscosity is strongly dependent upon the temperature of the magma. In order to avoid the birth of a volcano, one should devise strategies to reduce the temperature of the bounded magma. The phenomenon of heat transfer in two-dimensional magma flow will be studied in the coming paper.

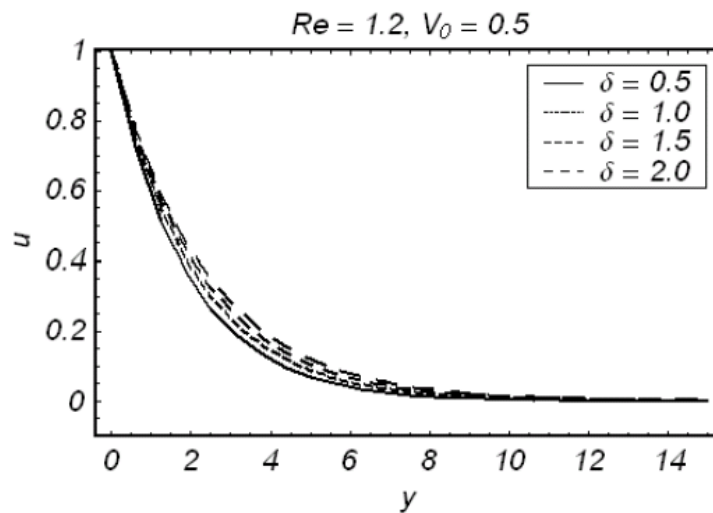


Fig. 1. Velocity profile at different values of non-Newtonian parameter

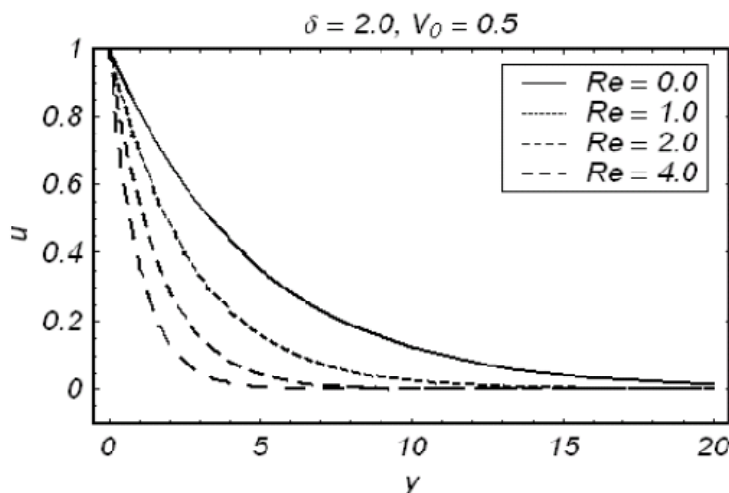


Fig. 2. Velocity profile at different values of Reynolds number

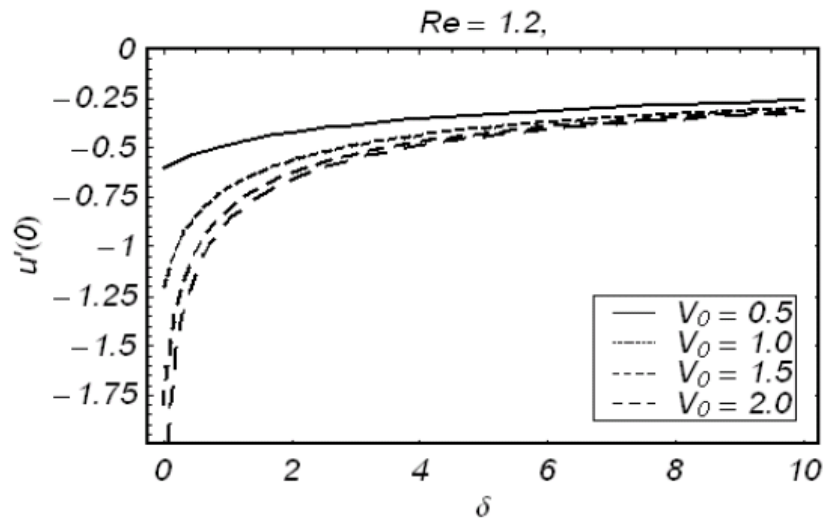


Fig. 3. Velocity gradients against viscoelastic parameter

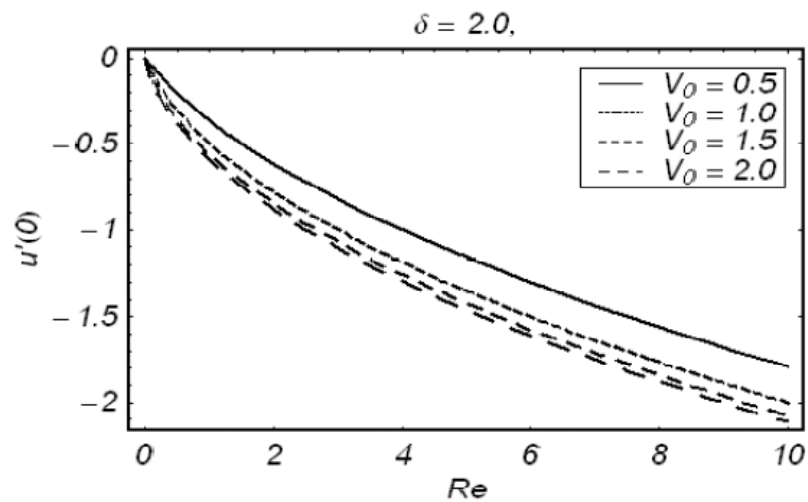


Fig. 4. Velocity gradients against the Reynolds number

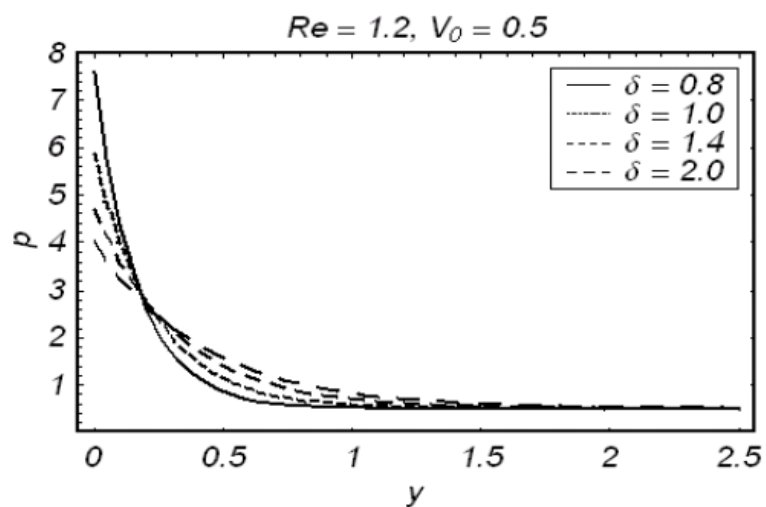


Fig. 5. Pressure distribution at different values of non-Newtonian parameter

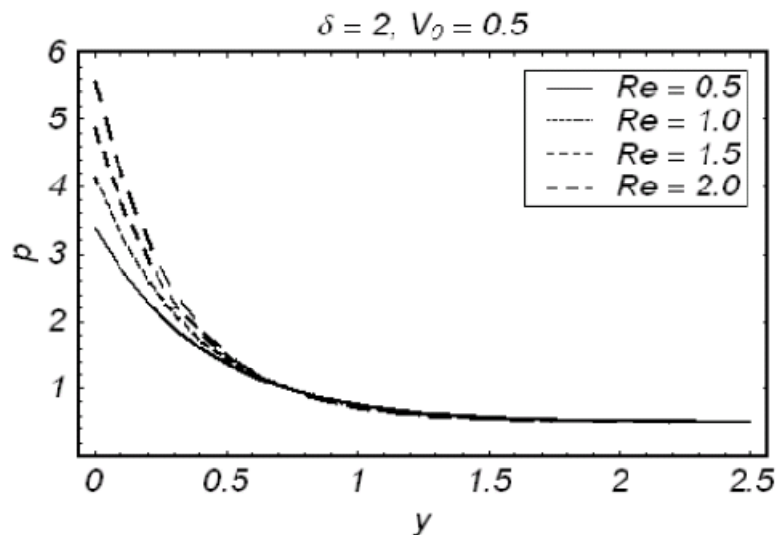


Fig. 6. Pressure graphs at different Reynolds number

3. CONCLUDING REMARKS

In the present investigation the flow of an incompressible magma fluid (represented by a second-grade model) has been studied. The method used to solve the nonlinear dynamic equations is the Lie-group method. The exact solution is obtained by using translational symmetries in the x- and y- directions. It is observed that the elastic parameter supports the enhancement of the velocity in the fluid and causes a reduction in the shear stress at the solid surface. On the other hand, strong Reynolds number causes large shear stress at the solid surface. Further, it is observed that the presence of strong viscoelastic properties in the magma help the bounding surface to sustain against the volcanic eruption. The magma pressure at the earth surface increases for large Reynolds number, which shows a strong dependence of magma pressure on the magma viscosity. Finally, the flow of magma at large Reynolds numbers and large seepage at the boundary results in the birth of a volcano.

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