

## JOHNSON AMENABILITY FOR TOPOLOGICAL SEMIGROUPS\*

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**Abstract** –A notion of amenability for topological semigroups is introduced. A topological semigroup  $S$  is called Johnson amenable if for every Banach  $S$ -bimodule  $E$ , every bounded crossed homomorphism from  $S$  to  $E^*$  is principal. In this paper it is shown that a discrete semigroup  $S$  is Johnson amenable if and only if  $\ell^1(S)$  is an amenable Banach algebra. Also, we show that if a topological semigroup  $S$  is Johnson amenable, then it is amenable, but the converse is not true.

**Keywords** – Amenability, crossed homomorphism, topological semigroup

### 1. INTRODUCTION

The Johnson's Theorem [1] asserts that a locally compact Hausdorff group  $G$  is amenable if and only if the Banach algebra  $L^1(G)$  is amenable. This is not true for discrete semigroups.

Duncan and Nomioka [2] showed that if  $\ell^1(S)$  is amenable, then  $S$  is amenable, and for a wide class of inverse semigroups  $S$ , they showed that  $\ell^1(S)$  fails to be amenable if  $E_S$  (the set of idempotent elements of  $S$ ) is infinite.

Amini [3] has recently introduced the notion of module amenability for Banach algebras and showed that, under some action for an inverse semigroup  $S$ ,  $\ell^1(S)$  is module amenable if and only if  $S$  is amenable.

In this paper we introduce the concept of Johnson amenability for topological semigroups. In particular we show that a discrete semigroup  $S$  is Johnson amenable if and only if  $\ell^1(S)$  is an amenable Banach algebra.

### 2. PRELIMINARIES

Let  $S$  be a topological semigroup, that is, a semigroup which is a topological space and the semigroup multiplication is separately continuous. A Banach space  $E$  is called a Banach  $S$ -bimodule, if there exists a two sided linear transitive action of  $S$  on  $E$  such that,

- $s \cdot (x \cdot t) = (s \cdot x) \cdot t$  for all  $s, t \in S$ ,  $x \in E$ ,
- if  $s_i \rightarrow s$  in  $S$  and  $x \in E$ , then  $s_i \cdot x \rightarrow s \cdot x$  and  $x \cdot s_i \rightarrow x \cdot s$  in the norm topology, and
- the action is bounded, that is, there is a  $M > 0$  such that for every  $x \in E$  and  $s \in S$ , we have

$$\|s \cdot x\| \leq M\|x\|, \quad \|x \cdot s\| \leq M\|x\|.$$

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We say that the right (respectively, left) action of  $S$  on  $E$  is trivial, if  $x \cdot s = x$  (respectively,  $s \cdot x = x$ ) for  $s \in S$  and  $x \in E$ ; Also, the right (respectively, left) action is called zero, if  $x \cdot s = 0$  (respectively,  $s \cdot x = 0$ ).

If  $E$  is a Banach  $S$ -bimodule, then the topological dual  $E^*$  of  $E$  is also an  $S$ -bimodule, where the action is defined by

$$\langle s \cdot f, x \rangle = \langle f, x \cdot s \rangle, \quad \langle f \cdot s, x \rangle = \langle f, s \cdot x \rangle \quad (s \in S, f \in E^*, x \in E).$$

Note that if  $s_i \rightarrow s$  in  $S$ ,  $f_i \rightarrow f$  in  $E^*$  in the weak\*-topology and  $\sup_i \|f_i\| < \infty$ , then  $s_i \cdot f_i \rightarrow s \cdot f$  and  $f_i \cdot s_i \rightarrow f \cdot s$  in the weak\*-topology and the dual action is also bounded.

Let  $C(S)$  be the Banach algebra of complex valued continuous bounded functions on  $S$ . Then  $C(S)$  is an  $S$ -bimodule via the following actions

$$a \cdot s(t) = a(st), \quad s \cdot a(t) = a(ts) \quad (s, t \in S, a \in C(S)).$$

We call these actions the right and the left function module actions respectively.

A function  $f \in C(S)$  is right uniformly continuous if  $\lim_i \|f \cdot s_i - f \cdot s\|_\infty = 0$  whenever  $s_i \rightarrow s$ . The Banach algebra of all right uniformly continuous functions on  $S$  is denoted by  $\mathbf{RUC}(S)$ . Similarly the Banach algebra of all left uniformly continuous functions on  $S$  is denoted by  $\mathbf{LUC}(S)$ .

Note that  $\mathbf{RUC}(S)$  (respectively,  $\mathbf{LUC}(S)$ ) is a Banach  $S$ -bimodule with the right (respectively, left) function module action and trivial left (respectively, right) action.

Let  $E$  be a linear subspace of  $C(S)$  which contains the constant function  $1_S$ . A mean on  $E$  is a functional  $m$  in  $E^*$ , such that  $m(1_S) = \|m\| = 1$ . Suppose that  $E$  is also closed under right function module action. Then the mean  $m$  is called left invariant if  $s \cdot m = m$  for all  $s \in S$ . Right invariant means are defined similarly.

**Definition 2.1.** A semigroup  $S$  is called left (respectively, right) amenable if there exists a left (respectively, right) invariant mean on  $\mathbf{RUC}(S)$  (respectively,  $\mathbf{LUC}(S)$ );  $S$  is called amenable if it is left and right amenable.

Recall that if  $S$  is left amenable with respect to some topology, then it is left amenable with respect to all topologies which are coarser than that. Thus a commutative topological semigroup is amenable, since it is amenable with discrete topology [4, page 16].

We now give the definition of amenability for Banach algebras. Recall that, for a Banach algebra  $A$ , a Banach space  $E$  is a Banach  $A$ -bimodule if  $E$  is an  $A$ -bimodule and there is a constant  $M$  such that  $\|a \cdot x\| \leq M \|a\| \|x\|$  and  $\|x \cdot a\| \leq M \|x\| \|a\|$  for each  $a$  in  $A$  and  $x$  in  $E$ .

If  $E$  is a Banach  $A$ -bimodule, then the dual space  $E^*$  is a Banach  $A$ -bimodule with the actions defined by  $\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle$  and  $\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle$ , for  $a$  in  $A$ ,  $x$  in  $E$  and  $f$  in  $E^*$ . A derivation of  $A$  into an  $A$ -bimodule  $E$  is a linear map  $D : A \rightarrow E$  such that  $D(ab) = a \cdot D(b) + D(a) \cdot b$ , for all  $a, b$  in  $A$ . For  $x$  in  $E$ , the map  $a \mapsto a \cdot x - x \cdot a$  is a derivation. Such derivations are called inner.

**Definition 2.2.** A Banach algebra  $A$  is amenable if for any Banach  $A$ -bimodule  $E$ , every continuous derivation  $D : A \rightarrow E^*$  is inner.

### 3. JOHNSON AMENABILITY

Let  $S$  be a topological semigroup and let  $E$  be a Banach  $S$ -bimodule. A bounded crossed homomorphism is a weak\*-continuous map  $D : S \rightarrow E^*$ , such that  $D(st) = s \cdot D(t) + D(s) \cdot t$  for every

$s, t \in S$  and  $\sup_{s \in S} \|D(s)\| < \infty$ . If  $f$  is in  $E^*$ , then  $d_f : S \rightarrow E^*$  defined by  $ad_f(s) = s \cdot f - f \cdot s$  is a bounded crossed homomorphism. Such bounded crossed homomorphisms are called principal.

**Definition 3.1.** Let  $S$  be a topological semigroup. Then  $S$  is called Johnson amenable if for every Banach  $S$ -bimodule  $E$ , every bounded crossed homomorphism from  $S$  to  $E^*$  is principal.

**Proposition 3.2.** For a topological semigroup  $S$  the following are equivalent.

- i.  $S$  is left amenable.
- ii. For every Banach  $S$ -bimodule  $E$  with trivial left action, any bounded crossed homomorphism  $D : S \rightarrow E^*$  is principal.

**Proof:** First, suppose that  $S$  is left amenable. Let  $E$  be a Banach  $S$ -bimodule and let  $D : S \rightarrow E^*$  be a bounded crossed homomorphism. For every  $x \in E$  we define  $\omega_x : S \rightarrow \mathbb{C}$  by  $\omega_x(s) = \langle D(s), x \rangle$ . Then  $\|\omega_x\|_\infty = \sup_{s \in S} |\omega_x(s)| \leq \sup_{s \in S} \|D(s)\| \|x\| \leq M \|x\|$ , where  $M > 0$  is a bound for  $D$ . Let  $s_\lambda \rightarrow s$  in  $S$ . Then  $D(s_\lambda) \rightarrow D(s)$  in the weak\*-topology. Thus  $\omega_x(s_\lambda) \rightarrow \omega_x(s)$ , that is,  $\omega_x$  is continuous. Let  $t_\lambda \rightarrow t$  in  $S$ . Then

$$\begin{aligned} \|\omega_x \cdot t_\lambda - \omega_x \cdot t\|_\infty &= \sup_{s \in S} |\omega_x(t_\lambda s) - \omega_x(ts)| \\ &= \sup_{s \in S} |\langle D(t_\lambda s), x \rangle - \langle D(ts), x \rangle| \\ &\leq \langle D(t_\lambda) - D(t), x \rangle + \sup_{s \in S} |\langle D(s), x \cdot t_\lambda - x \cdot t \rangle|. \end{aligned}$$

Since the net  $D(t_\lambda)$  is weak\* convergent to  $D(t)$ , then  $|\langle D(t_\lambda) - D(t), x \rangle| \rightarrow 0$ . Also,  $\sup_{s \in S} |\langle D(s), x \cdot t_\lambda - x \cdot t \rangle| \leq M \|x \cdot t_\lambda - x \cdot t\|$ . Therefore,  $\sup_{s \in S} |\langle D(s), x \cdot t_\lambda - x \cdot t \rangle| \rightarrow 0$  since  $x \cdot t_\lambda \rightarrow x \cdot t$  in norm and so  $\|\omega_x \cdot t_\lambda - \omega_x \cdot t\|_\infty \rightarrow 0$ , which implies that  $\omega_x \in \mathbf{RUC}(S)$ . Now, let  $m$  be a left invariant mean on  $\mathbf{RUC}(S)$ . Define a linear functional  $f$  on  $E$  by  $\langle f, x \rangle = m(\omega_x)$  for every  $x \in E$ . Then we have,

$$\|f\| = \sup_{\|x\| \leq 1} |\langle f, x \rangle| = \sup_{\|x\| \leq 1} |m(\omega_x)| \leq \sup_{\|x\| \leq 1} \|\omega_x\|_\infty \leq M.$$

Thus  $f \in E^*$ . For all  $x \in E$  and  $s, t \in S$ , we have

$$\begin{aligned} \omega_{x \cdot s}(t) &= \langle D(t), x \cdot s \rangle \\ &= \langle s \cdot D(t), x \rangle \\ &= \langle D(st), x \rangle - \langle D(s), x \rangle \\ &= \omega_x(st) - \langle D(s), x \rangle 1_S(t). \end{aligned}$$

Therefore,  $\omega_{x \cdot s} = \omega_x \cdot s - \langle D(s), x \rangle 1_S$ . This implies that

$$\begin{aligned} \langle f - s \cdot f, x \rangle &= \langle f, x \rangle - \langle f, x \cdot s \rangle \\ &= m(\omega_x - \omega_{x \cdot s}) \\ &= m(\omega_x - \omega_x \cdot s + \langle D(s), x \rangle 1_S) \\ &= \langle D(s), x \rangle, \end{aligned}$$

for every  $x \in E$  and  $s \in S$ . Thus  $D(s) = f - s \cdot f$  for all  $s \in S$ , and  $D$  is principal.

Conversely, consider the Banach  $S$ -bimodule  $E = \mathbf{RUC}(S)$  with trivial left action. Let  $F = E / \mathbb{C}1_S$ . Then  $F$  is a Banach  $S$ -bimodule and  $F^*$  is canonically isometrically isomorphic with the submodule  $L = \{f \in E^* : \langle f, 1_S \rangle = 0\}$  of  $E^*$ . In particular,  $L$  is the dual of a Banach  $S$ -bimodule. Let

$f \in E^* \setminus L$  be arbitrary (note that by the Hahn-Banach theorem  $E^* \setminus L \neq \emptyset$ ). Define  $D : S \rightarrow L$  by  $D(s) = s \cdot f - f$ . Clearly  $D$  is a bounded crossed homomorphism. Thus  $D$  is principal and so for some  $g \in L$  we have  $D(s) = s \cdot g - g$ . Thus for  $h = g - f$ , we have  $h \neq 0$  and  $s \cdot h = h$  for every  $s \in S$ . Since  $\text{RUC}(S)$  is a commutative  $C^*$ -algebra, there exists a compact Hausdorff space  $\Delta$  with a canonical left action of  $S$ , such that  $C(\Delta)$  and  $\text{RUC}(S)$  are isometrically  $*$ -isomorphic  $C^*$ -algebras and isomorphic  $S$ -modules. Thus one can consider  $h$  as a  $S$ -invariant complex Borel regular measure on  $\Delta$ . Now,  $|h| / \|h\|(\Delta)$  is an invariant mean for  $S$ , where  $|h|$  denotes total variation measure of  $h$ .

With the same argument of Proposition 3.2 one can prove that,  $S$  is right amenable, if and only if for every Banach  $S$ -bimodule  $E$ , with trivial right action, every bounded crossed homomorphism from  $S$  to  $E^*$  is principal. Thus we have,

**Theorem 3.3.** Let  $S$  be a topological semigroup. If  $S$  is Johnson amenable,  $S$  is amenable.

**Lemma 3.4.** Let  $S$  be a topological semigroup with a unit element  $e$  and let  $E$  be a Banach  $S$ -bimodule with either zero right action or zero left action. Then any bounded crossed homomorphism from  $S$  to  $E^*$  is principal.

**Proof:** Suppose that the right action is zero. Then the left action of  $S$  on  $E^*$  is zero. If  $D : S \rightarrow E^*$  is a bounded crossed homomorphism, then for every  $s \in S$  we have,  $D(s) = D(es) = e \cdot D(s) + D(e) \cdot s = D(e) \cdot s = D(e) \cdot s - s \cdot D(e) = \text{ad}_{-D(e)}(s)$ . Proof for the other case is similar.

Let  $S$  be a topological semigroup with a unit element  $e$ . A Banach  $S$ -bimodule  $E$  is called left-unital (respectively, right-unital) if  $e \cdot x = x$  (respectively,  $x \cdot e = x$ ) for all  $x \in E$ . Also,  $E$  is called unital if it is both left and right-unital.

**Lemma 3.5.** Let  $S$  be a topological semigroup with a unit element  $e$  and let  $E$  be a Banach  $S$ -bimodule. Let  $F = \{e \cdot x : x \in E\}$  (respectively,  $G = \{x \cdot e : x \in E\}$ ). Then  $F$  (respectively,  $G$ ) is a left-unital (respectively, right-unital) closed submodule of  $E$ . Also, if every bounded crossed homomorphism from  $S$  to  $F^*$  (respectively,  $G^*$ ) is principal, then every bounded crossed homomorphism from  $S$  to  $E^*$  is principal.

**Proof:** We prove the Lemma for  $F$ , the other case is similar. Note that  $F$  is a left unital submodule of  $E$ . Let  $x \in E$  be an accumulation point of  $F$ . Then there exists a net  $(e \cdot x_i)$  in  $F$  such that  $e \cdot x_i \rightarrow x$  in norm. Thus  $e \cdot x_i = e \cdot (e \cdot x_i) \rightarrow e \cdot x$  in norm. This implies  $x = e \cdot x \in F$ , thus  $F$  is closed. Now, suppose every bounded crossed homomorphism from  $S$  to  $F^*$  is principal. Let  $D : S \rightarrow E^*$  be a bounded crossed homomorphism and let  $\pi : E^* \rightarrow F^*$  be the restriction map. Then  $\pi$  is a bimodule homomorphism and thus  $\pi \circ D$  is a bounded crossed homomorphism from  $S$  to  $F^*$ . Thus there exists  $f \in F^*$  such that  $\pi \circ D = \text{ad}_f$ . Let  $\bar{f} \in E^*$  be such that  $\pi(\bar{f}) = f$  and let  $\tilde{D} = D - \text{ad}_{\bar{f}}$ . Then for every  $s \in S$ , the functional  $\tilde{D}(s)$  vanishes on  $F$ . By identification  $\{f \in E^* : f|_F = 0\} \cong (E/F)^*$ , we can suppose that  $\tilde{D}$  is a bounded crossed homomorphism from  $S$  to  $(E/F)^*$ . On the other hand, the left action of  $S$  on  $E/F$  is zero. Thus by Lemma 3.4,  $\tilde{D} = \text{ad}_g$  for some  $g \in E^*$ . Therefore we have  $D = \text{ad}_{f+g}$ .

**Proposition 3.6.** Let  $S$  be a topological semigroup with a unit element  $e$ . Then the following are equivalent.

i.  $S$  is Johnson amenable.

ii. For every unital Banach  $S$ -bimodule  $E$ , any bounded crossed homomorphism from  $S$  to  $E^*$  is principal.

**Proof:** (i)  $\Rightarrow$  (ii) is trivial.

(ii)  $\Rightarrow$  (i) Let  $E$  be a Banach  $S$ -bimodule and  $F_1 = \{x \cdot e : x \in E\}$ . By Lemma 3.5,  $F_1$  is a closed right-unital submodule of  $E$ . Also, let  $F_2 = \{e \cdot x \cdot e : x \in E\}$ . By Lemma 3.5,  $F_2$  is a closed (two sided) unital submodule of  $F_1$  and hence any bounded crossed homomorphism from  $S$  to  $F_2^*$  is principal, thus by Lemma 3.5, any bounded crossed homomorphism from  $S$  to  $F_1^*$  is principal. Again by Lemma 3.5, any bounded crossed homomorphism from  $S$  to  $E^*$  is principal. This completes the proof.

Recall that for a topological group  $G$ , the map  $(g, h) \mapsto gh^{-1}$  from  $G \times G$  to  $G$  is jointly continuous.

**Theorem 3.7.** Let  $G$  be a topological group. Then the following are equivalent.

- i.  $G$  is amenable.
- ii.  $G$  is Johnson amenable.

**Proof:** (ii)  $\Rightarrow$  (i) Follows from Theorem 3.3.

Suppose that  $G$  is amenable. By Proposition 3.6, it is enough to prove that if  $E$  is a unital Banach  $G$ -bimodule and  $D : G \rightarrow E^*$  is a bounded crossed homomorphism, then  $D$  is principal. Let  $E_{\#}$  be a Banach  $G$ -bimodule with underlying space  $E$  and trivial left action, and the right action defined by  $x * g := g^{-1} \cdot x \cdot g$  for  $x \in E$  and  $g \in G$ . Then the dual action of  $G$  on  $E_{\#}^*$  becomes,  $g * f = g \cdot f \cdot g^{-1}$  and  $f * g = f$  for  $f \in E^*$ . Define  $D_{\#} : G \rightarrow E_{\#}^*$  by  $D_{\#}(g) = D(g) \cdot g^{-1}$  ( $g \in G$ ). Then for any  $g, h \in G$ , we have

$$\begin{aligned} D_{\#}(gh) &= D(gh) \cdot (gh)^{-1} \\ &= D(g) \cdot (h h^{-1} g^{-1}) + g \cdot D(h) \cdot (h^{-1} g^{-1}) \\ &= D_{\#}(g) + g * D_{\#}(h) \\ &= D_{\#}(g) * h + g * D_{\#}(h). \end{aligned}$$

Thus  $D_{\#}$  is a bounded crossed homomorphism and by Proposition 3.2,  $D_{\#}$  is principal. Thus for some  $f \in E^*$  and every  $g \in G$ , we have  $D(g) \cdot g^{-1} = D_{\#}(g) = g * f - f = g \cdot f \cdot g^{-1} - f$ , that implies  $D(g) = g \cdot f - f \cdot g$ . Thus  $D$  is principal.

**Theorem 3.8.** Let  $G$  be a locally compact Hausdorff group. Then the following are equivalent.

- i.  $G$  is amenable.
- ii.  $G$  is Johnson amenable.
- iii.  $L^1(G)$  is an amenable Banach algebra.

**Proof:** (i) and (iii) are equivalent by Johnson's Theorem [1]. (i) and (ii) are equivalent by Theorem 3.7.

**Theorem 3.9.** Let  $S$  be a discrete semigroup. Then  $S$  is Johnson amenable if and only if  $\ell^1(S)$  is an amenable Banach algebra.

**Proof:** Suppose that  $\ell^1(S)$  is amenable. Let  $E$  be a Banach  $S$ -bimodule and let  $D : S \rightarrow E^*$  be a bounded crossed homomorphism. Then  $E$  by the actions

$$a \cdot x = \sum_{s \in S} a(s)(s \cdot x), \quad x \cdot a = \sum_{s \in S} a(s)(x \cdot s) \quad (a \in \ell^1(S), x \in E)$$

is a Banach  $\ell^1(S)$ -bimodule and  $D$  can be canonically extended to a bounded derivation  $\bar{D} : \ell^1(S) \rightarrow E^*$ , defined by  $\bar{D}(\delta_s) = D(s)$ . Thus there is  $f \in E^*$  such that  $\bar{D}(a) = a \cdot f - f \cdot a$  for all  $a \in \ell^1(S)$ . Thus  $D = ad_f$  and  $S$  is Johnson amenable.

Conversely, suppose that  $S$  is Johnson amenable. Let  $E$  be a Banach  $\ell^1(S)$ -bimodule and let  $D : \ell^1(S) \rightarrow E^*$  be a bounded derivation. Then  $E$  is a Banach  $S$ -bimodule by the actions

$$s \cdot x = \delta_s \cdot x, \quad x \cdot s = x \cdot \delta_s \quad (s \in S, x \in E).$$

Also,  $\tilde{D} : S \rightarrow E^*$  defined by  $\tilde{D}(s) = D(\delta_s)$  is a bounded crossed homomorphism. Thus there is  $f \in E^*$  such that for all  $s \in S$ ,  $\tilde{D}(s) = s \cdot f - f \cdot s$ . This implies that  $D(a) = a \cdot f - f \cdot a$  for all  $a \in \ell^1(S)$ , and thus  $D$  is an inner derivation.

#### 4. HEREDITARY PROPERTIES

**Proposition 4.1.** Let  $S$  and  $T$  be topological semigroups and let  $\phi : T \rightarrow S$  be a continuous semigroup homomorphism with dense range. If  $T$  is Johnson amenable, then so is  $S$ .

**Proof:** Suppose that  $T$  is Johnson amenable. Let  $E$  be a Banach  $S$ -bimodule and  $D : S \rightarrow E^*$  be a bounded crossed homomorphism. Then  $E$  is a Banach  $T$ -bimodule by the action,

$$t \cdot x = \phi(t) \cdot x, \quad x \cdot t = x \cdot \phi(t) \quad (t \in T, x \in E).$$

Also,  $D \circ \phi : T \rightarrow E^*$  is a bounded crossed homomorphism. Thus there exists  $f \in E^*$  such that for all  $t \in T$ ,  $D(\phi(t)) = t \cdot f - f \cdot t$ . Since  $\phi(T)$  is dense in  $S$  and  $D$  is continuous, we have  $D(s) = s \cdot f - f \cdot s$  for all  $s \in S$ .

**Corollary 4.2.** Let  $S$  be a topological semigroup and  $T$  be a dense topological subsemigroup of  $S$ . Then, if  $T$  is Johnson amenable, then so is  $S$ .

**Proof:** Apply Proposition 4.1 with the identity continuous homomorphism  $id : T \rightarrow S$ .

Let  $G$  be a locally compact Hausdorff non compact group, and  $S = G \cup \{\infty\}$  be its one point compactification. Extend the semigroup operation of  $G$  to  $S$  by putting  $g\infty = \infty g = \infty\infty = \infty$  ( $g \in G$ ). Then  $S$  becomes a compact Hausdorff topological semigroup which is not a group and has  $G$  as a dense subsemigroup. Thus by Theorem 3.7 and Corollary 4.2, if  $G$  is an amenable group, then  $S$  is Johnson amenable.

**Corollary 4.3.** Let  $S$  be a semigroup and let  $\tau$  and  $\tau'$  be two topologies on  $S$  for which  $S$  is a topological semigroup, such that  $\tau \subset \tau'$ . If  $S$  is Johnson amenable with topology  $\tau'$ , then  $S$  is Johnson amenable with  $\tau$ .

**Proof:** Apply Proposition 4.1 with the identity continuous homomorphism  $id : (S, \tau') \rightarrow (S, \tau)$ .

**Corollary 4.4.** Let  $\{S_i\}_{i \in I}$  be a class of topological semigroups. Consider the topological semigroup  $\prod_{i \in I} S_i$ , with product topology. If  $\prod_{i \in I} S_i$  is Johnson amenable, then so is  $S_i$  for every  $i \in I$ .

**Proof:** For every  $j \in I$ , consider the canonical projection  $\prod_{i \in I} S_i \rightarrow S_j$  and apply Proposition 4.1.

**Proposition 4.5.** Let  $S$  be a topological semigroup and let  $(I, <)$  be a directed set. Suppose that for any  $i \in I$ ,  $S_i$  is a topological subsemigroup of  $S$  such that

- i. if  $i < j$ , then  $S_i \subset S_j$ ,
- ii.  $S_0 = \cup_{i \in I} S_i$  is dense in  $S$ , and
- iii. there exists a  $K > 0$  such that for every  $i \in I$ , for every Banach  $S_i$ -bimodule  $E$  and for each bounded crossed homomorphism  $D : S_i \rightarrow E^*$ , there exists  $f \in E^*$  with  $D(s) = s \cdot f - f \cdot s$  ( $s \in S_i$ ) and  $\|f\| \leq K \sup_{s \in S_i} \|D(s)\|$ . Then  $S$  is Johnson amenable.

**Proof:** By Corollary 4.2, it is enough to prove that  $S_0$  is Johnson amenable. Let  $E$  be a Banach  $S_0$ -bimodule and  $D : S_0 \rightarrow E^*$  be a bounded crossed homomorphism. For every  $i \in I$ , let  $f_i$  be in  $E^*$  such that  $D|_{S_i}(s) = s \cdot f_i - f_i \cdot s$  ( $s \in S_i$ ) and  $\|f_i\| \leq K \sup_{s \in S_i} \|D(s)\| \leq K \sup_{s \in S_0} \|D(s)\|$ . Then  $(f_i)_{i \in I}$  is a bounded net in  $E^*$ , and thus has a weak\*-accumulation point  $f$ . By passing to a subnet we may assume  $f_i \xrightarrow{w^*} f$ . Now, if  $s \in S_0$ , then for some  $i_0$ , we have  $s \in S_i$  for all  $i \geq i_0$ , and thus for every  $x \in E$ ,

$$\begin{aligned} \langle D(s), x \rangle &= \langle s \cdot f_i - f_i \cdot s, x \rangle \\ &= \langle f_i, x \cdot s - s \cdot x \rangle \\ &\rightarrow \langle f, x \cdot s - s \cdot x \rangle \\ &= \langle s \cdot f - f \cdot s, x \rangle. \end{aligned}$$

Thus for every  $s \in S_0$ ,  $D(s) = s \cdot f - f \cdot s$  and  $D$  is principal. This completes the proof.

**Theorem 4.6.** Let  $S$  and  $T$  be topological semigroups with unit elements. If  $S$  and  $T$  are Johnson amenable, then so is  $S \times T$ .

**Proof:** Let  $e$  and  $e'$  denote the units of  $S$  and  $T$ , respectively. Consider topological subsemigroups  $\hat{S} = S \times \{e'\}$  and  $\hat{T} = \{e\} \times T$  of  $S \times T$ . Clearly,  $\hat{S}$  and  $\hat{T}$  are Johnson amenable. Let  $E$  be a Banach  $S \times T$ -bimodule and  $D : S \times T \rightarrow E^*$  be a bounded crossed homomorphism. Then  $E$  is canonically a Banach  $\hat{S}$ -bimodule and a Banach  $\hat{T}$ -bimodule.

Consider the bounded crossed homomorphism  $D|_{\hat{S}} : \hat{S} \rightarrow E^*$ . By Johnson amenability of  $\hat{S}$ , there is some  $f_0 \in E^*$ , such that for all  $s \in S$ ,

$$D(s, e') = (s, e') \cdot f_0 - f_0 \cdot (s, e') \quad (1)$$

Now, consider bounded crossed homomorphism  $\tilde{D} := D - \text{ad}_{f_0}$  from  $S \times T$  to  $E^*$ . Then  $\tilde{D}|_{\hat{S}} = 0$  and for all  $(s, t) \in S \times T$  we have,

$$\begin{aligned} \tilde{D}(s, t) &= \tilde{D}((s, e')(e, t)) \\ &= \tilde{D}(s, e') \cdot (e, t) + (s, e') \cdot \tilde{D}(e, t) \\ &= (s, e') \cdot \tilde{D}(e, t). \end{aligned}$$

Similarly,  $\tilde{D}(s, t) = \tilde{D}(e, t) \cdot (s, e')$ . Thus if  $F = \{f \in E^* : (s, e') \cdot f = f \cdot (s, e') \text{ for all } s \in S\}$ , then the range of  $\tilde{D}$  is in  $F$ . On the other hand if  $L$  is the closed linear span of  $\{(s, e') \cdot x - x \cdot (s, e') : s \in S, x \in E\}$ , then  $L$  is a Banach  $\hat{T}$ -submodule of  $E$ . Thus  $F$  is the dual of a Banach  $\hat{T}$ -bimodule, since  $F$  is identical with  $(E/L)^*$ . Therefore,  $\tilde{D}|_{\hat{T}} : \hat{T} \rightarrow F$  is a bounded crossed homomorphism and thus for some  $f_1 \in F \subset E^*$ , we have  $\tilde{D}|_{\hat{T}} = \text{ad}_{f_1}|_{\hat{T}}$ , or equivalently, for all  $t \in T$ ,

$$\tilde{D}(e, t) = (e, t) \cdot f_1 - f_1 \cdot (e, t), \quad (2)$$

and for all  $s \in S$ ,

$$(s, e') \cdot f_1 = f_1 \cdot (s, e'). \quad (3)$$

Now from (1), (2) and (3) we have for all  $(s, t) \in S \times T$ ,

$$\begin{aligned} D(s, t) &= D((s, e')(e, t)) \\ &= D(s, e') \cdot (e, t) + (s, e') \cdot D(e, t) \\ &= ((s, e') \cdot f_0 - f_0 \cdot (s, e')) \cdot (e, t) \\ &\quad + (s, e') \cdot ((e, t) \cdot f_1 - f_1 \cdot (e, t) + (e, t) \cdot f_0 - f_0 \cdot (e, t)) \\ &= (s, t) \cdot f_1 - f_1 \cdot (s, t) + (s, t) \cdot f_0 - f_0 \cdot (s, t) \\ &= ad_{f_0+f_1}(s, t). \end{aligned}$$

This completes the proof.

## 5. SOME EXAMPLES AND APPLICATIONS

Let  $A$  be a Banach algebra. By a *structural semigroup* of  $A$ , we mean a subset  $S$  of  $A$ , such that

- i.  $S$  is closed under multiplication,
- ii. the linear span of  $S$  is norm dense in  $A$ , and
- iii.  $\sup_{s \in S} \|s\| < \infty$ .

We consider  $S$  as a topological semigroup with topology induced by the norm of  $A$ .

**Theorem 5.1.** Let  $A$  be a Banach algebra and let  $S$  be a structural semigroup of  $A$ . If  $S$  is Johnson amenable, then  $A$  is amenable.

**Proof:** Let  $E$  be a Banach  $A$ -bimodule and let  $D : A \rightarrow E^*$  be a bounded derivation. Then it is easily checked that  $E$  is a Banach  $S$ -bimodule with the same action as  $A$ , and the map  $D|_S : S \rightarrow E^*$  is a crossed homomorphism. Since  $D$  is a bounded derivation and  $\sup_{s \in S} \|s\| < \infty$ , then  $\sup_{s \in S} \|D|_S(s)\| < \infty$ . Also,  $D|_S$  is continuous in the weak\*-topology of  $E^*$ , since it is continuous in the norm topology of  $E^*$ . Therefore,  $D|_S$  is a bounded crossed homomorphism. Since  $S$  is Johnson amenable, then there is  $f \in E^*$  such that  $D(s) = s \cdot f - f \cdot s$  for all  $s \in S$ . Since  $A$  is the closed linear span of  $S$ , we have  $D(a) = a \cdot f - f \cdot a$  for all  $a \in A$ , and the proof is complete.

Let  $A$  be a non-amenable commutative Banach algebra. Let  $S$  be a structural semigroup of  $A$  defined by,

$$S = \{a \in A : \|a\| < 1\}.$$

Then  $S$  is an amenable topological semigroup since  $S$  is commutative. But by Theorem 5.1,  $S$  is not Johnson amenable.

Let  $B$  be a unital commutative C\*-algebra and  $G$  be the unitary group of  $B$ . Then  $G$  is a structural semigroup of  $B$ . On the other hand,  $G$  is abelian and thus an amenable group. Then by Theorem 3.7,  $G$  is Johnson amenable. Thus by Theorem 5.1,  $B$  is an amenable Banach algebra.

A Banach algebra  $A$  is called *dual*, if there exists a closed submodule  $A_*$  of  $A^*$  such that  $A = (A_*)^*$ , see [5, Section 4.4]. Let  $A$  be a dual Banach algebra. In what follows, we shall therefore suppose that  $A$  always comes with a fixed  $A_*$ . It is easily checked that the multiplication of  $A$  is separately weak\*-continuous. Let  $E$  be a Banach  $A$ -bimodule. We call  $E$  *pre normal*  $A$ -bimodule if for each  $x \in E$ , the maps  $a \mapsto a \cdot x$  and  $a \mapsto x \cdot a$  from  $A$  to  $E^*$  are weak\*-continuous.

The dual Banach algebra  $A$  is called *Connes-amenable*, if for every pre normal Banach  $A$ -bimodule  $E$ , every weak\*-continuous derivation  $D : A \rightarrow E^*$  is inner. For more details on Connes-amenability, we refer the reader to [5-8].

For the dual Banach algebra  $A$ , we call a subset  $S$  of  $A$ , *dual structural semigroup* of  $A$ , if it satisfies (i) and (iii) of the definition of structural semigroup and also satisfies

ii'. the linear span of  $S$  is weak\*-dense in  $A$ .

We always consider the dual structural semigroup  $S$  as a topological semigroup with induced weak\*-topology of the dual Banach algebra  $A$ .

The proof of the following is similar to the proof of Theorem 5.1.

**Theorem 5.2.** Let  $A$  be a dual Banach algebra and  $S$  be a dual structural semigroup of  $A$ . If  $S$  is Johnson amenable, then  $A$  is Connes-amenable.

In [8] it was shown that for any locally compact group  $G$ , the measure algebra  $M(G)$  is Connes-amenable if and only if  $G$  is amenable. Now, we can prove the "if" part of this result by our method.

**Theorem 5.3.** Let  $G$  be an amenable locally compact group. Then  $M(G)$  is Connes-amenable.

**Proof:** Let  $\delta : G \rightarrow M(G)$  be the usual pointmass measure map. Then  $\delta$  is a continuous homomorphism in the weak\*-topology and convolution product of  $M(G)$ . Thus by Theorem 3.7 and Proposition 4.1, the topological semigroup  $\delta(G)$  is Johnson amenable. Also,  $\delta(G)$  is a dual structural semigroup for dual Banach algebra  $M(G)$ . Thus by Theorem 5.2,  $M(G)$  is Connes-amenable.

The following is a Hann-Banach theorem. This is similar to Proposition 2 of [9] in the Banach algebra case, see also [10]. We call an element  $x$  of  $S$ -module  $E$  *symmetric* if for every  $s$  in  $S$ ,  $s \cdot x = x \cdot s$ .

**Theorem 5.4.** Let  $S$  be a Johnson amenable topological semigroup. Let  $E$  be a Banach  $S$ -bimodule and  $F$  be a Banach submodule of  $E$ . Then any symmetric functional in  $F^*$  has an extension to a symmetric functional in  $E^*$ .

**Proof:** The quotient Banach space  $Y = E / F$  is a Banach  $S$ -bimodule by the actions  $s \cdot (x + F) = s \cdot x + F$  and  $(x + F) \cdot s = x \cdot s + F$  for  $s$  in  $S$  and  $x$  in  $E$ . Let  $f$  be a symmetric element of  $F^*$  and  $\hat{f}$  be any continuous extension of  $f$  on  $E$ . For every  $s$  in  $S$ ,  $s \cdot \hat{f} - \hat{f} \cdot s$  is in  $F^\perp$ , the Banach space of all functional in  $E^*$  that vanish on  $F$ . Let  $Q$  be the canonical isometry from  $F^\perp$  onto  $(E / F)^*$ . Then the map  $\delta(s) = Q(s \cdot \hat{f} - \hat{f} \cdot s)$  is a bounded crossed homomorphism, since  $Q$  is weak\*-weak\* continuous and  $S$ -bimodule homomorphism. Since  $S$  is Johnson amenable, there exists  $h$  in  $F^\perp$  such that  $\delta(s) = s \cdot Q(h) - Q(h) \cdot s$ , for all  $s$  in  $S$ . It follows that  $\bar{f} = \hat{f} - h$  is a symmetric extension of  $f$  on  $E$ .

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